Unit 2: Applications in Algebra and Calculus (H7X1 77)

Applying Algebraic and Calculus Skills to Properties of Functions

Domain/Range Restrictions:

Revision from Higher:

- $h(x) = \frac{f(x)}{g(x)} \Rightarrow g(x) \neq 0, x \in \mathbb{R}$
- $h(x) = \sqrt{g(x)} \Rightarrow g(x) \geq 0, x \in \mathbb{R}$
- $h(x) = \log_n x \Rightarrow x > 0, x \in \mathbb{R}$
- $h(x) = \sin x$ or $h(x) = \cos x \Rightarrow -1 \leq h(x) \leq 1, x \in \mathbb{R}$

Bk1 P97 Ex1 Q1

Bk1 P102 Ex4 Q1-3

From Advanced Higher *(over to you)*:

- $h(x) = x! \Rightarrow x \geq 0, x \in \mathbb{N}_0$
- $h(x) = \sec x \Rightarrow$ ______________________________
- $h(x) = \operatorname{cosec} x \Rightarrow$ ______________________________
- $h(x) = \cot x \Rightarrow$ ______________________________
- $h(x) = |g(x)| \Rightarrow$ ______________________________
- $h(x) = \sin^{-1} x$ or $h(x) = \cos^{-1} x \Rightarrow$

  ____
- $h(x) = \tan^{-1} x \Rightarrow$ ______________________________

Inverse Functions:

Revision from Higher:

- Let $f(x) = y$ — $f(x) = 2x + 1 \Rightarrow y = 2x + 1$
- Make x the subject — $x = \frac{y-1}{2}$
- Re-write using $f^{-1}(x)$ notation — $f^{-1}(x) = \frac{x-1}{2}$

Bk1 P100 Ex3 Q1 & 2

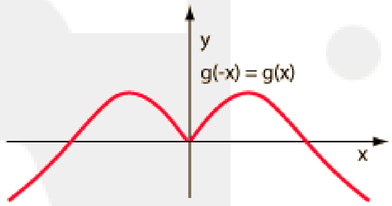
Transformation of Functions (2nd Reminder):

$f(x) \rightarrow$	Transformation	Change in (x, y)
$-f(x)$	Reflection in x-axis	$(x, y) \rightarrow (x, -y)$
$f(-x)$	Reflection in y-axis	$(x, y) \rightarrow (-x, y)$
$kf(x)$	Vertical stretch (k>1)	$(x, y) \rightarrow (x, ky)$
$f(kx)$	Horizontal compression (k>1)	$(x, y) \rightarrow (\frac{x}{k}, y)$
$f^{-1}(x)$	Reflection in $y = x$	$(x, y) \rightarrow (y, x)$
$f'(x)$	SP's become roots	SP $(x, y) \rightarrow (x, 0)$

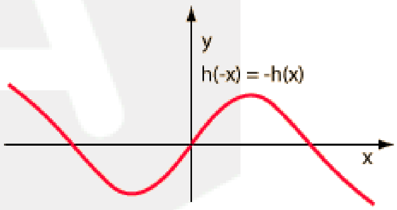
Odd and Even Functions:

If $g(-x) = g(x)$ then $g(x)$ is said to be **EVEN** i.e. reflection in y-axis. $(x, y) \rightarrow (-x, y)$

If $h(-x) = -h(x)$ then $h(x)$ is said to be **ODD** i.e. half turn symmetry about the origin. $(x, y) \rightarrow (-x, -y)$



Even function



Odd function

❶ Is $f(x) = 2x^2 + 5$ an odd or even function?

Step 1: Substitute $-x$ for x: $f(-x) = 2(-x)^2 + 5$

Step 2: Simplify: $f(-x) = 2x^2 + 5$

Step 3: State conclusion:

$f(-x) = f(x)$ *so* $f(x)$ *is an* ***EVEN*** *function.*

❷ Is $f(x) = 3x^5 + 7x^3 - 4x$ an odd or even function?

$$f(-x) = 3(-x)^5 + 7(-x)^3 - 4(-x)$$
$$f(-x) = -3x^5 - 7x^3 + 4x$$
$$f(-x) = -(3x^5 + 7x^3 - 4x)$$

$f(-x) = -f(x)$ so $f(x)$ is an __ODD__ function.

