# Integration by Parts

Inverse of the product rule:

$f(x) = uv$ so $f'(x) = u'v + uv'$

$$\frac{d}{dx}(uv) = u'v + uv'$$

Integrating both sides gives: $\int \frac{d}{dx}(uv)dx = \int u'vdx + \int uv'dx$

Simplifying gives: $uv = \int u'vdx + \int uv'dx$

Re-arranging gives: $\int u'vdx = uv - \int uv'dx$ OR $\int uv'dx = uv - \int u'vdx$

❶ $\int x\cos x\, dx$

*Let* $u' = \cos x$ *so* $u' = -\sin x$

*Let* $v = x$ *so* $v' = 1$

$\int x\cos x\, dx = x\sin x - \int \sin x \times 1\, dx = x\sin x + \cos x + c$

This makes the second integration easier

❷ $\int (3x+2)e^x\, dx$

*Let* $u' = e^x$ *so* $u = e^x$

$v = 3x + 2$ *so* $v' = 3$

$\int (3x+2)e^x\, dx = e^x(3x+2) - \int e^x \times 3\, dx$ - making 2nd integral easier

$$e^x(3x+2) - 3\int e^x\, dx = e^x(3x+2) - 3e^x + c$$

$$= e^x(3x + 2 - 3) + C = e^x(3x-1) + c$$

❸ $\int x\sec^2 x\, dx$

*Let* $u' = \sec^2 x$ *so* $u = \tan x$ *since* $\sec^2 x$ *is hard to differentiate but easy to integrate*

$v = x$ *so* $v' = 1$

$$\int x\sec^2 x\, dx = x\tan x - \int \tan x\, dx = x\tan x - \int \frac{\sin x}{\cos x} dx$$

*Let* $u = \cos x$ *so* $\frac{du}{dx} = -\sin x$ *so* $dx = -\frac{du}{\sin x}$ *i.e. integration by substitution*

$$x\tan x - \int \frac{\sin x}{\cos x} dx = x\tan x - \int \frac{du}{u} = x\tan x - \ln|u| + c$$

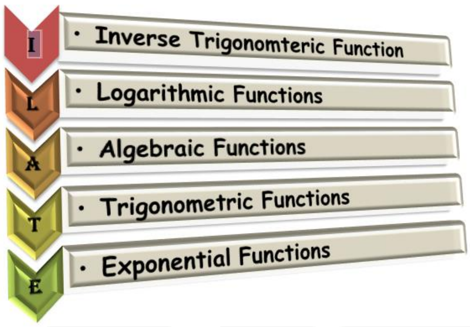
$$= x\tan x - \ln|\cos x| + c$$

❹ $\int \sin x \ln|\cos x|\, dx$ Let $u' = \sin x$ so $u = -\cos x$

$v = \ln|\cos x|$ so $v' = \frac{1}{\cos x} \times -\sin x = -\tan x$

$$\int \sin x \ln|\cos x|\, dx = -\cos x \ln|\cos x| - \int -\cos x \times -\tan x\, dx$$

$$= -\cos x \ln|\cos x| - \int \sin x\, dx == -\cos x \ln|\cos x| + \cos x + c$$




P69 Ex4
Q1c, 1f, 2c, 2f, 3, 4

## Repeated Use & Cyclic Use of Integration by Parts

Certain functions require more than one application of Integration by Parts and others can end up going round in circles. At unit level, you are only required to apply the formula once. However, at assessment level you could be required to apply the formula more than once:

❺ $\int x^2 \sin x\, dx$ Let $u' = \sin x$ so $u = -\cos x$

$v = x^2$ so $v' = 2x$

$\int x^2 \sin x\, dx = -x^2 \cos x + 2\int x \cos x\, dx$ Let $u' = \cos x$ so $u = \sin x$

$v = x$ so $v' = 1$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2\sin x - 2\int \sin x\, dx$$

$$= -x^2 \cos x + 2\sin x + 2\cos x + c$$

You could also be required to solve a cyclic function i.e. one that doesn't simplify and ends up back at the beginning:

❻ $\int e^x \sin x\, dx$ Let $u' = e^x$ so $u = e^x$

$v = \sin x$ so $v' = \cos x$

$\int e^x \sin x\, dx = e^x \sin x - \int e^x \cos x\, dx$ Let $u' = e^x$ so $u = e^x$

$v = \cos x$ so $v' = -\sin x$

$\int e^x \sin x\, dx = e^x \sin x - e^x \cos x - \int e^x \sin x\, dx$ - back where we started!!

