Unit 3: Geometry, Proof and Systems of Equations (H7X3 77) - Matrices

A **matrix** is simply a **rectangular array of numbers**.

A matrix with *m* rows and *n* columns is said to have **order** $m \times n$.

The **entry** (or element) in row *i* and column *j* of the matrix *A* is denoted by a_{ij}.

$A=\begin{pmatrix}2 & 4 & -1\\3 & 1 & 2\end{pmatrix}$ is a 2×3 matrix. $B=\begin{pmatrix}2 & 0\\-1 & 1\end{pmatrix}$ is a 2×2 matrix.

$C=\begin{pmatrix}1\\-2\\2\end{pmatrix}$ is a 3×1 matrix. $D=\begin{pmatrix}1 & 3 & 5 & -1\end{pmatrix}$ is a 1×4 matrix.

ADDITION AND SUBTRACTION OF MATRICES

Only matrices of the same order can be added/subtracted

❶ $$\begin{pmatrix}3 & 4 & 5\\-2 & 1 & 3\end{pmatrix}+\begin{pmatrix}2 & -1 & 2\\3 & 1 & -6\end{pmatrix}=\begin{pmatrix}3+2 & 4+(-1) & 5+2\\-2+3 & 1+1 & 3+(-6)\end{pmatrix}=\begin{pmatrix}5 & 3 & 7\\1 & 2 & -3\end{pmatrix}$$

❷ $$\begin{pmatrix}5 & 1\\-2 & 3\end{pmatrix}-\begin{pmatrix}1 & -2\\2 & 4\end{pmatrix}=\begin{pmatrix}5-1 & 1-(-2)\\-2-2 & 3-4\end{pmatrix}=\begin{pmatrix}4 & 3\\-4 & -1\end{pmatrix}$$

$$A+B=B+A$$
$$(A+B)+C=A+(B+C)$$

SCALAR MULTIPLICATION

If *k* is a **scalar** (number), the matrix kA is formed by multiplying each entry of the matrix *A* by *k*.

❸ $$3\begin{pmatrix}2 & 1\\-3 & 5\end{pmatrix}=\begin{pmatrix}3\times 2 & 3\times 1\\3\times(-3) & 3\times 5\end{pmatrix}=\begin{pmatrix}6 & 3\\-9 & 15\end{pmatrix}$$

❹ Given the matrices $A=\begin{pmatrix}1 & 2\\3 & -1\end{pmatrix}$, $B=\begin{pmatrix}2 & 0\\-1 & 2\end{pmatrix}$ and $C=\begin{pmatrix}-2 & 4\\5 & 0\end{pmatrix}$, find the matrix $2A+3B-2C$

$$2A+3B-2C=2\begin{pmatrix}1 & 2\\3 & -1\end{pmatrix}+3\begin{pmatrix}2 & 0\\-1 & 2\end{pmatrix}-2\begin{pmatrix}-2 & 4\\5 & 0\end{pmatrix}=\begin{pmatrix}2 & 4\\6 & -2\end{pmatrix}+\begin{pmatrix}6 & 0\\-3 & 6\end{pmatrix}-\begin{pmatrix}-4 & 8\\10 & 0\end{pmatrix}$$
$$=\begin{pmatrix}2+6-(-4) & 4+0-8\\6+(-3)-10 & -2+6-0\end{pmatrix}=\begin{pmatrix}12 & -4\\-7 & 4\end{pmatrix}$$

[*Note that the distributive law applies for scalar multiplication, i.e.* $k(A+B)=kA+kB$ *for any matrices* A *and* B *of the same order, where* k *is a scalar.*]

ZERO MATRIX: All entries are zero $O=\begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}$

TRANSPOSE MATRIX: Denoted A^T or A'. Rows and columns swap i.e the first row of A becomes the first column of A'

$$(a_{ij})'_{m\times n} = (a_{ji})_{n\times m}$$

$$(A')' = A$$

If $A = \begin{pmatrix} 3 & 4 & 5 \\ 2 & -1 & 3 \end{pmatrix}$, then $A' = \begin{pmatrix} 3 & 2 \\ 4 & -1 \\ 5 & 3 \end{pmatrix}$.

NB Matrix *A* is of order 2×3, whereas the matrix A' is of order 3×2.

<u>A matrix A is said to be **symmetric** if</u> $A' = A$.