❸ Is $f(x) = 2x^2 + x$ an odd or an even function?

$$f(-x) = 2(-x)^2 + (-x)$$
$$f(-x) = 2x^2 - x$$

So $f(x)$ is neither an odd nor an even function.

❹ Is $f(x) = \sin x$ an odd or an even function?

$$f(-x) = \sin(-x) = -\sin x$$

$f(-x) = -f(x)$ so $f(x) = \sin x$ is an __ODD__ function.

❺ Is $f(x) = \cos x$ an odd or an even function?

$$f(-x) = \cos(-x) = \cos x$$

$f(-x) = f(x)$ so $f(x) = \cos x$ is an __EVEN__ function.

Bk 1 P108 Ex8

Q1, 2 & 3(1st col)

__Sketching Polynomials__: Reminder

1. **Find the zeros i.e. where it**
   a. **cuts the y-axis** $x = 0 \Rightarrow f(0)$
   b. **cuts the x-axis (roots)** $y = 0 \Rightarrow f(x) = 0$
2. **Find the Stationary Points** $f'(x) = 0$
3. **Identify the nature of the SPs** $f''(x)$/nature table
4. **Sketch the graph**

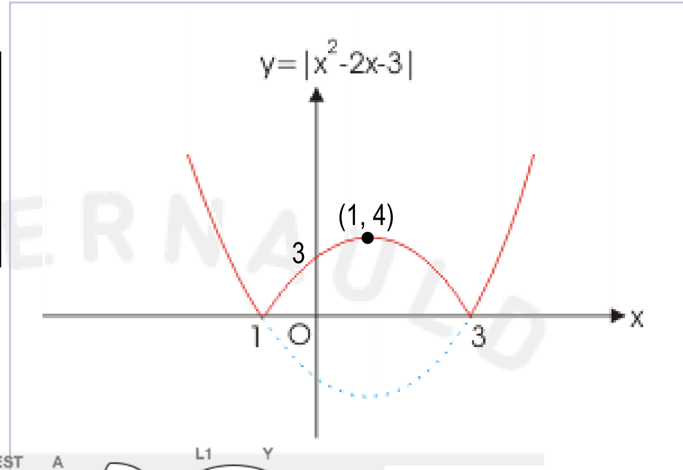
Bk 1 P103/4 Ex5

Q1a, d & 2a, d

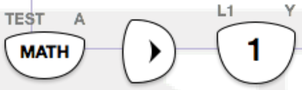
Modulus Function

Reflect any negative parts of the graph in the x-axis.

$(x, y) \to (x, |y|)$



All the key points should be indicated on the graph (i.e. zeros and SPs)

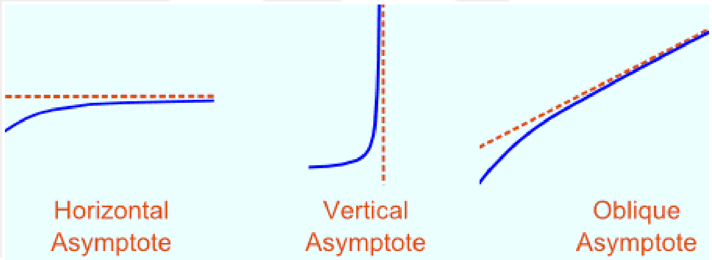
Use the graphics calculator: 

Bk1 P99 Ex2
Odd Numbers

Asymptotes:

An asymptote is a line (or curve) that the graph of a function approaches, as it heads towards infinity.