However, if we re-arrange the equation by collecting $\int e^x \sin x\, dx$ terms we get:

$$2\int e^x \sin x\, dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x\, dx = \frac{1}{2}(e^x \sin x - e^x \cos x) + c$$

Bk2 P71 Ex5A
Q1 - 1st column
Q2a, 2g, 2f

# **Differential Equations** - Variables Separable

Remember this type of question from Higher Mathematics?

**10.** Given that

- $\frac{dy}{dx} = 6x^2 - 3x + 4$, and
- $y = 14$ when $x = 2$,

express $y$ in terms of $x$.

| Question | Generic scheme | Illustrative scheme | Max mark |
|---|---|---|---|
| 10. | •¹ know to and integrate one term | •¹ eg $2x^3 \ldots$ | 4 |
| | •² complete integration | •² eg $\ldots - \frac{3}{2}x^2 + 4x + c$ | |
| | •³ substitute for $x$ and $y$ | •³ $14 = 2(2)^3 - \frac{3}{2}(2)^2 + 4(2) + c$ | |
| | •⁴ state equation | •⁴ $y = 2x^3 - \frac{3}{2}x^2 + 4x - 4$ stated explicitly | |

This is known as a Differential equation. They arise in modelling of physical situations such as electric circuits and vibrating systems.

- $2x^3 - \frac{3}{2}x^2 + 4x + c$ is known as the GENERAL solution to the equation
- $2x^3 - \frac{3}{2}x^2 + 4x - 4$ is known as the PARTICULAR solution to the equation

Bk2 P74 Ex6
Q1 - 1st column
Q2a, 2c, 2e
Q3

At AH level the differential equation can be implicit and/or involve more variables e.g. $x$ and $y$ in the equation:

❼ Find the general solution of the differential equation $\frac{dy}{dx} = y$

*Re-arranging to separate the variables gives:* $\frac{dy}{y} = dx$

*Now integrate both sides wrt its own variable:* $\int \frac{dy}{y} = \int 1.dx$

$\ln y = x + c$

*Take exponential of both sides:* $e^{\ln y} = e^{x+c}$

$y = e^x \times e^c$

*Since $e^c$ is still a constant, we can simplify* $y = ke^x$

❽ Find the general solution of the differential equation $\frac{dy}{dx} = \frac{3}{\sqrt{3y+1}}$

*Separating variables:* $\int (3y+1)^{\frac{1}{2}} dy = \int 3.dx$

*Integrating both sides:* $\frac{(3y+1)^{\frac{3}{2}}}{\frac{3}{2}\times 3} = 3x + C$

$$\frac{2}{9}(3y+1)^{\frac{3}{2}} = 3x + C$$

*Make y the subject:* $(3y+1)^{\frac{3}{2}} = \frac{9}{2}(3x + C)$

*Square and Cube root both sides:* $3y + 1 = \frac{9}{2}(3x + C)^{\frac{2}{3}}$

$$y = \frac{1}{3}\left(\frac{9}{2}(3x + C)^{\frac{2}{3}} - 1\right)$$

Bk2 P76 Ex7
Q1a, 1e, 2a, 2e

❾ Find the general solution to the equation $(2x+1)\,\frac{dy}{dx} = y^2$

$$\int \frac{dy}{y^2} = \int \frac{dx}{2x+1} \Rightarrow \frac{-1}{y} = \frac{1}{2}\ln|2x+1| + c$$