If $A = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{pmatrix}$, then $A' = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 2 & -1 \\ 5 & -1 & 7 \end{pmatrix}$. $A' = A$, so A is symmetric.

Note that a symmetric matrix is a **<u>square</u>** matrix which is symmetrical along the leading diagonal (the diagonal running from the top-left corner to the bottom-right corner of the matrix.)

<u>A matrix A is said to be **skew-symmetric** if</u> $A' = -A$.

If $A = \begin{pmatrix} 0 & 3 & -5 \\ -3 & 0 & 1 \\ 5 & -1 & 0 \end{pmatrix}$, then $A' = \begin{pmatrix} 0 & -3 & 5 \\ 3 & 0 & -1 \\ -5 & 1 & 0 \end{pmatrix}$. $A' = -A$, so A is skew-symmetric.

Note that a skew-symmetric matrix must be a **<u>square</u>** matrix with all entries in the leading diagonal equal to **<u>zero</u>**.

Bk3 P4 Ex1 Q6 & 8(a)

$$(a_{ij})'_{m\times n} = (a_{ji})_{n\times m}$$

$$(AB)' = B'A'$$

MATRIX MULTIPLICATION

❺ Let $P = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & -2 \end{pmatrix}$.

The elements in first row of P is multiplied by the elements in the first column of Q and the answers added: $4\times2+3\times6 = 26$. This is repeated as shown below:

$$PQ = \begin{pmatrix} 4 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 4\times2+3\times6 & 4\times4+3\times1 & 4\times5+3\times(-2) \\ -1\times2+2\times6 & -1\times4+2\times1 & -1\times5+2\times(-2) \end{pmatrix}$$

$$= \begin{pmatrix} 26 & 19 & 14 \\ 10 & -2 & -9 \end{pmatrix}$$

The matrix product PQ can only be formed if the number of columns of matrix P is the same as the number of rows of matrix Q.

If matrix P is $m \times n$ and the order of matrix Q is $n \times r$. The order of the matrix product PQ is then $m \times r$.

$$\begin{matrix} P & Q & & PQ \\ (m\times n) & (n\times p) & \rightarrow & m\times p \end{matrix}$$

Think of the n's "cancelling out"

The matrix product PQ can be formed since P is of order 2×2 and Q is of order 2×3
The matrix PQ will then be of order 2×3.

Note that the matrix products AB and BA are **not equal**. This is true in general for matrices A and B.

The order in which matrices are multiplied is therefore crucial. In the matrix product AB, we say that A pre-multiplies B or that B post-multiplies A.

❻ Simplify $A(A+B)-B(B-A)$

$$A(A+B)-B(B-A) = AA + AB - BB + BA = A^2 + AB - B^2 + BA$$

This cannot be simplified further as $AB \neq BA$ in most cases.

If the matrix products AB and BA are equal, the matrices A and B are said to **commute**.

❼ Given the 2×2 matrix $M = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}$, find M^2, M^3 and M^4.

$$M^2 = MM = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 3\times 3+(-2)\times 0 & 3\times(-2)+(-2)\times 3 \\ 0\times 3+3\times 0 & 0\times(-2)+3\times 3 \end{pmatrix} = \begin{pmatrix} 9 & -12 \\ 0 & 9 \end{pmatrix}$$

$$M^3 = MM^2 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 9 & -12 \\ 0 & 9 \end{pmatrix} = \begin{pmatrix} 3\times 9+(-2)\times 0 & 3\times(-12)+(-2)\times 9 \\ 0\times 9+3\times 0 & 0\times(-12)+3\times 9 \end{pmatrix} = \begin{pmatrix} 27 & -54 \\ 0 & 27 \end{pmatrix}$$

[The matrix M^3 can also be found by forming the matrix product M^2M .]

$$M^4 = MM^3 = \begin{pmatrix} 3 & -2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 27 & -54 \\ 0 & 27 \end{pmatrix} = \begin{pmatrix} 3\times 27+(-2)\times 0 & 3\times(-54)+(-2)\times 27 \\ 0\times 27+3\times 0 & 0\times(-54)+3\times 27 \end{pmatrix} = \begin{pmatrix} 81 & -216 \\ 0 & 81 \end{pmatrix}$$

[The matrix M^4 can also be found by forming the matrix product M^3M or M^2M^2.]

