Unit 2: Applications in Algebra and Calculus (H7X1 77)

Applying Algebraic Skills to Sequences and Series

Sequence: list of terms following a given rule

e.g. $3, 8, 13, 18, \ldots.$ $n \to 5n - 2$

Series: sum of these terms e.g. $3 + 8 + 13 + 18 + \ldots..$

Arithmetic Sequence:

An Arithmetic sequence is created when the same amount is added to u_n to produce u_{n+1}. This constant amount is known as the common difference and is denoted by d.

The first term of any sequence is denoted by a.

General Arithmetic Sequence: $a, a + d, a + d + d, \ldots = a, a + d, a + 2d, \ldots.$

Thus $u_1 = a, u_2 = a + d, u_3 = a + 2d, u_4 = a + 3d, \ldots.. ., u_n = a + (n - 1)d$

$u_n = a + (n - 1)d$ is the general formula for an arithmetic sequence

❶ Find the formula for the n^{th} term of the sequence 6, 11, 16 and hence find u_{10}.

$$a = 6, d = 5 \Rightarrow u_n = 6 + (n - 1) \times 5 \Rightarrow u_n = 5n + 1$$

$$u_{10} = 5 \times 10 + 1 = 51$$

❷ Find the arithmetic sequence whose third term is 9 and 7th term is 17.

$u_3 = a + 2d \Rightarrow a + 2d = 9$ *Simultaneous equations*

$u_7 = a + 6d \Rightarrow a + 6d = 17$ *Subtract*

$\Rightarrow 4d = 8 \Rightarrow d = 2 \Rightarrow a = 5$ *First term is 2 then add 5*

Sequence is: 5, 7, 9, 11, · · · ·

❸ Find the position of term 62 in the sequence of $2, 8, 14, 20, \ldots.$

$a = 2, d = 6 \Rightarrow u_n = 2 + (n - 1) \times 6 \Rightarrow u_n = 6n - 4$

$6n - 4 = 62 \Rightarrow n = 11$ *so 62 is the 11th term in the sequence*

Bk 2 P117 Ex2A

Q2(1st col), 3-6

Sum to n terms of an Arithmetic Series - S_n

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+(n-2)d) + (a+(n-1)d)$$

Reversing this calculation gives

$$S_n = (a+(n-1)d) + (a+(n-2)d) + \ldots + (a+2d) + (a+d) + a$$

Adding both together vertically gives

$$2S_n = (2a+(n-1)d) + (2a+(n-1)d) + \ldots + (2a+(n-1)d) + (2a+(n-1)d)$$

$2S_n = n(2a+(n-1)d)$ since there are n lots:-

$$\boxed{S_n = \frac{n}{2}(2a+(n-1)d)}$$

This is the general formula for the sum to n terms of an Arithmetic Series and is given in the formula sheet

❹ Find the sum of the first 15 terms of the sequence 3, 8, 13, 18

$$a = 3, d = 5 \Rightarrow S_{15} = \frac{15}{2}(2 \times 3 + (15-1) \times 5) = 570$$

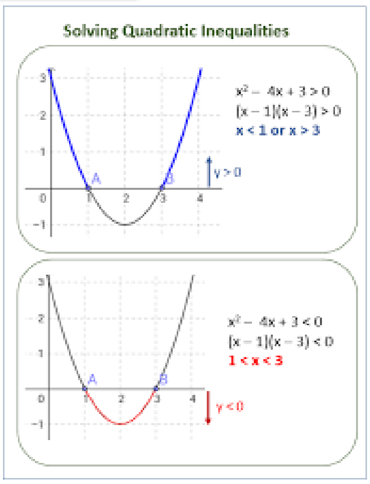
❺ When does the series $2 + 10 + 18 + 26 + \cdots$ first exceed 300?

$$a = 2, d = 8 \Rightarrow S_n = \frac{n}{2}(2 \times 2 + (n-1) \times 8) = \frac{n}{2}(8n-4)$$

$$S_n > 300 \Rightarrow \frac{n}{2}(8n-4) > 300 \Rightarrow 4n^2 - 2n > 300$$

$$\Rightarrow 4n^2 - 2n - 300 > 0 \Rightarrow 2n^2 - n - 150 > 0$$

Remember Quadratic Inequalities from Higher?