*Re-write c as* $\ln k$ $\Rightarrow \frac{-1}{y} = \frac{1}{2}\ln|2x+1| + \ln k$

$$\Rightarrow \frac{-1}{y} = \frac{1}{2}\ln\left|k(2x+1)^{\frac{1}{2}}\right|$$

$$y = -\frac{1}{\ln\left|k(2x+1)^{\frac{1}{2}}\right|}$$

❶❿ Find the particular solution to the equation $x\frac{dy}{dx} - y^2 - 1 = 0$
Given that $y = 1$ when $x = 1$

*Make* $\frac{dy}{dx}$ *the subject:* $\frac{dy}{dx} = \frac{1+y^2}{x} \Rightarrow \frac{dy}{1+y^2} = \frac{dx}{x}$

*Integrate both sides:* $\int \frac{dy}{1+y^2} = \int \frac{dx}{x}$

$$\tan^{-1} y = \ln|x| + c$$

$$y = \tan(\ln|x| + c)$$

$$1 = \tan(\ln|1| + c)$$

$$1 = \tan(c) \Rightarrow c = \tan^{-1}(1) = \frac{\pi}{4}$$

$$y = \tan\left(\ln|x| + \frac{\pi}{4}\right)$$

Bk2 P77 Ex8
Q1 - 1st column
Q2a, 5a, 5b

# First Order Differential Equations

A first order linear differential equation is of the form: $\frac{dy}{dx} + P(x)y = Q(x)$

To solve a 1st order differential equation we must:

1. Identify $P(x)$ and $Q(x)$ $\quad \frac{dy}{dx} + \frac{2}{x}y = x \quad P(x) = \frac{2}{x}$ and $Q(x) = x$
2. Integrate $P(x)$ $\quad \int \frac{2}{x} dx = 2\ln x = \ln x^2$
3. Find the Integrating Factor (IF) $e^{\int P(x)dx}$ $\quad e^{\ln x^2} = x^2$
4. Multiply both sides by the IF – in this case $x^2$

   *NB the LHS always ends up the exact differential of* $e^{\int P(x)dx}y$
   $$x^2\left(\frac{dy}{dx} + \frac{2}{x}y\right) = x^2 \times x$$
5. Simplify $\quad \frac{d}{dx}(x^2y) = x^3$
6. Integrate both sides $\quad x^2y = \int x^3 dx = \frac{x^4}{4} + c$
7. Make $y$ the subject $\quad y = \frac{x^2}{4} + \frac{c}{x^2}$

❶❶ Find the solution to the differential equation $\frac{dy}{dx} - 2xy = 3x$

1. Identify $P(x)$ and $Q(x)$ $\quad P(x) = -2x$ and $Q(x) = 3x$
2. Integrate $P(x)$ $\quad \int(-2x)dx = -x^2$
3. Find the Integrating Factor (IF) $e^{\int P(x)dx}$ $\quad e^{-x^2}$
4. Multiply both sides by the IF – in this case $x^2$
   $$e^{-x^2}\left(\frac{dy}{dx} - 2xy\right) = e^{-x^2} \times 3x$$
5. Simplify $\quad \frac{d}{dx}\left(e^{-x^2}y\right) = 3xe^{-x^2}$
6. Integrate both sides $\quad e^{-x^2}y = \int 3xe^{-x^2}dx$

   $\int 3xe^{-x^2}dx = \int 3xe^{-u}\frac{du}{2x}$ using the substitution $u = x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$
   $$= \frac{3}{2}\int e^{-u}du = -\frac{3}{2}e^{-u} + c = -\frac{3}{2}e^{-x^2} + C$$
7. Make $y$ the subject $\quad e^{-x^2}y = -\frac{3}{2}e^{-x^2} + C$
   $$y = -\frac{3}{2} + \frac{C}{e^{-x^2}} \Rightarrow y = Ce^{x^2} - \frac{3}{2}$$

