Unit 1: Methods in Algebra and Calculus (H7X2 77)

Integration – New rules and Functions

In Higher Maths the formula sheet contained only two 'Standard integrals'. You were also expected to learn that:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{(n+1)a} + c$$ *NB linear function only*

Higher

Standard integrals	
$f(x)$	$\int f(x)dx$
$\sin ax$	$-\frac{1}{a}\cos ax + c$
$\cos ax$	$\frac{1}{a}\sin ax + c$

P70 Ex1A & 1B All

Advanced Higher

Standard integrals	
$f(x)$	$\int f(x)dx$
$\sec^2(ax)$	$\frac{1}{a}\tan(ax) + c$
$\frac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{a^2 + x^2}$	$\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^{ax}	$\frac{1}{a}e^{ax} + c$

No $\frac{1}{a}$!

Examples:

❶ $\int \cos(2x+1)dx = \frac{1}{2}\sin(2x+1) + c$

❷ $\int (5x+3)^4 dx = \frac{(5x+3)^5}{5\times 5} + c = \frac{1}{25}(5x+3)^5 + c$

❸ $\int (3x+5)^{-2} dx = \frac{(3x+5)^{-1}}{3\times -1} + c = -\frac{1}{3(3x+5)} + c$

❹ $\int_0^1 e^{3x} dx = \left[\frac{1}{3}e^{3x}\right]_0^1 = \left(\frac{1}{3}e^3\right) - \left(\frac{1}{3}e^0\right) = \frac{1}{3}e^3 - \frac{1}{3} = \frac{1}{3}(e^3 - 1)$

❺ $\int \frac{4}{x} dx = 4\int \frac{1}{x} dx = 4(ln|x|) + c = 4ln|x| + c$

NB: the constant can be removed from an integration as a factor.

❻ $\int \tan^2 x\, dx$ *NB* $\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{1-\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}$

$$\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + c$$

❼ $\int_{1.5}^3 \frac{dx}{\sqrt{9-x^2}} = \int_{1.5}^3 \frac{dx}{\sqrt{3^2 - x^2}} = \left[\sin^{-1}\left(\frac{x}{3}\right)\right]_{1.5}^3 = \sin^{-1}(1) - \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

P72 Ex2A & 2B All

Bk2 P61 Ex1A Q2(a) &(e)
P62 Ex1B 1st column

Integration by Substitution

Inverse of the chain rule – look for product of functions where one function closely resembles the derivative of the other.

❽ $\int x(x^2+3)^3\,dx$

Let $u = x^2+3$ *so* $\frac{du}{dx} = 2x$ *re-arranging gives* $dx = \frac{du}{2x}$

$\int x(x^2+3)^3\,dx = \int xu^3\frac{du}{2x}$ replacing x^2+3 and dx

$$= \int \frac{u^3}{2}du = \frac{1}{2}\int u^3 du = \frac{1}{2}\times\frac{u^4}{4} + c = \frac{1}{8}u^4 + c$$

Replace u with x^2+3 to give $\frac{1}{8}(x^2+3)^4 + c$

❾ $\int 3x^2(x^3-4)^5\,dx$ *Let* $u = x^3-4$ *so* $\frac{du}{dx} = 3x^2 \Rightarrow dx = \frac{du}{3x^2}$

$$\int 3x^2(x^3-4)^5\,dx = \int 3x^2u^5\frac{du}{3x^2} = \int u^5\,du = \frac{u^6}{6} + c$$

Then back to x: $\frac{1}{6}(x^3-4)^6 + c$

❶⓿ $\int 8\cos x\sin^3 x\,dx$ *Let* $u = \sin x$ *so* $\frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$

$$= 8\int \cos x\, u^3\frac{du}{\cos x} = 8\int u^3 du = 8\times\frac{1}{4}u^4 + c = 2u^4 + c$$

