## Unit 3: Geometry, Proof and Systems of Equations (H7X3 77)

### Applying Algebraic and Geometric skills to Methods of Proof

In mathematics, a **STATEMENT** is either **TRUE** or **FALSE** but not both.

❶ The sum of the angles in a triangle is always $180°$ - TRUE

❷ $x = 3 \Rightarrow x^2 = 9$ - TRUE

❸ $x^2 = 9 \Rightarrow x = 3$ - FALSE since $x = -3$ is another solution

❹ $n^2 + n$ is even for **all** positive integers $n$

When $n = 1$: $n^2 + n = 1^2 + 1 = 2$ - TRUE
When $n = 2$: $n^2 + n = 2^2 + 2 = 6$ - TRUE
When $n = 3$: $n^2 + n = 3^2 + 3 = 12$ - TRUE

Certainly, $n^2 + n$ is even for $n = 1, 2, 3$ however this is not a proof it is even for all positive integers $n$. The above calculations are no guarantee it will be even for other values of $n$.

There are methods for rigorously proving that statements are true beyond any doubt and you will learn some simple methods of proof in this section. Firstly, however, you will need to learn some new notation and how to prove that a statement is FALSE.

# Notation

By convention, particular symbols are reserved for the most important sets of numbers:

$\mathbb{R}$ – set of real numbers
$\mathbb{Q}$ – set of rational numbers (from quotient – numbers that can be written as a fraction e.g. $^2/_3$)
$\mathbb{C}$ – set of complex numbers (met this already)
$\mathbb{Z}$ – set of integers (from Zahl which is german for number)
$\mathbb{N}$ – set of natural/whole numbers $\mathbb{W}/\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$

There is no universal symbol for irrational numbers (e.g. $\pi, \sqrt{2}$) although $\mathbb{I}$ is used in some texts.

| | | |
|---|---|---|
| $\forall$ - for all | $\exists$ - there exists | $iff$ – if and only if |
| $\in$ - is a member of | $\Rightarrow$ - implies | $\Leftrightarrow$ - equivalence |
| $\Leftarrow$ - converse | $\neg$ - negation | $p\|a$ - $p$ divides into $a$ |

There are 5 types of statements:

__UNIVERSAL statement__ refers to all items in the defined set:

$2x + 3x = 5x$ for all $x \in \mathbb{R}$ can be re-written as $2x + 3x = 5x, \forall x \in \mathbb{R}$.

__EXISTENTIAL statement__ says that there is at least one item in the given set that has the property in the statement:

There exists an $x$ value such that $x^2 + 1$ is even can be
re-written as $\exists x$ s. t. $x^2 + 1$ is even.

__NEGATION of a statement__ is formed by putting **NOT** in front of the verb:

"There exists an $x$ value such that $P(x)$ holds" $(\exists x)(P(x))$ becomes
"For all $x$ the property $P(x)$ does not hold" $(\forall x)(\neg P(x))$

__COMPOUND statement__ is a combination of statements linked using if/and/or/then:

If $a + b = c$ and/or $b + c > 10$ then . . .

__IMPLICATION/CONDITIONAL__ statement uses the if/then and breaks it down into 3 types. For "If the shape is a square then ($\Rightarrow$) it has 4 sides":

1. INVERSE – "If the shape is not a square then it does not have 4 sides".
2. CONVERSE – "If the shape has 4 sides then ($\Leftarrow$) it is a square".
3. CONTRAPOSITIVE – If the shape does not have 4 sides then it is not a square".

NB – if the original implication is true then the contrapositive is true, but the inverse and converse need not be true.

# Counter-Examples

A statement can be disproved (proved to be false) by providing a counter-example.

❺ $2^n + n$ is divisible by 3 for all natural numbers ($\forall\, n \in \mathbb{N}$)

When $n = 1$: $2^n + n = 2^1 + 2 = 3$ - TRUE
When $n = 2$: $2^n + n = 2^2 + 2 = 6$ - TRUE
When $n = 3$: $2^n + n = 2^3 + 3 = 11$ - FALSE

$2^n + n$ is NOT divisible by 3 when $n = 3$ so we have found a counter-example which proves that the statement is false.