[If required, the matrix product ABC can be found by considering the matrix product $(AB)C$ or $A(BC)$. Note that the order $A \rightarrow B \rightarrow C$ must be preserved.]

IDENTITY MATRICES

The matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the 2×2 **identity matrix**.

All the entries on the main diagonal are 1 and all other entries are zero in this identity matrix.

Pre-multiplying or post-multiplying any matrix by the identity matrix will not change the original matrix.

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1a+0c & 1b+0d \\ 0a+1c & 0b+1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$.

Also, $AI = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1a+0b & 0a+1b \\ 1c+0d & 0c+1d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$. $IA = A$ and $AI = A$

The 3×3 identity matrix is $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and behaves in the same way.

Powers of matrices (A^n) can be written in the form $pA + qI$

❽ Given the 2×2 matrix $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$, find the values of the integers p and q such that $A^2 = pA + qI$

$$A^2 = AA = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 2\times 2 + (-1)\times 3 & 2\times(-1) + (-1)\times 5 \\ 3\times 2 + 5\times 3 & 3\times(-1) + 5\times 5 \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix}$$

$$A^2 = pA + qI \quad \Rightarrow \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = p\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 2p & -p \\ 3p & 5p \end{pmatrix} + \begin{pmatrix} q & 0 \\ 0 & q \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 1 & -7 \\ 21 & 22 \end{pmatrix} = \begin{pmatrix} 2p+q & -p \\ 3p & 5p+q \end{pmatrix}$$

Equating entries: $-p = -7 \quad \Rightarrow \quad p = 7$

$2p + q = 1 \quad \Rightarrow \quad 2(7) + q = 1$

$\Rightarrow \quad q = -13$

$A^n = pA + qI$

Hence $p = 7$ *and* $q = -13$ *so* $A^2 = 7A - 13I$

This technique can be extended repeatedly to find the matrices A^3, A^4, ... in the form $xA + yI$

❾ $A^3 = AA^2 = A(7A - 13I)$

$= 7A^2 - 13A$ [since $AI = A$]

$= 7(7A - 13I) - 13A$

$= 49A - 91I - 13A$

$= 36A - 91I$

Bk3 P10 Ex3 Q2(i)

Bk3 P13 Ex4B Q1, 3, 4, 6, 7, 13, 15

Hence $A^3 = 36A - 91I$.

If $A^T A = I$ then A is said to be **ORTHOGONAL**

Inverse of a 2×2 Matrix

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then the inverse matrix A^{-1} is given by:

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$ad - bc$ is known as the **determinant** of matrix A and can be denoted :

$$\det(A) \quad \text{or} \quad \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If $\det(A) = 0$ then no inverse exists and the matrix is called **singular**

Bk3 P16 Ex5 Q1 & 2

❶⓿ Find the inverse of the matrix $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

$\det(A) = 6 - 4 = 2$ so inverse exists

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -2 \\ -\frac{1}{2} & 1 \end{pmatrix}$$

❶❶ Given $B^2 = 3B - 2I$ show that $B^{-1} = \frac{3}{2}I - \frac{1}{2}B$

$$BB = 3B - 2I \Rightarrow B^{-1}BB = B^{-1}3B - B^{-1}2I$$

$$\Rightarrow IB = 3B^{-1}B - 2B^{-1}I \Rightarrow B = 3I - 2B^{-1}$$

$$\Rightarrow B^{-1} = \frac{3}{2}I - \frac{1}{2}B$$

Bk3 P19 Ex6A
Q2, 3(1st col), 4, 7, 8, 9

Linear Equations

The equation $3x + 2y = 5$ is a linear equation and has an infinite set of solutions: e.g. $\{(1,1), (9, -11), \ldots..\}$

The general solution, for any value of x, is $\left(x, \frac{5-3x}{2}\right)$ [make y the subject].

All solutions, when joined plotted on the Cartesian plane, lie in a straight line.

2×2 Linear Equations

$2x + 3y = 21$
$3x + 2y = 19$

There are two equations and two unknowns (variables).