Asymptotes exist when we are dealing with rational functions $h(x) = \frac{f(x)}{g(x)}$



Vertical Asymptotes

If $g(a) = 0$ then the rational function $h(x)$ is not defined at $x = a$.
Thus, as $x \to a, h(x) \to \pm\infty$, the function is said to be **discontinuous** at a and $x = a$ is a vertical asymptote of the function.

Non-Vertical Asymptotes

Reminder: An improper rational function occurs when the degree of the numerator **is equal to or greater** than the degree of the denominator.

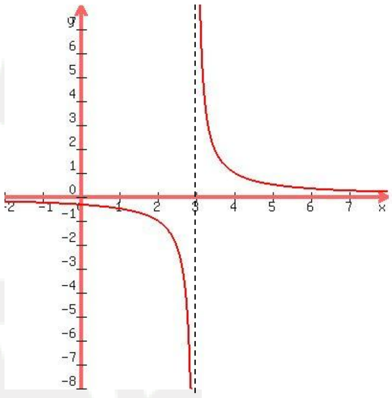
An improper function should be reduced, by polynomial division, to the sum of a quotient function and a **proper** rational fraction, i.e.:

$$h(x) = \frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

- If $q(x) = a$ (a constant) then $y = a$ is a horizontal asymptote.
- If $q(x) = mx + c$ then $y = mx + c$ is an oblique asymptote, etc.

❻ Identify the vertical asymptote of $f(x) = \frac{1}{x-3}$ and describe the behaviour of the function as it approaches its asymptote from both sides.

Denominator is 0 when $x = 3$ *so this is vertical asymptote.*



As $x \to 1$ *from left,* $x - 3 < 0$ *so* $\frac{1}{x-3} < 0$

i.e. $f(x) \to \infty^-$

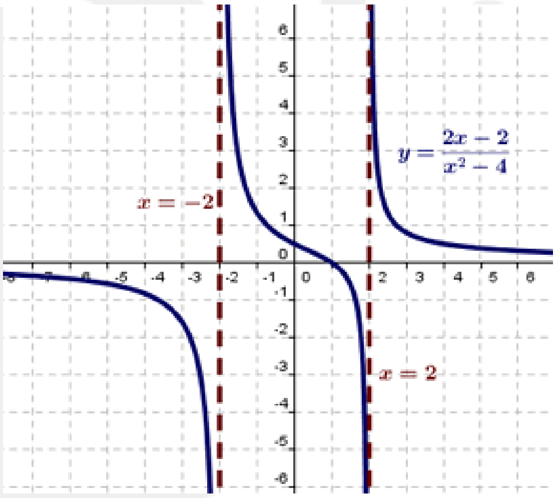
As $x \to 1$ *from right,* $x - 3 > 0$ *so* $\frac{1}{x-3} > 0$

i.e. $f(x) \to \infty^+$

NB Numerator $\neq 0$ *so* $y = 0$ *is a horizontal asymptote.*

❼ Identify the vertical asymptotes of $f(x) = \frac{2x-2}{x^2-4}$ and describe the behaviour of the function as it approaches its asymptotes from both sides.

Denominator is 0 when $x^2 - 4 = 0 \Rightarrow x = \pm 2$ *so* $x = 2$ *and* $x = -2$ *are the vertical asymptotes.*



As $x \to 2$ *from right,* $2x - 2 > 0$ *and* $x^2 - 4 > 0$ *i.e.* $f(x) \to \infty^+$

As $x \to 2$ *from left,* $2x - 2 > 0$ *and* $x^2 - 4 < 0$ *i.e.* $f(x) \to \infty^-$

As $x \to -2$ *from right,* $2x - 2 < 0$ *and* $x^2 - 4 < 0$ *i.e.* $f(x) \to \infty^+$

As $x \to -2$ *from left,* $2x - 2 < 0$ *and* $x^2 - 4 > 0$ *i.e.* $f(x) \to \infty^-$