Bk2 P3/4 Ex1A
Q7, 8

P8 Ex1B Q5a-d

# Direct Proof

Reminder from finding a formula:

| | |
|---|---|
| • 1, 2, 3, 4, . . . . . | $n$ (counting numbers) |
| • 2, 4, 6, 8, . . . . . | $2n$ (even numbers) |
| • 1, 3, 5, 7, . . . . . . | $2n-1$ (odd numbers) |
| • 3, 6, 9, 12, . . . . . | $3n$ (multiples of 3) |
| • 4, 7, 10, 13, . . . . | $3n+1$ |
| • 1, 4, 9, 16, . . . . . | $n^2$ (square numbers) |

| | | |
|---|---|---|
| Also: | $(n-1), n, (n+1), (n+2), (n+3)$ | consecutive numbers |
| | $(2n-2), 2n, (2n+2)$ | consecutive even numbers |
| | $(2n+1), (2n+3), (2n+5)$ | consecutive odd numbers |

❻ Let $n$ be a natural number. Prove that $n^2+3n$ is always divisible by 2.

In notation form: $\forall\, n \in \mathbb{N},\ 2|n^2+3n$

2 cases to consider:

- $n$ is even i.e. $n = 2k$

$$n^2+3n = (2k)^2+3(2k) = 4k^2+6k = 2(2k^2+3k)$$

$$\Rightarrow n^2+3n = 2\text{a} \Rightarrow 2|a^2+3a\ \forall \text{ even numbers}$$

- $n$ is odd i.e. $n = 2k-1$

$$n^2+3n = (2k-1)^2+3(2k-1) = 4k^2-4k+1+6k-3 = 2(2k^2+k-1)$$

$$\Rightarrow n^2+3n = 2a^2+a-1 \Rightarrow 2|2a^2+a-1\ \forall \text{ odd numbers}$$

Thus $2|n^2+3n,\ \forall\, n \in \mathbb{N}$

❼ Prove that the product of 3 consecutive whole numbers $(n+1)(n+2)$, is always divisible by 3

There are 3 cases to consider. When a whole number is divided by 3 we can have remainder 0, remainder 1 or remainder 2 i.e. $n = 3k$, $n = 3k+1$ or $n = 3k+2$.

- $n$ is divisible by 3

$\Rightarrow n = 3a, a \in \mathbb{N}$

$\Rightarrow n(n+1)(n+2) = 3a(3a+1)(3a+2) = 3[a(3a+1)(3a+2)]$

$\Rightarrow n(n+1)(n+2) = 3k$ where $k = a(3a+1)(3a+2)$

$\Rightarrow 3|n(n+1)(n+2)$, $\forall n$ divisible by 3

- $n$ has a remainder of 1 when divided by 3

$\Rightarrow n = 3a+1, a \in \mathbb{N}$

$\Rightarrow n(n+1)(n+2) = (3a+1)(3a+2)(3a+3) = 3[(3a+1)(3a+2)(a+1)]$

$\Rightarrow n(n+1)(n+2) = 3k$ where $k = (3a+1)(3a+2)(a+1)$

$\Rightarrow 3|n(n+1)(n+2)$, $\forall n$ which have a remainder of 1 when divided by 3

- $n$ has a remainder of 2 when divided by 3

$\Rightarrow n = 3a+2, a \in \mathbb{N}$

$\Rightarrow n(n+1)(n+2) = (3a+2)(3a+3)(3a+4) = 3[(3a+2)(a+1)(3a+4)]$

$\Rightarrow n(n+1)(n+2) = 3k$ where $k = (3a+2)(a+1)(3a+4)$

$\Rightarrow 3|n(n+1)(n+2)$, $\forall n$ which have a remainder of 2 when divided by 3