❶❷ Solve the equation $x^2\frac{dy}{dx} - x^3 + xy = 0$

*Must be in the form* $\frac{dy}{dx} + P(x)y = Q(x)$ *so ÷ through by* $x^2$

$$\frac{dy}{dx} + \frac{1}{x}y = x$$

1. $P(x) = \frac{1}{x}$ and $Q(x) = x$
2. $\int P(x)dx = \int\left(\frac{1}{x}\right)dx = \ln|x|$
3. (IF) $e^{\int P(x)dx} = e^{\ln x} = |x|$
4. $|x|\left(\frac{dy}{dx} + \frac{1}{x}y\right) = |x| \times x$ $|x|$ *appears on both sides so* | | *unnecessary*
5. $\frac{d}{dx}(xy) = x^2$
6. $xy = \int x^2 dx$ *leading to* $xy = \frac{x^3}{3} + C$
7. $y = \frac{x^2}{3} + \frac{C}{x}$

❶❸ Solve the equation $x\frac{dy}{dx} - y = x^2$ given that when $x = 1, y = 0$

$$\frac{dy}{dx} - \frac{y}{x} = x$$

- $P(x) = -\frac{1}{x}$ and $Q(x) = x$
- $\int P(x)dx = \int\left(-\frac{1}{x}\right)dx = -\ln|x| = \ln|x|^{-1} = \ln\left|\frac{1}{x}\right|$
- (IF) $e^{\int P(x)dx} = e^{\ln\left|\frac{1}{x}\right|} = \left|\frac{1}{x}\right|$
- $\left|\frac{1}{x}\right|\left(\frac{dy}{dx} - \frac{y}{x}\right) = \left|\frac{1}{x}\right| \times x$ | | *now unnecessary*
- $\frac{d}{dx}\left(\frac{1}{x}y\right) = 1$
- $\frac{y}{x} = \int 1dx$ *leading to* $\frac{y}{x} = x + C$
- $y = x^2 + Cx$
- *Using* $x = 1, y = 0$ we get $C = -1$ so $y = x^2 - x$

| General Solution<br>Bk 3 P114 Ex1<br>Q3 - 2nd column | Particular Solution<br>Bk 3 P116 Ex2<br>Q1a, 1c & 1e |
|---|---|

# <u>Second Order Differential Equations</u> - Homogeneous

A second order linear homogeneous differential equation is of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

There are 3 types, one of which requires the use of complex numbers so you need to know that:

- $i$ is a number such that $i^2 = -1$, $i \in ℂ$   $i^2 = -1 \Rightarrow i = \sqrt{-1}$
- $ℂ$ is the set of Complex Numbers (similar to $R$ is the set of Real Numbers)
- $z$ denotes a complex number and is made up of two parts, a real part and an imaginary part
- $z = a + bi$ where $a = Re(z)$ and $b = Im(z)$

Remember the quadratic formula: $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

❶❹ Solve the equation: $z^2 - 2z + 5 = 0$

$a = 1$, $b = -2$, $c = 5$ so $b^2 - 4ac = -16$ so $z = \frac{2\pm\sqrt{-16}}{2} = \frac{2\pm\sqrt{16}\sqrt{-1}}{2}$

Bk2 P90 Ex1 Q3

$$z = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

Type 1 – Real and distinct roots has general solution: $y = Ae^{px} + Be^{qx}$

❶❺ Solve the differential equation: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

1. Create an AUXILIARY equation $am^2 + bm + c = 0$

   *For the above example:* $m^2 - 5m + 6 = 0$
2. Solve the auxiliary equation: $m = 2$, $m = 3$
3. These roots are the $p$ and $q$ values for the general solution

$$y = Ae^{2x} + Be^{3x}$$

Type 2 – Real repeated roots has general solution: $y = Ae^{kx} + Bxe^{kx}$

❶❻ Solve the differential equation: $9\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0$