Using the quadratic formula $x = \frac{1 \pm \sqrt{1-4\times2\times(-150)}}{4}$

$$x = \frac{1 \pm \sqrt{1201}}{4} \Rightarrow x = -8.4 \; or \; 8.9$$

$$n < -8.4 \; or \; n > 8.9$$

$n = 9$ since it is a Natural number

Bk2 P120 Ex3A Q3(1st col), 4-8

S_n can also be found via:

$$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}\big(a + (a + (n-1)d)\big) = \frac{n}{2}(a + l)$$

where $l = last\ term = a + (n-1)d$

Geometric Sequence:

A Geometric sequence is created when the previous term is multiplied by the same factor to get the next term. This multiplying factor is known as the common ratio and is denoted r.

$$r = \frac{u_2}{u_1} = \frac{u_3}{u_2} =$$

General Geometric Sequence: $a, a \times r, a \times r \times r, \ldots = a, ar, ar^2, \ldots.$

Thus $u_1 = a, u_2 = ar, u_3 = ar^2, u_4 = ar^3, \ldots\ldots, u_n = ar^{(n-1)}$

$u_n = ar^{(n-1)}$ is the general formula for an geometric sequence

❻ Find the formula for the nth term and hence the 10th term of the sequence $3, 12, 48, \ldots$

$a = 3, r = \frac{u_2}{u_1} = \frac{12}{3} = 4$ so $u_n = 3 \times 4^{(n-1)}$

$u_{10} = 3 \times 4^{(10-1)} = 3 \times 4^{(9)} = 786432$

❼ Find the geometric sequence whose 3rd term is 18 and 8th term is 4374.

$u_3 = 18 \Rightarrow ar^2 = 18$ $\qquad$ $u_8 = 4374 \Rightarrow ar^7 = 4374$

Divide: $\frac{u_8}{u_3} = \frac{ar^7}{ar^2} = \frac{4374}{18} \Rightarrow r^5 = 243 \Rightarrow r = 3$

$$ar^2 = 18 \Rightarrow 9a = 18 \Rightarrow a = 2$$

Sequence is 2, 6, 18, 54, · · ·

Bk 2 P120 Ex4A

Q1(1st col), 2-5

Sum to n terms of a Geometric Series - S_n

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

Multiply through by r

$$rS_n = ar + ar^2 + ar^3 + ar^4 + \ldots + ar^{n-1} + ar^n$$

Subtracting gives

$$S_n - rS_n = a - ar^n \Rightarrow S_n(1-r) = a(1-r^n)$$

$$\boxed{S_n = \frac{a(1-r^n)}{(1-r)}}$$

This is the general formula for the sum to n terms of a Geometric Series and is given in the formula sheet

❽ For the sequence 12, 15, 18.75, . . find the smallest value of n s.t. $S_n > 100$

$a = 12, r = \frac{15}{12} = \frac{5}{4}$ so $S_n = \frac{12\left(1-\left(\frac{5}{4}\right)^n\right)}{\left(1-\frac{5}{4}\right)} = -48\left(1-\left(\frac{5}{4}\right)^n\right)$

$S_n > 100 \Rightarrow = -48\left(1-\left(\frac{5}{4}\right)^n\right) > 100 \Rightarrow 1-\left(\frac{5}{4}\right)^n < -\frac{25}{12}$ NB > reversed

$\Rightarrow -\left(\frac{5}{4}\right)^n < -\frac{37}{12} \Rightarrow \left(\frac{5}{4}\right)^n > \frac{37}{12}$ < reversed again!

Bk2 P127 Ex5A
Q1&2 (1st column)
Q3-5

$$n\ln\frac{5}{4} > \ln\frac{37}{12} \Rightarrow n > \frac{\ln\frac{37}{12}}{\ln\frac{5}{4}} \Rightarrow n > 5.046$$

So 6 terms would be added for the series to exceed 100.

❾ Find the sequence whose 1st 3 terms add to 14 and first 6 terms add to 126

$$S_3 = \frac{a(1-r^3)}{(1-r)} = 14 \qquad S_6 = \frac{a(1-r^6)}{(1-r)} = 126$$

Divide: $\frac{S_6}{S_3} = \frac{126}{14} = \frac{\frac{a(1-r^6)}{(1-r)}}{\frac{a(1-r^3)}{(1-r)}} = \frac{a(1-r^6)}{(1-r)} \times \frac{(1-r)}{a(1-r^3)} = \frac{(1-r^6)}{(1-r^3)} = 9$

X-multiply and re-arrange: $9 - 9r^3 = 1 - r^6 \Rightarrow r^6 - 9r^3 + 8 = 0$

Factorise: $(r^3-8)(r^3-1) = 0 \Rightarrow r = 2 \text{ or } r = 1 \quad r \neq 1\ (a, a, a, etc)$

$r = 2 \Rightarrow \frac{a(1-2^3)}{(1-2)} = 14 \Rightarrow a = 2$ so 2, 4, 8, 16

Sum to infinity of a Geometric Series - S_∞

We know $S_n = \frac{a(1-r^n)}{(1-r)}$ If $r > 1$ then as $n \to \infty$, r^n becomes dominant.