NB Numerator can $= 0$ *so* $y = 0$ *is not a horizontal asymptote this time.*

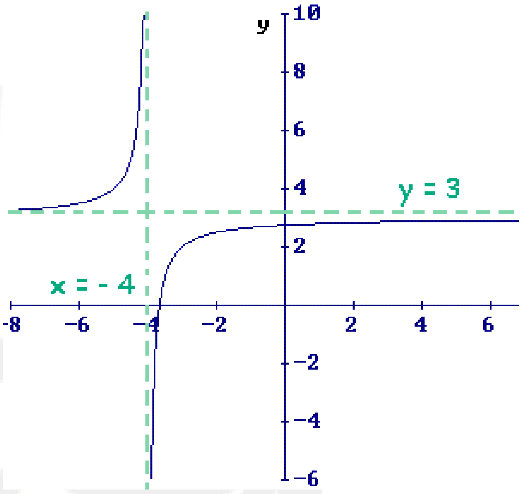
Bk 1 P109 Ex9
Q1(1st col)

❽ Identify the asymptotes of $f(x) = \frac{3x+11}{x+4}$ and describe the behaviour of the function as it approaches these asymptotes.

Denominator is 0 when $x = -4$ *so this is vertical asymptote.*

As $x \to -4$ *from left,* $3x + 11 < 0$ *and* $x + 4 < 0$ *i.e.* $f(x) \to \infty^+$

As $x \to -4$ *from right,* $3x + 11 < 0$ *and* $x + 4 > 0$ *i.e.* $f(x) \to \infty^-$



By polynomial division $\frac{3x+11}{x+4} = 3 - \frac{1}{x+4}$

NB - SQA marking instructions stipulate:

Points of expected response	**Illustrative scheme**
•[1] rewrite equation in quotient/remainder form	$y = 3 - \dfrac{1}{x+4}$
•[2] state what happens as $x \to \pm\infty$ and give equation of non-vertical asymptote	As $x \to \pm\infty$, $y \to 3$ and $\frac{1}{x+4} \to 0$ $\therefore y = 3$
Notes: An answer of $y = 3$ only gains 1 mark.	

so $y = 3$ *is horizontal asymptote*

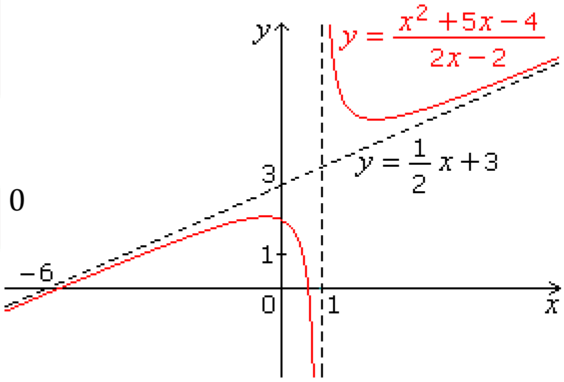
As $x \to \infty^+$, $y \to 3 - 0^+$ *so asymptote approaches from below.*

As $x \to \infty^-$, $y \to 3 - 0^-$ *so asymptote approaches from above.*

❾ Identify the asymptotes of $f(x) = \frac{x^2+5x-4}{2x-2}$ and describe the behaviour of the function as it approaches these asymptotes.

Denominator is 0 when $x = 1$ *so this is vertical asymptote.*

As $x \to 1$ *from left,* $x^2 + 5x - 4 < 0$ *and* $2x - 2 > 0$ *i.e.* $f(x) \to \infty^-$



As $x \to 1$ *from right,* $x^2 + 5x - 4 > 0$ *and* $2x - 2 > 0$ *i.e.* $f(x) \to \infty^+$

By polynomial division $\frac{x^2+5x-4}{2x-2} = \frac{1}{2}x + 3 + \frac{2}{2x-2}$

As $x \to \pm\infty$, $y \to \frac{1}{2}x + 3$ and $\frac{2}{2x-2} \to 0 \therefore y = \frac{1}{2}x + 3$ *is an oblique asymptote*

