Unit 1: Methods in Algebra and Calculus (H7X2 77)

Partial Fractions

In National 5 Maths we learned how to combine 2 algebraic fractions into a single fraction, e.g.

$$\frac{3x}{x+7}+\frac{2}{x-3}=\frac{3x(x-3)+2(x+7)}{(x+7)(x-3)}=\frac{3x^2-9x+2x+14}{x^2-3x+7x-21}=\frac{3x^2-7x+14}{x^2+4x-21}$$

The aim of this chapter is to reverse the change i.e. convert a single fraction into a sum or difference of fractions.

Examples: ❶ Convert $\frac{4x-5}{x^2-x-2}$ to partial fractions.

Firstly, we must factorise the denominator (if not already done)

$$x^2-x-2=(x-2)(x-1)$$

Let $\frac{4x-5}{(x-2)(x+1)}=\frac{A}{x-2}+\frac{B}{x+1}$ *then multiply through by* $(x-2)(x+1)$

$$\frac{4x-5}{(x-2)(x+1)}=\frac{A(x+1)+B(x-2)}{(x-2)(x+1)}$$

Equating numerators we get $4x-5= A(x+1)+B(x-2)$

Let $x=-1$ $\quad 4(-1)-5= A(0)+B(-3)$

$-9=-3B$ *so* $B=3$

Let $x=2$ $\quad 4(2)-5= A(3)+B(0)$

$3=3A$ *so* $A=1$

Hence $\frac{4x-5}{(x-2)(x+1)}=\frac{1}{x-2}+\frac{3}{x+1}$

❷ Express $\frac{5x^2-2x-1}{(x+1)(x^2+1)}$ in partial fractions.

Denominator factorised but we have a linear factor and a <u>*quadratic*</u> *factor*

$\frac{5x^2-2x-1}{(x+1)(x^2+1)}=\frac{A}{x+1}+\frac{Bx+C}{x^2+1}$ *Numerator for the quadratic denominator is linear*

Multiplying through: $\frac{5x^2-2x-1}{(x+1)(x^2+1)}=\frac{A(x^2+1)}{x+1}+\frac{(Bx+C)(x+1)}{x^2+1}$

Equating numerators $5x^2-2x-1= A(x^2+1)+(Bx+C)(x+1)$

Let $x = -1$ $\quad 5 + 2 - 1 = A(2) + (Bx + C)(0)$

$6 = 2A$ *so* $A = 3$

Let $x = 0$ $\quad 0 - 0 - 1 = A(1) + C(1)$

$-1 = 3 + C$ *so* $C = -4$

We now have $5x^2 - 2x - 1 = 3(x^2 + 1) + (Bx - 4)(x + 1)$

$= 3x^2 + 3 + Bx^2 + Bx - 4x - 4$

Equating x^2 *coefficients* $\quad 5 = 3 + B$

$B = 2$

Hence $\quad \frac{5x^2-2x-1}{(x+1)(x^2+1)} = \frac{3}{x+1} + \frac{2x-4}{x^2+1}$

P18 Ex2
1st Column

❸ Resolve $\frac{x+4}{(x+1)(x-2)^2}$ into its partial fractions.

Denominator factorised but we have a linear factor and a *repeated factor*

$$\frac{x+4}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

Multiplying through: $\quad \frac{x+4}{(x+1)(x-2)^2} = \frac{A(x-2)^2+B(x+1)(x-2)+C(x+1)}{(x+1)(x-2)^2}$

Equating numerators $\quad x + 4 = A(x-2)^2 + B(x+1)(x-2) + C(x+1)$

Let $x = 2$ $\quad 6 = A(0)^2 + B(0) + C(3)$

$6 = 3C$ *so* $C = 2$

Let $x = -1$ $\quad 3 = A(9) + B(0) + C(0)$

$9A = 3$ *so* $A = \frac{1}{3}$

Let $x = 0$ $\quad 4 = \frac{1}{3}(-2)^2 + B(1)(-2) + 2(1)$

$2B = -\frac{2}{3}$ *so* $B = -\frac{1}{3}$

Hence $\quad \frac{x+4}{(x+1)(x-2)^2} = \frac{1}{3(x+1)} - \frac{1}{3(x-2)} + \frac{2}{(x-2)^2}$

P19 Ex3
1st Column

❹ Reduce $\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)}$ into its partial fractions.