1. Auxiliary equation $9m^2 - 6m + 1 = 0$
2. Solve the auxiliary equation: $(3m-1)^2 = 0$ so $m = \frac{1}{3}$
3. This root is the $k$ value for the general solution

$$y = Ae^{\frac{1}{3}x} + Bxe^{\frac{1}{3}x} = e^{\frac{1}{3}x}(A + Bx)$$

Type 3 – Complex (Non-Real) roots has general solution:

$$y = e^{rx}(A\cos sx + B\sin sx) \text{ where } r = Re \text{ and } s = Im$$

❶❼ Find the particular solution to the differential equation:

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0 \text{ if } x = 0 \text{ and } y = 2 \text{ when } \frac{dy}{dx} = 0$$

1. Auxiliary equation $m^2 + 4m + 13 = 0$
2. Use the quadratic formula: $m = \frac{-4\pm\sqrt{-36}}{2} = \frac{-4\pm 6i}{2} = -2 \pm 3i$
3. The general solution is: $y = e^{-2x}(A\cos 3x + B\sin 3x)$
4. Substitute for $x$ and $y$ $0 = e^0(A\cos 0 + B\sin 0)$ gives $A = 2$
5. Differentiate: $\frac{dy}{dx} = -2e^{-2x}(A\cos 3x + B\sin 3x) + e^{-2x}(-3A\cos 3x + 3B\sin 3x)$
6. Substitute for $x$ and $\frac{dy}{dx}$: $0 = -2A + 3B \Rightarrow 0 = -4 + 3B \Rightarrow B = \frac{4}{3}$

Particular solution is $y = e^{-2x}\left(2\cos 3x + \frac{4}{3}\sin 3x\right)$

❶❽ Find the particular solution to the differential equation:

$$2\frac{d^2y}{dx^2} + 7\frac{dy}{dx} - 4y = 0 \text{ if } x = 0 \text{ and } y = 1 \text{ when } \frac{dy}{dx} = 2$$

1. $2m^2 + 7m - 4 = 0$
2. $(2m-1)(m+4) = 0 \Rightarrow m = \frac{1}{2}$ or $m = -4$
3. $y = Ae^{\frac{1}{2}x} + Be^{-4x}$
4. Substitute for $x$ and $y$ $1 = A + B$
5. Differentiate: $\frac{dy}{dx} = \frac{1}{2}Ae^{\frac{1}{2}x} - 4Be^{-4x}$
6. Substitute for $x$ and $\frac{dy}{dx}$: $2 = \frac{1}{2}A - 4B \Rightarrow A = \frac{4}{3} \Rightarrow B = -\frac{1}{3}$ by sim equations

Particular solution is $y = \frac{4}{3}e^{\frac{1}{2}x} - \frac{1}{3}e^{-4x}$

Real & Distinct
Bk 3 P119 Ex3
Q1a, 2a

Real & Repeated
Bk 3 P120 Ex4
Q1a, 2a

Complex
Bk 3 P122 Ex5A
Q1a, 2a
Bk 3 P122 Ex5B
Q1a, 2

# <u>Second Order Differential Equations</u> – Non-Homogeneous

A second order linear homogeneous differential equation is of the form:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q(x) \text{ where } Q(x) \text{ is a function of } x$$

The solution is $y=CF+PI$ where $CF$ is the Complementary Function and $PI$ is the Particular Integral

<u>Examples</u>:

❶❾ Solve $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=15x-7$

*The $CF$ is what we have been doing already:* $m^2-5m+6=0$

*This gives:* $(m-2)(m-3)=0 \Rightarrow m=2, m=3$ *i.e. 2 real roots*

*$CF$ is therefore:* $y=Ae^{2x}+Be^{3x}$

*The RHS of the original equation $Q(x)$ is linear ($15x-7$) so we use a linear function as the PI:* $y=Cx+D$

*Now find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ then substitute these into the original equation to find $C$ and $D$:* $\frac{dy}{dx}=C$ *and* $\frac{d^2y}{dx^2}=0$ *so for* $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=15x-7$