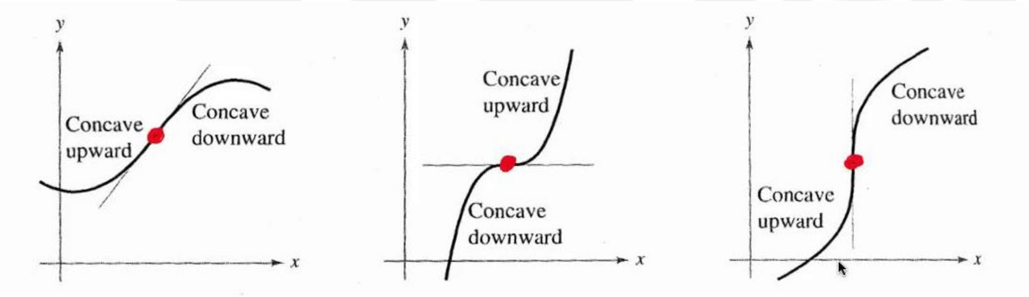
As $x \to \infty^+$, $y \to \frac{1}{2}\infty^+ + 3 + 0^+$ *so asymptote approaches from above.*

As $x \to \infty^-$, $y \to \frac{1}{2}\infty^- + 3 + 0^-$ *so asymptote approaches from below.*

Bk1 P110/111 Ex10
Q1(1st column), 2 & 3

Concavity and Points of Inflexion

In mathematics we call a region in which the gradient is increasing **CONCAVE UP**, and one in which the gradient is decreasing **CONCAVE DOWN**.



A point of inflexion occurs where a function changes concavity.

The first derivative, $f'(x)$, measures the rate of change of the function $f(x)$ with respect to x.

The second derivative, $f''(x)$, measures the rate of change of the gradient with respect to x.

- When this is positive, i.e. $f''(x) > 0$, then the curve is **concave up**.
- When this is negative, i.e. $f''(x) < 0$, then the curve is **concave down**.
- When neither, i.e. $f''(x) = 0$, then it is **INCONCLUSIVE**.

Some Stationary Points are POINTS OF INFLEXION ($f''(x) = 0$) but not all POINTS OF INFLEXION are Stationary Points ($f'(x) \neq 0$).

Points of inflexion occur where $f''(x) = 0$ (or where $f''(x)$ does not exist) and require a nature table with $f''(x)$ to confirm there has been a change in concavity.

❶⓿ Find the stationary point of $f(x) = x^4$ and determine its nature.

$f'(x) = 4x^3$ SPs @ $f'(x) = 0$ so $4x^3 = 0 \Rightarrow x = 0$. SP @ $(0, 0)$

$f''(x) = 12x^2 \Rightarrow f''(0) = 0$ so PI???

Table of signs:

x	$\rightarrow$	0	$\rightarrow$
$f''(x)$	$+$	0	$+$
Concavity	up		up

No change in concavity so not a point of inflexion!

$f''(a) = 0$ is not enough evidence to show PI.

❶❶ Find the stationary point of $f(x) = x^5$ and determine its nature

$f'(x) = 5x^4$ SPs @ $f'(x) = 0$ so $5x^4 = 0 \Rightarrow x = 0$. SP @ $(0, 0)$

$f''(x) = 20x^3 \Rightarrow f''(0) = 0$ so PI?

Table of signs:

x	$\rightarrow$	0	$\rightarrow$
$f''(x)$	$-$	0	$+$
Concavity	down		up

Change in concavity so point of inflexion.

❶❷ Identify the points of inflexion of the curve $y = \cos x$ over the interval $0 \le x \le 2\pi$.

$\frac{dy}{dx} = -\sin x \Rightarrow \frac{d^2y}{dx^2} = -\cos x$ Set $\frac{d^2y}{dx^2} = 0$ so $-\cos x = 0 \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

Table of signs:

x	$\rightarrow$	$\frac{\pi}{2}$	$\rightarrow$	$\frac{3\pi}{2}$	$\rightarrow$
$f''(x)$	$-$	0	$+$	0	$-$
Concavity	down		up		down

NB - these are PIs but not SPs

Bk1
P106/107
Ex7
Q3&5

Change in concavity so points of inflexion at $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$.