This system of equations produces a unique solution:

$$\begin{array}{ll} 2x + 3y = 21 & \times 3 \\ 3x + 2y = 19 & \times 2 \end{array} \Rightarrow \begin{array}{l} 6x + 9y = 63 \\ 6x + 4y = 38 \end{array} \quad \text{Subtracting gives:} \quad \begin{array}{r} 5y = 25 \\ y = 5 \end{array}$$

Substituting $y = 5$ into either of the original equations gives $x = 3$ so $(3,5)$

Not all pairs of 2×2 Linear Equations have a unique solution!

$$\begin{array}{ll} 3x + 3y = 6 & \times 4 \\ 4x + 4y = 8 & \times 3 \end{array} \Rightarrow \begin{array}{l} 12x + 12y = 24 \\ 12x + 12y = 24 \end{array} \quad \text{Subtracting gives:} \quad 0 = 0$$

When this occurs, one of the equations is said to be **redundant** and, in fact, there are an infinite number of solutions. For any value of x, $(x, 2 - x)$ is a solution. (Collinear Lines)

$$\begin{array}{ll} x + 4y = 6 & \times 2 \\ 2x + 8y = 10 & \times 1 \end{array} \Rightarrow \begin{array}{l} 2x + 8y = 12 \\ 2x + 8y = 10 \end{array} \quad \text{Subtracting gives:} \quad 0 = 2!!$$

When this occurs, the equations are said to be **inconsistent** and, in fact, have no solution. (Parallel Lines)

Bk1 P121 Ex1
All

Using Matrices to solve Systems of Equations

❶❷ The system of equations $\begin{matrix} 2x + 4y = 42 \\ x + 5y = 57 \end{matrix}$ can be represented by the matrices:

$$\begin{pmatrix} 2 & 4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 42 \\ 57 \end{pmatrix}$$

We then combine these matrices to form an **augmented matrix:**

$$\left(\begin{array}{cc|c} 2 & 4 & 42 \\ 1 & 5 & 57 \end{array}\right)$$

We now convert the LHS of the matrix to an identity matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We use **elementary row operations – EROs** – to achieve this:

These EROs are:

- Rows can be interchanged
- A row can be multiplied by a constant
- A row can be added/subtracted to another row

$$\left(\begin{array}{cc|c} 2 & 4 & 42 \\ 1 & 5 & 57 \end{array}\right) \quad R1 \to R1 \div 2 \quad \left(\begin{array}{cc|c} 1 & 2 & 21 \\ 1 & 5 & 57 \end{array}\right) \quad R2 \to R2 - R1 \quad \left(\begin{array}{cc|c} 1 & 2 & 21 \\ 0 & 3 & 36 \end{array}\right)$$

$$R2 \to R2 \div 3 \quad \left(\begin{array}{cc|c} 1 & 2 & 21 \\ 0 & 1 & 12 \end{array}\right) \quad R1 \to R1 - 2R2 \quad \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 12 \end{array}\right)$$

Solution can be read from the matrix: $x = -3$ and $y = 12$

Bk1 P124 Ex3
All

Gaussian Elimination

This technique of solving a system of equations is used to solve three equations with three variables (unknowns).

❶❸ Use Gaussian elimination to obtain the solution to the system of equations:

$$\begin{aligned} x + 2y + z &= 4 \\ 2x - y - z &= 0 \\ 3x + 2y + z &= 6 \end{aligned}$$

- Express as **augmented matrix**: $\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -1 & -1 & 0 \\ 3 & 2 & 1 & 6 \end{array}\right)$

- Reduce the matrix to **upper triangular form**: $\left(\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \end{array}\right)$

$$\begin{array}{c} R1 \\ R2-2R1 \\ R3-3R1 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -5 & -3 & -8 \\ 0 & -4 & -2 & -6 \end{array}\right)$$

$$\begin{array}{c} R1 \\ R2 \div -1 \\ 5R3-4R2 \end{array} \quad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 5 & 3 & 8 \\ 0 & 0 & 2 & 2 \end{array}\right) \text{ i.e. upper triangular form}$$

- Use **back substitution** to find the values of the variables:

$$\begin{array}{ll} 2z = 2 & \Rightarrow z = 1 \\ 5y + 3z = 8 \Rightarrow 5y + 3 = 8 & \Rightarrow y = 1 \\ x + 2y + z = 4 \Rightarrow x + 2 + 1 = 4 & \Rightarrow z = 1 \end{array}$$