*This gives:* $0-5C+6(Cx+D)=15x-7$

$-5C+6Cx+6D=15x-7$

*Equating co-efficients:* $6C=15 \Rightarrow C=\frac{5}{2}$

*Equating constants:* $-5C+6D=-7 \Rightarrow D=\frac{11}{12}$

*The $PI$ is therefore:* $y=\frac{5}{2}x+\frac{11}{12}$

*Giving a general solution of:* $y=Ae^{2x}+Be^{3x}+\frac{5}{2}x+\frac{11}{12}$

<u>Note:</u>

- for $Q(x)=ax^2+bx+c$, use a $PI$ of the form : $y=Cx^2+Dx+E$
- for $Q(x)=ae^{kx}$ use a $PI$ of the form : $y=Ce^{kx}$
- for $Q(x)=a\sin x \pm b\cos x$ use a $PI$ of the form : $y=C\sin x+D\cos x$

**20** Find the particular solution to the equation $\frac{d^2y}{dx^2} + 4y = 26e^{3x}$ given that $y = 3$ when $x = 0$ and $y = 1$ when $x = \frac{\pi}{4}$

$CF$: $m^2 + 4 = 0 \Rightarrow m = \mp 2i$ i.e. complex roots

$y = e^0(A\cos 2x + B\sin 2x) = A\cos 2x + B\sin 2x$

$PI$: $Q(x) = e^{3x}$ *so use* $y = Ce^{3x}$

$\frac{dy}{dx} = 3Ce^{3x}$ *and* $\frac{d^2y}{dx^2} = 9Ce^{3x}$

*so for:* $\frac{d^2y}{dx^2} + 4y = 26e^{3x} \Rightarrow 9Ce^{3x} + 4Ce^{3x} = 26e^{3x} \Rightarrow C = 2$

*General solution:* $y = A\cos 2x + B\sin 2x + 2e^{3x}$

*Substitute* $y = 3$ *and* $x = 0$ $3 = A\cos 0 + B\sin 0 + 2e^0 \Rightarrow A = 1$

*Substitute* $y = 1$ *and* $x = \frac{\pi}{4}$ $1 = \cos\frac{\pi}{2} + B\sin\frac{\pi}{2} + 2e^{\frac{3\pi}{4}} \Rightarrow B = 1 - 2e^{\frac{3\pi}{4}}$

*Particular solution:* $y = \cos 2x + \left(1 - 2e^{\frac{3\pi}{4}}\right)\sin 2x + 2e^{3x}$

**Special Case for the $PI$ (usually for exponentials):**

When a $PI$ (of the same format as $Q(x)$) has already appeared in the $CF$ then try $xQ(x)$ or $x^2Q(x)$e.g.

1. $CF$ is $y = Ae^{2x} + Be^{3x}$ and $Q(x) = 2e^{3x}$

   $Q(x)$ is an exponential so we try $y = Ce^{3x}$

   As $Ce^{3x}$ appears in the $CF$ we try $y = xCe^{3x}$

   As $y = xCe^{3x}$ does not appear in the $CF$, it is okay to use.

2. $CF$ is $y = Ae^{2x} + Bxe^{3x}$ and $Q(x) = 3e^{2x}$

   $Q(x)$ is an exponential so we try $y = Ce^{2x}$

   As $Ce^{2x}$ appears in the $CF$ we try $y = xCe^{2x}$

   As $xCe^{2x}$ appears in the $CF$ we try $y = x^2Ce^{2x}$

   As $y = x^2Ce^{2x}$ does not appear in the $CF$, it is okay to use.

3. $CF$ is $y = Ae^{2x} + Be^{3x}$ and $Q(x) = 2e^{4x}$

   $Q(x)$ is an exponential so we try $y = Ce^{4x}$ ***(note different power)***

   As $y = Ce^{4x}$ does not appear in the $CF$, it is okay to use.

PI form given
Bk 3 P124 Ex6
Q1

PI form not given
Bk 3 P126 Ex7A
Q1a, c, e, i, 2, 3a, c

PI form not given
Bk 3 P127 Ex7A
Q1a