- At $x = \frac{\pi}{2}$, gradient of tangent is $-\sin\frac{\pi}{2} = -1$
- At $x = \frac{3\pi}{2}$, gradient of tangent is $-\sin\frac{3\pi}{2} = +1$

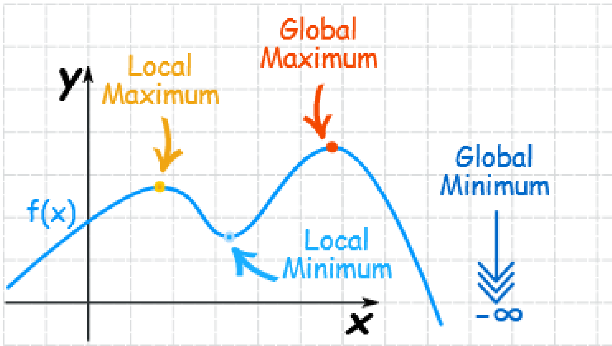
Critical Points

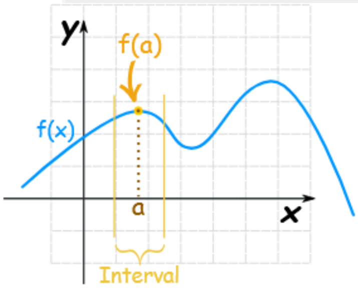
Assuming that a is in the domain of a function f, then a CRITICAL POINT of the function is any point $(a, f(a))$ where $f'(a) = 0$ (SPs) or $f'(a)$ is undefined (e.g $f(x) = x - \sqrt{x}\ (x \geq 0) \Rightarrow f'(x) = 1 - \frac{1}{2\sqrt{x}}$, $x = 0$ is in the domain but $f'(0)$ does not exist).

If $(a, f(a))$ is a critical point of a function then a is referred to as a CRITICAL NUMBER and $f(a)$ as a CRITICAL VALUE of the function.

Extreme Values of a Function (Extrema)

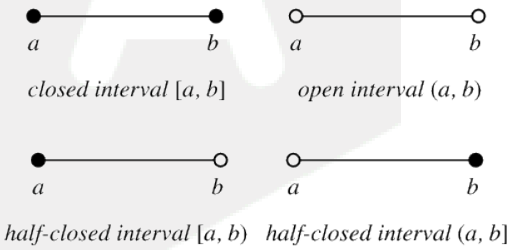
Maxima and **minima** are critical points where a function reaches a highest or lowest value, respectively. There are three kinds of **extrema** (a word meaning maximum *or* minimum): **global** (absolute), **local** (relative) and **end-point**. A global maximum is a point that takes the largest value on the entire range of the function, while a global minimum is the point that takes the smallest value on the range of the function.





Local extrema are the largest or smallest values of the function in a given domain or interval.

Reminder (from Higher):



Extrema can occur at interior points or end-points of an interval.

Extrema that occur at end-points are called ENDPOINT EXTREMA.

Bk1 P56/57 Ex2
Q3b, d, f, h

Bk1 P60/61 Ex3
Q1a, c, e, 4

Sketching Graphs – remember you have a graphics calculator!!!

When sketching curves, gather as much information as possible from the list below:

1. Zeros (intercepts)
    a. y-intercept – sub $x = 0$
    b. x-intercept (roots) – make $y = 0$ usually requires factorising.

2. Asymptotes
    a. Vertical asymptotes – make denominator $= 0$ and solve, check for $\infty^{\pm}$ to the left and right of vertical asymptote.
    b. Non-vertical asymptote – polynomial division, $y = q(x)$ is non-vertical asymptote, check above/below for $x \to \infty^{\pm}$

3. Stationary Points
    a. Make 1st derivative $= 0$ and solve (numerator $= 0$ for rational functions)
    b. Use 2nd derivative to determine concavity (i.e. min/max TP)
    c. If 2nd derivative $= 0$ then use a table of signs as INCONCLUSIVE.