Bk1 P127 Ex4A
1st column

$(1,1,1)$

Once the matrix is in upper triangular form, it is possible to continue the EROs until the LHS is a 3×3 identity matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & p \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & r \end{array}\right)$$

No need for back substitution with this, the solution is (p, q, r)

This method is also available on the TI-83 calculator under "rref" (reduced-row echelon form) in the [MATRIX] menu

❶❹ Use Gaussian elimination to obtain the solution to the system of equations:

$$\begin{array}{c} 3x + y = 5 \\ x + 2y - 3z = -12 \\ x + 2z = 10 \end{array}$$

- Express as **augmented matrix**: $\left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 1 & 2 & -3 & -12 \\ 1 & 0 & 2 & 10 \end{array}\right)$

- Reduce the matrix to **upper triangular form**:

$$\begin{array}{c} R1 \\ R1-3R2 \\ R3-3R1 \end{array} \left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 0 & -5 & 9 & 41 \\ 0 & -2 & 5 & 22 \end{array}\right) \qquad \begin{array}{c} R1 \\ R2 \\ 5R3-2R2 \end{array} \left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 0 & -5 & 9 & 41 \\ 0 & 0 & 7 & 28 \end{array}\right)$$

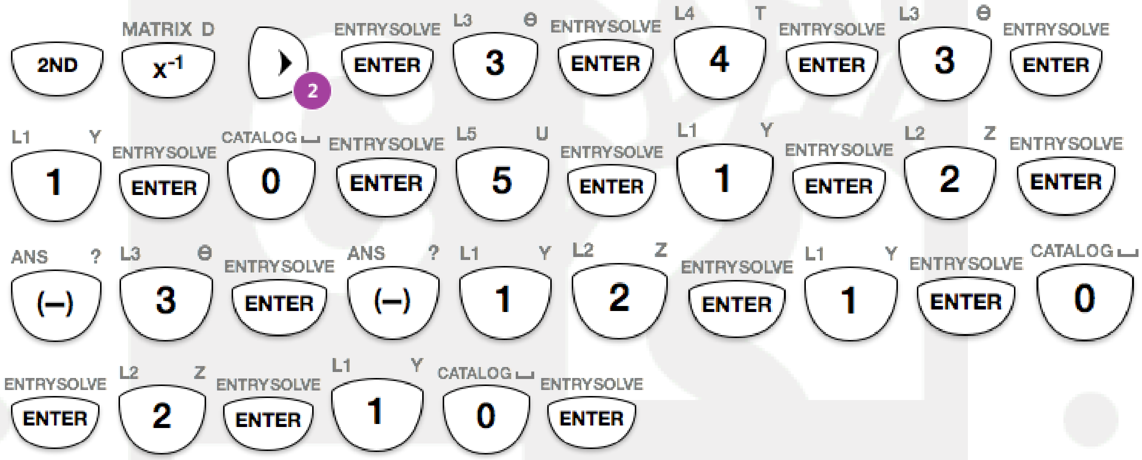
- Continue until the LHS is a <u>3×3 identity matrix</u>:

$$\begin{matrix} 5R1+R2 \\ 7R2-9R3 \\ R3 \div 7 \end{matrix}\left(\begin{array}{ccc|c} 15 & 0 & 9 & 5 \\ 0 & -35 & 0 & 35 \\ 0 & 0 & 1 & 4 \end{array}\right) \qquad \begin{matrix} R1-9R3 \\ R2 \div -35 \\ R3 \end{matrix}\left(\begin{array}{ccc|c} 15 & 0 & 0 & 30 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

$$\begin{matrix} R1 \div 15 \\ R2 \\ R3 \end{matrix}\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{array}\right)$$

- read off solution $(2,\ -1,\ 4)$

The TI-83 can be used to check your answer:

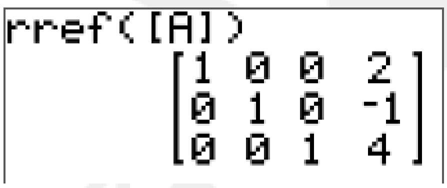


This stores the above example as a matrix in the calculator's memory.