Then back to x: $2\sin^4 x + c$

❶❶ $\int \frac{4\ln x}{x}dx$ *Let* $u = \ln x$ *so* $\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

$$\int \frac{4\ln x}{x}dx = 4\int\frac{u}{x}x\,dx = 4\int u\,du = 4\times\frac{1}{2}u^2 + c = 2u^2 + c$$

Then back to x: $2(\ln x)^2 + c$

❶❷ Use the substitution $x = \tan\theta$ to determine $\int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}}$

Let $x = \tan\theta$ *so* $\frac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\,d\theta$

New limits needed: *Upper* - $\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Lower - $\tan\theta = 0 \Rightarrow \theta = 0$

NB: Less obvious questions will have the substitution given.

$$\int_0^1 \frac{dx}{(1+x^2)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{(1+\tan^2\theta)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{(\sec^2\theta)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta d\theta}{\sec^3\theta}$$

$$= \int_0^{\frac{\pi}{4}} \frac{d\theta}{\sec\theta} = \int_0^{\frac{\pi}{4}} \cos\theta\, d\theta = [\sin\theta]_0^{\frac{\pi}{4}} = \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}}$$

P74 Ex3
Q1-15 - 1st column

P75 Ex4A
Q1-5 - 1st column

P76 Ex4B
Q4-9 - 1st column

❶❸ $\int_4^9 \frac{x+1}{x^2+2x-7}dx$ Let $u = x^2+2x-7$ so $\frac{du}{dx} = 2x+2 \Rightarrow dx = \frac{du}{2x+2}$

New limits: Upper - $x = 9 \Rightarrow 9^2 + 2\times 9 - 7 = 92$

Lower - $x = 4 \Rightarrow 4^2 + 2\times 4 - 7 = 17$

$$\int_4^9 \frac{x+1}{x^2+2x-7}dx = \int_{17}^{92} \frac{x+1}{u} \times \frac{du}{2(x+1)} = \frac{1}{2}\int_{17}^{92} \frac{du}{u} = \frac{1}{2}[\ln|u|]_{17}^{92}$$

$$= \frac{1}{2}(\ln 92 - \ln 17) = \frac{1}{2}\ln\frac{92}{17}$$

NB: No need to go back to x when we have created new limits

P77 Ex5A
Odd numbers (not Q3)

P77/78 Ex5B
All

Integrals of Rational Functions

Remember partial fractions!!

$\frac{ax+b}{(cx+d)(ex+f)} = \frac{A}{(cx+d)} + \frac{B}{(ex+f)}$ - Up to 3 linear factors

$\frac{ax^2+bx+c}{(dx+e)(fx+g)^2} = \frac{A}{(dx+e)} + \frac{B}{(fx+g)} + \frac{C}{(fx+g)^2}$ - Repeated linear factors

$\frac{ax^2+bx+c}{(dx+e)(fx^2+gx+h)} = \frac{A}{(dx+e)} + \frac{Bx+C}{(fx^2+gx+h)}$ - Linear & irreducible quadratic

$\frac{\text{degree} \geq \mathbf{n} \text{ polynomial}}{\text{degree } \mathbf{n} \text{ polynomial}}$ - Improper Rational Function (needs division)

Rational functions can be integrated by first converting them to partial fractions:

❶❹ $\int \frac{dx}{x^2-3x+2}$

Convert $\frac{1}{x^2-3x+2}$ *to partial fractions*

$$\frac{1}{(x-2)(x-1)}=\frac{A}{x-2}+\frac{B}{x-1}$$

$$A(x-1)+B(x-2)=1$$

Let $x=1 \Rightarrow A(0)+B(-1)=1 \Rightarrow B=-1$

Let $x=2 \Rightarrow A(1)+B(0)=1 \Rightarrow A=1$

$$=\int\left(\frac{1}{x-2}-\frac{1}{x-1}\right)dx$$

$$=\ln|x-2|-\ln|x-2|+c$$

$$=\ln\left(\left|\frac{x-2}{x-1}\right|\right)+c$$

❶❺ $\int \frac{x}{x^2-6x+9}dx$

Convert $\frac{x}{x^2-6x+9}$ *to partial fractions*

$$\frac{x}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}$$

$$A(x-3)+B=x$$

Let $x=3 \Rightarrow A(0)+B=3 \Rightarrow B=3$

Let $x=1 \Rightarrow A(-2)+3=1 \Rightarrow A=1$

$$=\int\left(\frac{1}{x-3}+\frac{3}{(x-3)^2}\right)dx$$

$$=\int\left(\frac{1}{x-3}+3(x-3)^{-2}\right)dx$$

$$=\ln|x-3|+\frac{3(x-3)^{-1}}{-1}+c$$

$$=\ln|x-3|-\frac{3}{x-3}+c$$

❶❻ *2015 Q17 of 18:* Find $\int \frac{2x^3-x-1}{(x-3)(x^2+1)}dx,\ x>3.$ *9 marks*

Question	Expected Answer/s	Max Mark	Additional Guidance
17	$x^3-3x^2+x-3\overline{)2x^3+0x^2-x-1}$ quotient 2; $2x^3-6x^2+2x-6$; remainder $6x^2-3x+5$	9	•[1] for knowing to divide and starting division •[2] correct division[1].
	$\int\frac{2x^3-x-1}{(x-3)(x^2+1)}dx=\int\left(2+\frac{6x^2-3x+5}{(x-3)(x^2+1)}\right)dx$		
	$=\int 2+\frac{A}{x-3}+\frac{Bx+C}{x^2+1}dx$		•[3] for correct form of PFs[5].
	$6x^2-3x+5=A(x^2+1)+(Bx+C)(x-3)$		•[4] creating correct equation
	$x=0 \quad 5=A-3C$ $x=3 \quad 50=10A \quad \Rightarrow A=5$ $C=0$		•[5] for any two values[4].
	$x=1 \quad 8=2A-2B-2C$ $8=10-2B \quad \Rightarrow B=1$		•[6] for third value[4].
	$\int\frac{2x^3-x-1}{(x-3)(x^2+1)}dx=\int 2+\frac{5}{x-3}+\frac{x}{x^2+1}dx$ $=2x+5\ln\|x-3\|+\frac{1}{2}\ln(x^2+1)+k$		•[7] for putting into integral *and* any one term correctly integrated[3]. •[8] for any second term. •[9] for third term and $+k$[2]

NB: $\int\frac{x}{x^2+1}$ *is found using integration by substitution where* $u=x^2+1$:

Not all rational functions require the use of partial fractions:

❶❼ $\int \frac{x+3}{x^2+4}\,dx = \int \frac{x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx$

Split the fraction similar to Higher work

$\int \frac{x}{x^2+4}\,dx$ *Let* $u =$ *so* $\frac{du}{dx} = 2x$ *re-arranging gives* $dx = \frac{du}{2x}$

$$\int \frac{x}{x^2+4}\,dx = \int \frac{x}{u} \times \frac{du}{2x} = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + c = \frac{1}{2}\ln|x^2+4\,| + c$$

$$\int \frac{3}{x^2+4}\,dx = 3\int \frac{1}{x^2+2^2}\,dx = 3 \times \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$$

Thus: $\int \frac{x}{x^2+4}\,dx + \int \frac{3}{x^2+4}\,dx = \frac{1}{2}\ln|x^2+4\,| + \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$

Bk2 P64 Ex2 Q5(a) & (d) and Q6(a) & (d)
P66 Ex3A Q3(a), (c) and (e)
P67 Ex3B 1st column

<u>Useful Integrals:</u>

- Using the chain rule:

$$\frac{d}{dx}\left[\frac{1}{2}(f(x))^2\right] = f(x) \times f'(x) \text{ so } \int [f(x)f'(x)]dx = \frac{1}{2}(f(x))^2 + c$$

- Similarly:

$$\frac{d}{dx}[\ln|f(x)|] = \frac{1}{f(x)} \times f'(x) \text{ so } \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$