4. Check for any Points of Inflexion
    a. Make 2nd derivative $= 0$ and solve
    b. Check there has been a change in concavity using a table of signs.

5. Closed/Open Interval – include end-points

❶❸ Sketch the graph of $y = \frac{x^2-x-2}{x-3}$

y-intercept: $x = 0 \Rightarrow y = \frac{-2}{-3} = \frac{2}{3} \Rightarrow (\frac{2}{3}, 0)$

x-intercept: $y = 0 \Rightarrow y = x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0$

$\Rightarrow x = 2 \; or \; x = -1 \Rightarrow (-1, 0) and \; (2, 0)$

Vertical asymptote: $x - 3 = 0 \Rightarrow x = 3$

As $x \to 3$ *from left,* $x^2 - x - 2 > 0$ *and* $x - 3 < 0$ *i.e.* $f(x) \to \infty^-$

As $x \to 3$ *from right,* $x^2 - x - 2 > 0$ *and* $x - 3 > 0$ *i.e.* $f(x) \to \infty^+$

Non-vertical asymptote: $y = \frac{x^2-x-2}{x-3} = x+2+\frac{4}{x-3} \Rightarrow y = x+2$

As $x \to \infty^+,\ y \to \infty^+ + 2 + 0^+$ *so asymptote approaches from above.*

As $x \to \infty^-,\ y \to \infty^- + 2 + 0^-$ *so asymptote approaches from below.*

Stationary Points: $\frac{dy}{dx} = 1 - \frac{4}{(x-3)^2}$ *(use divided version)*

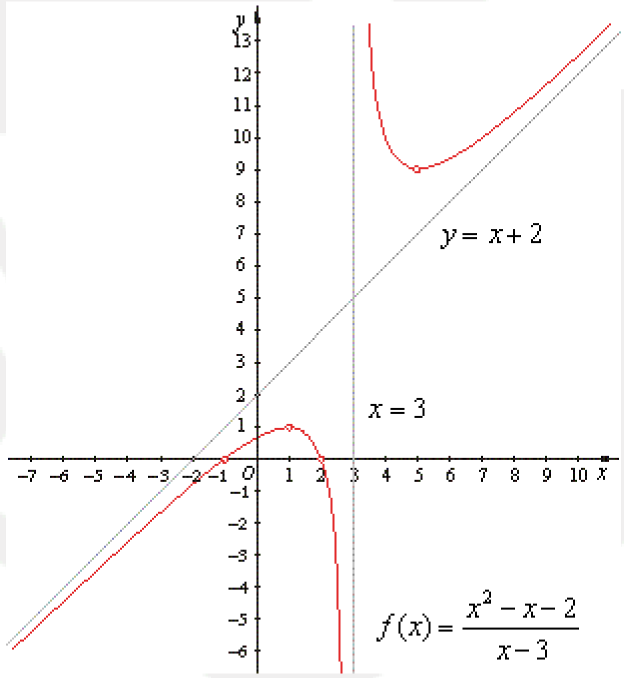
$$1 - \frac{4}{(x-3)^2} = 0 \Rightarrow (x-3)^2 = 4$$

$$x - 3 = \pm 2 \Rightarrow x = 1 \text{ and } x = 5 \text{ so SPs @ } (1,1) \text{ and } (5,9)$$

$\frac{d^2y}{dx^2} = \frac{8}{(x-3)^3}$, *at* $x = 1$ $\frac{d^2y}{dx^2} = -1 \Rightarrow$ *concave down so Max TP*

at $x = 5$ $\frac{d^2y}{dx^2} = 1 \Rightarrow$ *concave up so Min TP*

Points of inflexion: $8 \neq 0$ *so* $\frac{d^2y}{dx^2} \neq 0$ *so no points of inflexion*



<u>NB – make sure that intercepts of asymptotes are also labelled.</u>

Bk1 P112 Ex11
Q1(1st col), 2, 4