If $|r| < 1$, then as $n \to \infty$, $r^n \to 0$ and so $S_n = \frac{a(1-0)}{(1-r)}$

i.e. $S_\infty = \frac{a}{1-r}$ when $|r| < 1$

❶⓿ Calculate 24 + 12 + 6 + 3 + . . . i.e. S_∞

$a = 24, r = \frac{u_2}{u_1} = \frac{12}{24} = 0.5$ so $|r| < 1$ so limit exists.

$$S_\infty = \frac{a}{1-r} = \frac{24}{1-0.5} = 48$$

❶❶ Express 0.1212121212… as a vulgar fraction.

$0.1212121212\ldots = 0.12 + 0.0012 + 0.000012 + 0.00000012$

$a = 0.12, r = \frac{u_2}{u_1} = \frac{0.0012}{0.12} = 0.01$ so $|r| < 1$ so limit exists.

$$S_\infty = \frac{a}{1-r} = \frac{0.12}{1-0.01} = \frac{0.12}{0.99} = \frac{12}{99} = \frac{4}{33}$$

❶❷ Given that 12 and 3 are adjacent terms in an infinite geometric series with $S_\infty = 64$, find the first term.

$r = \frac{u_2}{u_1} = \frac{3}{12} = \frac{1}{4}$ so $|r| < 1$ so limit exists.

$$S_\infty = 64 = \frac{a}{1-\frac{1}{4}} = \frac{a}{\frac{3}{4}} \Rightarrow a = 48$$

❶❸ Obtain the infinite geometric series for $S_\infty = \frac{2}{2-3k}$ and find the values for r that make this series valid.

$$\frac{2}{2-3k} = \frac{2}{2\left(1-\frac{3}{2}k\right)} = \frac{1}{1-\frac{3}{2}k} \Rightarrow a = 1, r = \frac{3}{2}k$$

Bk2 P131 Ex6A
Q3, 5, 6

$$1 + \frac{3}{2}k + \frac{9}{4}k^2 + \frac{27}{8}k^3 + \cdots$$

$$\left|\frac{3}{2}k\right| < 1 \Rightarrow |k| < \frac{2}{3}$$

Sigma Notation $\sum$ - the sum of, (standard deviation, Binomial)

$$\sum_{r=1}^{n} f(r) = f(1) + f(2) + \cdots + f(n)$$

❶❹ Expand $\sum_{r=1}^{4} r!$

$$\sum_{r=1}^{4} r! = 1! + 2! + 3! + 4! = 33$$

If $f(r)$ contains more than one term then it can be split and constants removed (similar to Integration):

$$\boxed{\sum_{r=1}^{n} ar + b = \sum_{r=1}^{n} ar + \sum_{r=1}^{n} b = a\sum_{r=1}^{n} r + b \times n}$$

NB – To calculate part of a series, find the whole series subtract the start:

$$\boxed{\sum_{r=a}^{n} f(r) = \sum_{r=1}^{n} f(r) - \sum_{r=1}^{a-1} f(r)}$$

Sum to n terms of r, r^2 and r^3 - all given on the formula sheet

$$\sum_{r=1}^{n} r = 1 + 2 + \cdots + n$$

This is an Arithmetic series with $a = 1$ and $d = 1$ i.e.

$$S_n = \frac{n}{2}(2 \times 1 + (n-1) \times 1)$$

$$\boxed{\sum_{r=1}^{n} r = \frac{n}{2}(n+1)}$$

Also:

$$\boxed{\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}} \qquad \boxed{\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}}$$