Next step is to get to the home screen pressing:

Then press:

To show:



Bk1 P128 Ex4B
Q1

Redundancy/Inconsistency in a 3×3 System of Equations

Reminder: $0 = 0$ gives redundancy and $0 = a$ gives inconsistency

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$$\begin{aligned} x + 2y + 2z &= 11 \\ x - y + 3z &= 8 \\ 4x - y + 11z &= 35 \end{aligned}$$

produces the **augmented matrix**: $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 1 & -1 & 3 & 8 \\ 4 & -1 & 11 & 35 \end{array}\right)$

$$\begin{array}{r} R1 \\ R2 - R1 \\ R3 - 4R1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & -9 & 3 & -9 \end{array}\right) \qquad \begin{array}{r} R1 \\ R2 \\ R3 - 3R2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -3 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Redundancy $\Rightarrow$ no unique solution $\Rightarrow$ infinite number of solutions

Let $z = z$ and find y and x in terms of z using back substitution:

From $R2$: $-3y + z = -3 \Rightarrow y = \frac{z+3}{3}$

From $R3$: $x + 2y + 2z = 11 \Rightarrow x = 11 - 2y - 2z = 11 - 2\left(\frac{z+3}{3}\right) - 2z$

$$\Rightarrow x = \frac{33}{3} - \frac{2z+6}{3} - \frac{6z}{3} = \frac{27-8z}{3}$$

General solution is $\left(\frac{27-8z}{3}, \frac{z+3}{3}, z\right)$

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$$\begin{aligned} x + 2y + 2z &= 11 \\ 2x - y + z &= 8 \\ 3x + y + 3z &= 18 \end{aligned}$$

produces the **augmented matrix**: $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 8 \\ 3 & 1 & 3 & 18 \end{array}\right)$

$$\begin{array}{r} R1 \\ R2 - 2R1 \\ R3 - 3R1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & -5 & -3 & -15 \end{array}\right) \qquad \begin{array}{r} R1 \\ R2 \\ R3 - R2 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -14 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Inconsistency $\Rightarrow$ no solution

Bk1 P130 Ex5
1st column

Ill-Conditioned Equations

When a small change in any of the values in a system of equations produces a disproportionate change in the solutions then the equations are said to be ill-conditioned:

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$$\begin{aligned} x + 0.99y &= 1.99 \\ 0.99x + 0.98y &= 1.97 \end{aligned}$$

produces the augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 0.99 & 1.99 \\ 0.99 & 0.98 & 1.97 \end{array}\right)$$

$$\begin{array}{r} R1 \\ 0.99R1 - R2 \end{array} \quad \left(\begin{array}{cc|c} 1 & 0.99 & 1.99 \\ 0 & 0.0001 & 0.0001 \end{array}\right) \Rightarrow y = 1 \Rightarrow x = 1$$

If we now make a small change in the equations: $\begin{aligned} x + 0.99y &= 2.00 \\ 0.99x + 0.98y &= 1.97 \end{aligned}$

We get the augmented matrix: $\left(\begin{array}{cc|c} 1 & 0.99 & 2.00 \\ 0.99 & 0.98 & 1.97 \end{array}\right)$

$$\begin{matrix} R1 \\ R2 - 0.99R1 \end{matrix} \quad \left(\begin{array}{cc|c} 1 & 0.99 & 2.00 \\ 0 & -0.0001 & -0.01 \end{array}\right) \Rightarrow y = 100 \Rightarrow x = -97$$

From the above, it can be seen that a change of 0.001 (which is very small) in the RHS of one of the equations has produced a disproportionate change in the solutions.

Bk1 P136 Ex8
Q1 - 1st column
Q2, 3, 5

Determinant of a 3×3 Matrix

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ then $\det(A) = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$

Vector product

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

❶❽ $$\begin{vmatrix} 2 & 1 & 3 \\ 4 & -2 & 5 \\ 3 & 6 & 8 \end{vmatrix} = 2\begin{vmatrix} -2 & 5 \\ 6 & 8 \end{vmatrix} - 1\begin{vmatrix} 4 & 5 \\ 3 & 8 \end{vmatrix} + 3\begin{vmatrix} 4 & -2 \\ 3 & 6 \end{vmatrix}$$
$$= 2(-16 - 30) - 1(32 - 15) + 3(24 + 6)$$
$$= -92 - 17 + 90 = -19$$

Bk3 P25 Ex7
Q4a, 5a, 8c

Inverse of a 3×3 Matrix

❶❾ Find the inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 5 & 2 & 3 \end{pmatrix}$