*The quadratic factor is **IRREDUCIBLE** as its discriminant = -4*

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+2x+2}$$ *NB quadratic numerator is linear*

Multiplying through: $$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)} = \frac{A(x^2+2x+2)+(Bx+C)(x+1)}{(x+1)(x^2+2x+2)}$$

Equating numerators $3x^2 + 2x + 1 = A(x^2 + 2x + 2) + (Bx + C)(x + 1)$

P20 Ex4
1st Column

Let $x = -1$ $\quad 2 = A$

Let $x = 0$ $\quad 1 = 2A + C$ *so* $C = -3$

Equating x^2 *coefficients* $\quad 3 = A + B$ *so* $B = 1$

Hence $$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)} = \frac{2}{x+1} + \frac{x-3}{x^2+2x+2}$$

Long Division: $2210 \div 34$

		0	0	6	5		34 into 2 can't go
3	4	2	2	1	0		34 into 22 can't go
	-	2	0	4	↓		34 into 221 goes 6 times: 6 x 34 =204
			1	7	0		221 - 170=17 and bring down the zero
		-	1	7	0		34 into 170 goes 5 times, 5 x 34 = 170
					0		

Algebraic Long Division: $(x^3 + 3x^2 + x + 5) \div (x - 3)$

	x^2	$+6x$	$+19$		Use the x's to carry out the division
$x-3$	x^3	$+3x^2$	$+x$	$+5$	x into x^3 goes x^2 times
-	x^3	$-3x^2$	↓	↓	$x^2 \times (x-3) = x^3 - 3x$
		$6x^2$	$+x$	↓	$(x^3 + 3x^2) - (x^3 - 3x) = 6x^2$, bring down the $+x$
	-	$6x^2$	$-18x$	↓	$6x \times (x-3) = 6x^2 - 18x$
			$19x$	$+5$	$(6x^2 + x) - (6x^2 - 18x) = 19x$, bring down the $+5$
			$19x$	-57	$19 \times (x-3) = 19x - 57$
				62	$(19x + 5) - (19x - 57) = 62$ so remainder 62

$$(x^3 + 3x^2 + x + 5) \div (x - 3) = x^2 + 6x + 19 + \frac{62}{x-3}$$

<u>Handling Improper Rational Fractions:</u>

An improper fraction occurs when the degree of the numerator **is equal to or greater** than the degree of the denominator.

An improper fraction must be reduced, by algebraic division, to the sum of a polynomial function and a **proper** rational fraction. The fraction part is then converted to partial fractions.

❺ Resolve $\frac{x^3+2x^2-2x+2}{x^2+2x-3}$

Degree of the numerator is 3 and the denominator is 2 so improper.

Using algebraic division:

$$\begin{array}{r|llll} & & & x & \\ \hline x^2+2x-3 & x^3 & +2x^2 & -2x & +2 \\ - & x^3 & +2x^2 & -3x & \downarrow \\ \hline & & & x & +2 \end{array}$$

Using factorised denominator: $\frac{x^3+2x^2-2x+2}{(x-1)(x+3)} = x + \frac{x+2}{(x-1)(x+3)}$

Now convert the fraction part to partial fractions

$$\frac{x+2}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3}$$

$$\frac{x+2}{(x-1)(x+3)} = \frac{A(x+3)+B(x-1)}{(x-1)(x+3)}$$

$$x+2 = A(x+3)+B(x-1)$$

Let $x=-3$ $\quad -1 = A(0)+B(-4)$

$-1=-4B$ *so* $B=\frac{1}{4}$

Let $x=1$ $\quad 3 = A(4)+B(0)$

$3=4A$ *so* $A=\frac{3}{4}$

$$\frac{x+2}{(x-1)(x+3)} = x + \frac{3}{4(x+1)} + \frac{1}{4(x+3)}$$

P22 Ex6
1st column