❶❺ Write as a fully factorised formula: $\sum_{r=1}^{n}\left(r^2-\frac{1}{3}r\right)$

$$\sum_{r=1}^{n}\left(r^2-\frac{1}{3}r\right)=\sum_{r=1}^{n}r^2-\frac{1}{3}\sum_{r=1}^{n}r=\frac{n(n+1)(2n+1)}{6}-\frac{1}{3}\times\frac{n}{2}(n+1)$$

$$=\frac{n(n+1)[(2n+1)-1]}{6}=\frac{n^2(n+1)}{3}$$

❶❻ Write the series $18+28+38+\cdots+108$ in sigma notation

Arithmetic series so $u_n=a+(n-1)d$ *with* $a=18$ *and* $d=10$

$u_n=108=18+(n-1)10\Rightarrow n=10$ *i.e. upper limit.*

$$\sum_{r=1}^{10}(10r+8)$$

❶❼ Write the series $-26+22-18+\cdots+(-46)$ in sigma notation

Alternating $+/-$ *means we have* $(-1)^r$ *as a factor*

so $u_n=(-1)^n[a+(n-1)d]$ *with* $a=26$ *and* $d=-4$

$$u_n=-46=26+(n-1)(-4)\Rightarrow n=19$$

$$\sum_{r=1}^{19}(-1)^r(30-4r)$$

Expanding $(1-x)^{-1}$ and related functions

$$(1-x)^{-1}=\frac{1}{1-x}$$

If we compare this to the sum to infinity of a geometric series $|r|<1$:

$$S_\infty=\frac{a}{1-r}$$

We get the series with $a=1$ and $r=x$: $1+x+x^2+x^3+x^4+\cdots$

$$(1-x)^{-1}=1+x+x^2+x^3+x^4+\cdots$$

Similarly:

$$(1+x)^{-1}=\left(1-(-x)\right)^{-1}=1+(-x)+(-x)^2+(-x)^3+(-x)^4+\cdots$$

i.e.

$$(1+x)^{-1}=1-x+x^2-x^3+x^4+\cdots$$

❶❽ Expand $\frac{3}{1+x-2x^2}$

Step 1: Convert to partial fractions: $\frac{3}{1+x-2x^2} = \frac{A}{1+2x} + \frac{B}{1-x}$

$$A(1-x) + B(1+2x) = 3$$

$$x = 1 \Rightarrow B = 1 \qquad\qquad x = -0.5 \Rightarrow A = 2$$

Step 2: Substitute $\frac{3}{1+x-2x^2} = \frac{2}{1+2x} + \frac{1}{1-x} = 2(1+2x)^{-1} + (1-x)^{-1}$

Step 3: Expand:

$2(1+2x)^{-1} + (1-x)^{-1} = 2(1 - 2x + 4x^2 - 8x^3 + \cdots) + (1 + x + x^2 + x^3 + \cdots)$

Step 4: Simplify

$$= 3 - 3x + 9x^2 - 15x^3 + \cdots$$

Bk2 P134 Ex7A Q3

Bk2 P134 Ex7B Q2, 3, 5

Bk2 P137Ex8 Q1(1st col), 3, 5(1st col)

Maclaurin Expansions

Suppose that: $f(x) = a + bx + cx^2 + dx^3$

then $f'(x) = b + 2cx + 3dx^2$

$$f''(x) = 2c + 6dx$$

$$f'''(x) = 6d$$

From above: $f(0) = a$, $f'(0) = b$, $f''(0) = 2c \Rightarrow c = \frac{f''(0)}{2!}$ and $f'''(0) = 6d \Rightarrow d = \frac{f'''(0)}{3!}$

Hence $f(x) = a + bx + cx^2 + dx^3$ can be written as:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

Any series of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$ is called a **Power Series**. In many cases, the sum of such series becomes bigger and bigger as you add on successive terms, in which case the series is said to **diverge**.

On the other hand, some series are such that, as more terms are added, the sum approaches a particular limit (i.e. a single function), in which case it is said to **converge**: $\left|\frac{u_{n+1}}{u_n}\right| < 1$

Maclauren's Theorem states that, under certain circumstances, a function $f(x)$ is given by:-

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots + \frac{f^n(0)}{n!}x^n$$

The series can be found if $f^n(0)$ exists for all values of n and is given on the formula sheet. Some series converge to $f(x)$ for <u>all values</u> of x and others converge for a <u>limited range</u> of values of x. It can also be expressed using sigma notation:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}x^n$$

19 Write down the Maclaurin expansion of e^x.

$$f(x) = e^x \Rightarrow f(0) = 1$$
$$f'(x) = e^x \Rightarrow f'(0) = 1$$
$$f''(x) = e^x \Rightarrow f''(0) = 1$$
$$f'''(x) = e^x \Rightarrow f'''(0) = 1$$
$$f^{iv}(x) = e^x \Rightarrow f^{iv}(0) = 1$$