Case 1 **and** Case 2 **and** Case 3 $\Rightarrow \forall n\ 3|n(n+1)(n+2)$,

NB – If we wish to prove that $6|n(n+1)(n+2)$ we need to show:

- $\forall n\ 2|n(n+1)(n+2)$ **AND**
- $\forall n\ 3|n(n+1)(n+2)$

- If $p \Rightarrow q$ and $q \Rightarrow r$ then $p \Rightarrow r$

Bk2 P10 Ex2A
Q3 and 7

# Proof by Contradiction

1. Whatever we are trying to prove, we assume that the negation is true
2. Contradict the negation by a set of steps
3. Since all steps will be valid then the assumption must be false.
4. If the negation is false then the original statement must be true

❽ Prove that $\sqrt{2}$ is not rational.

Assume that the negation is true $\Rightarrow \sqrt{2} \in \mathbb{Q}$

$\Rightarrow \exists a, b \in \mathbb{Z}$ s.t. $a$ and $b$ have no common factors and $\sqrt{2} = \frac{a}{b}$

$\Rightarrow a = b\sqrt{2} \Rightarrow a^2 = 2b^2 \Rightarrow a^2$ is even $\Rightarrow a$ is even.

Let $a = 2k$ where $k \in \mathbb{N}$

$\Rightarrow 4k^2 = 2b^2 \Rightarrow b^2$ is even $\Rightarrow b$ is even

Both $a$ and $b$ are even $\Rightarrow a$ and $b$ have a common factor of 2.

This contradicts the original assumption so assumption is false.

Therefore the original statement must be true: $\sqrt{2} \notin \mathbb{Q}$

❾ Prove that $\frac{a+b}{2} \geq \sqrt{ab}\ \ \forall a, b \in \mathbb{N}$.

Assume that the negation is true $\Rightarrow \frac{a+b}{2} < \sqrt{ab}$ for some $a, b \in \mathbb{N}$.

$$\begin{aligned}\Rightarrow a + b < 2\sqrt{ab} \Rightarrow a + b < 4ab \Rightarrow a^2 + 2ab + b^2 &< 4ab\\ \Rightarrow a^2 - 2ab + b^2 &< 0\\ \Rightarrow (a - b)^2 &< 0\end{aligned}$$

This is not possible, so assumption is false.

Therefore the original statement must be true $\Rightarrow \frac{a+b}{2} \geq \sqrt{ab}\ \ \forall a, b \in \mathbb{N}$.

# Proof by Contrapositive

This proof depends on the fact that a statement and its contrapositive are equivalent.

❶⓿ Prove that if $x = 77$ then $x$ is not even.

STATEMENT: if $x$ is even then it cannot be divided by 2 without a remainder.

CONTRAPOSITIVE: If $x$ cannot be divided by 2 then $x$ is not even.

77 cannot be divided by 2 without a remainder.
$\Rightarrow$ 77 is not even $\Rightarrow x$ is not even.

❶❶ Prove that for $x \in \mathbb{Z}$, if $5x + 9$ is even then is $x$ odd

First we need to write down the contrapositive of the statement:

- For $x \in \mathbb{Z}$, if $x$ is not odd then $5x + 9$ is not even.

Or

- For $x \in \mathbb{Z}$, if $x$ is even then $5x + 9$ is odd

We now prove the contrapositive directly.

Proof: $x$ is even so $x = 2k$ for some integer $k$.

So $5x + 9 = 5(2k) + 9 = 10k + 9 = 10k + 8 + 1 = 2(5k + 4) + 1$.

This is odd since $2(5k + 4)$ produces an even number and when we add $1$ it becomes an odd number for all $k$.

As the contrapositive has been proven true this means the original statement is also true i.e. for $x \in \mathbb{Z}$, if $5x + 9$ is even then is $x$ odd

Bk2 P14 Ex3A
Q1, 6-8

"A common experience for people learning advanced mathematics is to come to the end of a proof and think, 'I understood how each line followed the previous one, but somehow I am none the wiser about *why* the theorem is true, or how anybody thought of this argument' ". – Timothy Gowers