Unit 1: Methods in Algebra and Calculus (H7X2 77)

Differentiation 2 – Implicit and Parametric Functions, Logarithmic Differentiation

Explicit function: y is expressed explicitly as a function of x i.e. y is the subject of the formula e.g. $y = 2x + 3$

Implicit function: y is not expressed explicitly as a function of x i.e. y is not the subject of the formula e.g. $y - 2x = 3$

Examples: ❶ Given $3x^2 + 7xy + 9y^2 = 6$, find $\frac{dy}{dx}$

Differentiate both sides with respect to x

$$\frac{d}{dx}(3x^2 + 7xy + 9y^2) = \frac{d}{dx}(6)$$

$$\frac{d}{dx}(3x^2) + \frac{d}{dx}(7xy) + \frac{d}{dx}(9y^2) = \frac{d}{dx}(6)$$

$$6x + 7\frac{d}{dx}(xy) + 9\frac{d}{dx}(y^2) = 0$$

- *The derivative of y is $\frac{dy}{dx}$*
- *Use the chain rule for $\frac{d}{dx}(y^2)$: $\frac{d}{dx}(y)^2 = 2y \times \frac{dy}{dx}$*
- *use the product rule for $\frac{d}{dx}(xy)$ – $u = x \Rightarrow u' = 1$ $\quad v = y \Rightarrow v' = \frac{dy}{dx}$*

$$\frac{d}{dx}(xy) = y + x\frac{dy}{dx}$$

This gives: $6x + 7\left(y + x\frac{dy}{dx}\right) + 9\left(2y\frac{dy}{dx}\right) = 0$

$$6x + 7y + 7x\frac{dy}{dx} + 18y\frac{dy}{dx} = 0$$

Now make $\frac{dy}{dx}$ the subject: $7x\frac{dy}{dx} + 18y\frac{dy}{dx} = -6x - 7y$

Take $\frac{dy}{dx}$ out as common factor: $\frac{dy}{dx}(7x + 18y) = -6x - 7y$

$$\frac{dy}{dx} = \frac{-6x - 7y}{7x + 18y}$$

Bk2 P36 Ex4A Q1 & 3

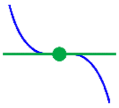
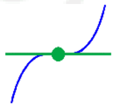
NB – Can't differentiate y with respect to x if no $x's$ in the function

- $\frac{d}{dx}y = \frac{dy}{dx}$, $\quad \frac{d}{dx}y^2 = 2y\frac{dy}{dx}$, $\quad \frac{d}{dx}4y^3 = 12y^2\frac{dy}{dx}$, etc.
- Treat the y as a bracket, derivative of outside × inside

Second derivative - $f''(x)$ or $\frac{d^2y}{dx^2}$

The 2nd derivative of a function can be used to determine the nature of an SP without the need for a Nature Table:

In general: $\frac{d^2y}{dx^2} > 0$ then min TP $\quad$ $\frac{d^2y}{dx^2} < 0$ then max TP

$\frac{d^2y}{dx^2} = 0$ then PI but is it  or  ??

❷ Find the nature of the SPs for the curve $y = 2x^3 - 2x^2 - 2x$

$\frac{dy}{dx} = 6x^2 - 4x - 2 = 2(3x+1)(x-1) = 0 \qquad$ so $SPs\ @\ x = -\frac{1}{3}\ or\ x = 1$

$\frac{d^2y}{dx^2} = 12x - 4 \qquad x = -\frac{1}{3}\ gives\ \frac{d^2y}{dx^2} = -8 \Rightarrow \max TP$

$$x = 1\ gives\ \frac{d^2y}{dx^2} = 8 \Rightarrow \min TP$$

Second derivative of Implicit Functions

❸ Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x and y only for $x^2 + 2xy = 1$

Differentiate both sides: $\frac{d}{dx}(x^2 + 2xy) = \frac{d}{dx}(1)$

$$2x + 2\left(x\frac{dy}{dx} + y\right) = 0$$

$$2x + 2x\frac{dy}{dx} + 2y = 0$$

$$2x\frac{dy}{dx} = -2x - 2y$$

$$\frac{dy}{dx} = \frac{-x-y}{x}$$

Use the quotient rule and substitute for $\frac{dy}{dx}$ *where needed:*

$$u = -x - y \Rightarrow u' = -1 - \frac{dy}{dx} = -1 + \frac{x+y}{x} = \frac{y}{x}$$

$$v = x \qquad \Rightarrow v' = 1$$

P38 Ex5
Q1 - 1st Column
Q3, 5, 7, 9

$$\frac{d^2y}{dx^2} = \frac{u'v - uv'}{v^2} = \frac{\frac{y}{x}x - (-x-y)}{x^2} = \frac{y + x + y}{x^2} = \frac{2y + x}{x^2}$$