Hence $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}$ for all $x \in \mathbb{R}$

NB For $e^{ax} = 1 + \frac{ax}{1!} + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \frac{(ax)^4}{4!} + \cdots$

20 Find the first 3 non-zero terms of the Maclaurin expansion of $f(x) = \sin x$ (in radians).

$$f(x) = \sin x \Rightarrow f(0) = 0$$
$$f'(x) = \cos x \Rightarrow f'(0) = 1$$
$$f''(x) = -\sin x \Rightarrow f''(0) = 0$$
$$f'''(x) = -\cos x \Rightarrow f'''(0) = -1$$
$$f^{iv}(x) = \sin x \Rightarrow f^{iv}(0) = 1$$

NB back to the "beginning"

Bk3 P91/92 Ex4
Q1, 6

Hence $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$ for all $x \in \mathbb{R}$

Expand:
1. $f(x) = \cos x$ as far as x^6
2. $f(x) = \tan x$ as far as x^3
3. $f(x) = \sin^{-1} x$ as far as x^3
4. $f(x) = \ln(1 - x)$ as far as x^4
5. $f(x) = (1 + x)^{\frac{1}{2}}$ as far as x^5
6. $f(x) = \tan 2x$ as far as x^5

Composite Functions

Two possible routes:

- Use the product/quotient/chain rules
- Expand each function separately if possible then multiply/divide results.

❷❶ Obtain the first three non-zero terms in the Maclaurin expansion of $f(x) = x\ln(2+x)$

$$f(x) = x\ln(2+x) \quad so \quad f(0) = 0$$

$$f'(x) = \frac{x}{2+x} + \ln(2+x) \quad so \quad f'(0) = \ln(2)$$

$$f''(x) = \frac{2}{(2+x)^2} + \frac{1}{2+x} \quad so \quad f''(0) = 1$$

$$f'''(x) = -\frac{4}{(2+x)^3} - \frac{1}{(2+x)^2} \quad so \quad f'''(0) = -\frac{3}{4}$$

Hence $f(x) = x\ln(2+x) = \frac{\ln(2)x}{1!} + \frac{x^2}{2!} - \frac{3}{4}\times\frac{x^3}{3!} = \ln(2)\,x + \frac{x^2}{2} - \frac{x^3}{8} + \cdots$

❷❷ Expand $f(x) = e^{-2x}\sin 3x$ in ascending powers of x as far as the term x^4.

We have seen above that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$ *and* $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!}$

So $e^{-2x} = 1 + \frac{(-2x)}{1!} + \frac{(-2x)^2}{2!} + \frac{(-2x)^3}{3!} + \frac{(-2x)^4}{4!} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 \ldots$

And $\sin 3x = \frac{(3x)}{1!} + \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} = 3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 \ldots.$

Hence $e^{-2x}\sin 3x = \left(1 - 2x + 2x^2 - \frac{4}{3}x^3 + \frac{2}{3}x^4 + \cdots\right)\left(3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 + \cdots\right)$

$$e^{-2x}\sin 3x = 3x - 6x^2 + 6x^3 - 4x^4 + \cdots - \frac{9}{2}x^3 + 9x^4$$

$$e^{-2x}\sin 3x = 3x - 6x^2 + \frac{3}{2}x^3 + 5x^4 + \cdots$$

Expand:

1. $f(x) = e^{\sin x}$ as far as x^4
2. $f(x) = \ln(1+\sin x)$ as far as x^4
3. $f(x) = e^x \sin x$ as far as x^5
4. $f(x) = \ln(1+e^x)$ as far as x^4

Answers to page 9:

1. $1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}$
2. $x+\frac{x^3}{3}$
3. $x+\frac{x^3}{6}$
4. $-x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$
5. $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\frac{7x^5}{256}$
6. $2x+\frac{8x^3}{3}+\frac{64x^5}{15}$

Answers to page 10:

1. $1+x+\frac{x^2}{2}-\frac{x^4}{8}$
2. $x-\frac{x^2}{2}+\frac{x^3}{6}-\frac{x^4}{12}$
3. $x+x^2+\frac{x^3}{3}-\frac{x^5}{30}$
4. $\ln(2)+\frac{x}{2}-\frac{x^2}{8}+\frac{x^4}{192}$