Using the graphics calculator: $|A| = 1 \Rightarrow A^{-1}$ *exists*

To create the inverse of a 3×3 matrix, we start with the expression AI then begin EROs to change the expression to IA^{-1}

$$AI = \begin{pmatrix} 1 & 1 & 1 \\ 2 & -3 & -1 \\ 5 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Reduce the matrix A to **upper triangular form**

$$\begin{matrix} R1 \\ 2R1 - R2 \\ 5R1 - R3 \end{matrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 5 & 3 \\ 0 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 5 & 0 & -1 \end{pmatrix}$$

$$\begin{matrix} R1 \\ 2R2 - 3R3 \\ 3R2 - 5R3 \end{matrix} \quad \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -11 & -2 & 3 \\ -19 & -3 & 5 \end{pmatrix}$$

- Continue until the LHS is a 3×3 identity matrix

$$\begin{matrix} R1 - R2 \\ R2 \\ R3 \div -1 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 12 & 2 & -3 \\ -11 & -2 & 3 \\ 19 & 3 & -5 \end{pmatrix}$$

$$\begin{matrix} R1 - R3 \\ R2 \\ R3 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -7 & -1 & 2 \\ -11 & -2 & 3 \\ 19 & 3 & -5 \end{pmatrix}$$

So $A^{-1} = \begin{pmatrix} -7 & -1 & 2 \\ -11 & -2 & 3 \\ 19 & 3 & -5 \end{pmatrix}$

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$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\det(AB) = \det(A) \times \det(B)$$

Transformation Matrices

Transformations are geometrical changes – reflection, rotation and dilatation (enlarging or reducing).

We find the new coordinates of points by pre-multiplying them by the transformation matrix – e.g. for the point $(2,5)$ to be transformed by the matrix $T = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, we get:

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \end{pmatrix} \Rightarrow (2,5) \to (9,10)$$

From Higher Maths:

$f(x) \to$	Transformation	Change in (x, y)	Transformation Matrix
$-f(x)$	Reflection in x-axis	$(x, y) \to (x, -y)$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$f(-x)$	Reflection in y-axis	$(x, y) \to (-x, y)$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
$kf(x)$	Vertical stretch (k>1)	$(x, y) \to (x, ky)$	$\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$
$f(kx)$	Horizontal compression (k>1)	$(x, y) \to (\frac{x}{k}, y)$	$\begin{pmatrix} \frac{1}{k} & 0 \\ 0 & 1 \end{pmatrix}$
$f^{-1}(x)$	Reflection in $y = x$	$(x, y) \to (y, x)$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

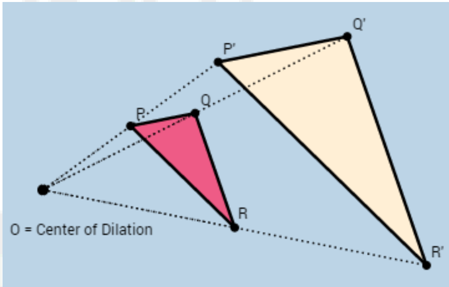
And for Advanced Higher

New function	Transformation	Change in (x, y)	Transformation Matrix
??	Anti-clockwise rotation of $\theta°$ about the origin.	$(x, y) \to (x\cos\theta - y\sin\theta,$ $x\sin\theta + y\cos\theta)$	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Combinations of these can be asked – multiply the transformation matrices together.

Dilatation is the enlargement of a shape that is centred at the origin:

The matrix associated with dilatation is $kI_{2\times2}$ where k is the scale factor of the enlargement $(k > 1)$ or reduction $(0 < k < 1)$



❷⓿ The triangle with vertices $A(1,2)$, $B(1,5)$ and $C(3,6)$ is enlarged by a scale factor of 2. Find the coordinates of the vertices of its image.

Setting up a transformation matrix T: $T = 2\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}$

$$\begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}\begin{pmatrix}1 & 1 & 3\\2 & 5 & 6\end{pmatrix} = \begin{pmatrix}2 & 2 & 6\\4 & 10 & 12\end{pmatrix} \Rightarrow A'(2,4),\ B'(2,10) \text{ and } C'(6,12)$$

NB:

- be aware of some confusion with A' notation being a transpose matrix and an image coordinate!
- $30°$ clockwise rotation $= 330°$ anti-clockwise rotation