Using the Logarithmic Function in Differentiation

❹ Find $\frac{dy}{dx}$ if $y = 4^x$

Take $\ln$ *of both sides:* $\ln y = \ln 4^x$

$\ln y = x \ln 4$

Differentiate both sides: $\frac{d}{dx}(\ln y) = \frac{d}{dx}(x \ln 4)$

$$\frac{1}{y}\frac{dy}{dx} = \ln 4$$

Make $\frac{dy}{dx}$ *the subject:* $\frac{dy}{dx} = y \ln 4$

Substitute for y: $\frac{dy}{dx} = 4^x \ln 4$

❺ $y = x^x$, Find $\frac{dy}{dx}$

$\ln y = \ln x^x$ so $\ln y = x \ln x$

$u = x \Rightarrow u' = 1$

$v = \ln x \Rightarrow v' = \frac{1}{x}$

$\frac{1}{y}\frac{dy}{dx} = \ln x + 1$ so $\frac{dy}{dx} = y(\ln x + 1)$ so $\frac{dy}{dx} = x^x(\ln x + 1)$

❻ Find $\frac{dy}{dx}$ if $y = \frac{x^2\sqrt{7x-3}}{1+x}$ [would normally need product/quotient rule]

$$\ln y = \ln\left(\frac{x^2\sqrt{7x-3}}{1+x}\right)$$

$$\ln y = \ln(x^2) + \ln\left(\sqrt{7x-3}\right) - \ln(1+x)$$

$$\ln y = 2\ln(x) + \frac{1}{2}\ln(7x-3) - \ln(1+x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{2} \times \frac{1}{7x-3} \times 7 - \frac{1}{1+x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{7}{2(7x-3)} - \frac{1}{1+x}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2 \times 2(7x-3)(1+x) + 7x(1+x) - 2x(7x-3)}{2x(7x-3)(1+x)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{4(7x^2+4x-3) + 7x + 7x^2 - 14x^2 + 6x}{2x(7x-3)(1+x)}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{21x^2+29x-12}{2x(7x-3)(1+x)}$$

$$\frac{dy}{dx} = y \times \frac{21x^2+29x-12}{2x(7x-3)(1+x)}$$

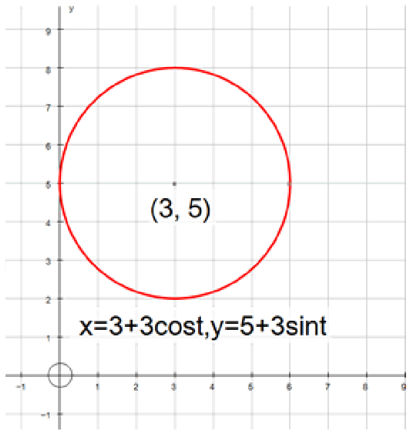
$$\frac{dy}{dx} = \frac{x^2(7x-3)^{\frac{1}{2}}}{1+x} \times \frac{21x^2+29x-12}{2x(7x-3)(1+x)}$$

$$\frac{dy}{dx} = \frac{x}{1+x} \times \frac{21x^2+29x-12}{2(7x-3)^{\frac{1}{2}}(1+x)}$$

$$\frac{dy}{dx} = \frac{x(21x^2+29x-12)}{2(7x-3)^{\frac{1}{2}}(1+x)^2}$$

P40 Ex6
Q1&2 1st column
Q3, 4, 5, 8

Parametric Equations



We are most familiar with Cartesian equations where x and y are linked e.g. $y = x^2 + 4$
Sometimes it is more convenient to involve a third variable, t or s or θ, and express both x and y in terms of this variable.
Equations $x = x(t)$ and $y = y(t)$ are referred to as **parametric equations** and t is referred to as the **parameter**.

We can convert from parametric equations to Cartesian (or rectangular) equations by eliminating t, s or θ.

This equation is called the corresponding **constraint equation**

❼ $x = 2t - 1$ and $y = 1 - t^2$

Change the subject of one equation to t*:* $2t = x + 1 \Rightarrow t = \frac{x+1}{2}$

Substitute for t *in the other equation:* $y = 1 - \left(\frac{x+1}{2}\right)^2 = 1 - \frac{x^2+2x+1}{4}$

Tidy up: $y = \frac{4}{4} - \frac{x^2+2x+1}{4} = \frac{4-x^2+2x+1}{4} = \frac{3-x^2+2x}{4}$

$$4y = 3 - x^2 + 2x$$

❽ $x = 2\sin 2\theta$ and $y = \cos\theta$

Square and add to eliminate sin/cos: $x^2 + y^2 = \sin^2 2\theta + \cos^2\theta$

$$x^2 + y^2 = (2\sin\theta\cos\theta)^2 + \cos^2\theta$$

$$x^2 + y^2 = 4\sin^2\theta\cos^2\theta + \cos^2\theta$$

$$x^2 + y^2 = 4(1 - \cos^2\theta)\cos^2\theta + \cos^2\theta$$

Since $y = \cos\theta$ *then* $x^2 = 4(1 - y^2)y^2 + y^2 - y^2$

$$x^2 = 4y^2 - 4y^4$$

P42 Ex7A
All

1st Derivative of Parametric Equations

We use the Chain Rule to differentiate Parametric Equations:

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} \quad \text{where} \quad \frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} \quad \text{i.e. invert } \frac{dx}{dt} \text{ to get } \frac{dt}{dx}$$

❾ Find $\frac{dy}{dx}$ when $x = 4 + 4t$ and $y = 3 - 3t^2$

$$\frac{dx}{dt} = 4 \Rightarrow \frac{dt}{dx} = \frac{1}{4} \qquad \frac{dy}{dt} = -6t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = -6t \times \frac{1}{4} = -\frac{3t}{2}$$

❶⓿ Find a formula for the gradient of the tangent to the curve whose points are given by $x = a(t - \sin t)$ and $y = a(1 - \cos t)$

$$\frac{dx}{dt} = a - a\cos t \Rightarrow \frac{dt}{dx} = \frac{1}{a - a\cos t} \qquad \frac{dy}{dt} = a\sin t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = a\sin t \times \frac{1}{a - a\cos t} = \frac{a\sin t}{a - a\cos t} = \frac{\sin t}{1 - \cos t}$$

❶❶ Find the coordinates of the points on the curve, $x = 1 - t^2$ and $y = t^3 + t$ at which the gradient = 2.

$$\frac{dx}{dt} = -2t \Rightarrow \frac{dt}{dx} = -\frac{1}{2t} \qquad \frac{dy}{dt} = 3t^2 + 1$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = (3t^2 + 1) \times -\frac{1}{2t} = -\frac{3t^2 + 1}{2t}$$

$$\frac{dy}{dx} = 2 \Rightarrow -\frac{3t^2+1}{2t} = 2 \qquad so \qquad 3t^2 + 4t + 1 = 0$$

$$(3t + 1)(t + 1) = 0 \Rightarrow t = -1, t = -\frac{1}{3}$$

For $t = -\frac{1}{3}$ *the* $x = \frac{8}{9}$ *and* $y = -\frac{10}{27}$ *so* $\left(\frac{8}{9}, -\frac{10}{27}\right)$

For $t = -1$ *the* $x = 0$ *and* $y = -2$ *so* $(0, -2)$

2^{nd} Derivative of Parametric Equations:

We make use of the Chain Rule again: $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$

❶❷ Find $\frac{d^2y}{dx^2}$ when $x = at^2$ and $y = 2at$

$\frac{dx}{dt} = 2at \Rightarrow \frac{dt}{dx} = \frac{1}{2at}$ $\qquad \frac{dy}{dt} = 2a$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2a \times \frac{1}{2at} = \frac{1}{t}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx} = \frac{d}{dt}\left(\frac{1}{t}\right) \times \frac{1}{2at} = -\frac{1}{t^2} \times \frac{1}{2at} = -\frac{1}{2at^3}$$

❶❸ Find $\frac{d^2y}{dx^2}$ when $x = \tan\theta$ and $y = \sin 2\theta$

$\frac{dx}{d\theta} = \sec^2\theta \Rightarrow \frac{d\theta}{dx} = \frac{1}{\sec^2\theta} = \cos^2\theta$ $\qquad \frac{dy}{d\theta} = 2\cos 2\theta$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = 2\cos 2\theta \times \cos^2\theta = 2\cos 2\theta \cos^2\theta$$

$$\frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx} = \frac{d}{d\theta}(2\cos 2\theta \cos^2\theta) \times \cos^2\theta$$

Using the product/chain rule:

$$\frac{d^2y}{dx^2} = \cos^2\theta\,(-4\sin 2\theta \times \cos^2\theta + 2\cos 2\theta \times 2\cos\theta \times -\sin\theta)$$

$$\frac{d^2y}{dx^2} = \cos^2\theta\,(-4\sin 2\theta\cos^2\theta - 2\cos 2\theta\sin 2\theta)$$

$$\frac{d^2y}{dx^2} = -2\sin 2\theta\cos^2\theta\,(2\cos^2\theta + \cos 2\theta)$$

$$\frac{d^2y}{dx^2} = -2\sin 2\theta\cos^2\theta\,(2\cos^2\theta + 2\cos^2\theta - 1)$$

$$\frac{d^2y}{dx^2} = -2\sin 2\theta\cos^2\theta\,(4\cos^2\theta - 1)$$

Bk 2 P44 Ex8A
Q1a, 1d, 2a ,2d, 3, 5

Velocity and Acceleration for parametric functions (Motion in a plane):

Reminder: $v = \frac{ds}{dt}$ *and* $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

For parametric functions: $Speed = |v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

The direction is given by an angle: $\tan\theta = \frac{y'(t)}{x'(t)}$

$x'(t) < 0$ $y'(t) > 0$	$x'(t) > 0$ $y'(t) > 0$
$x'(t) < 0$ $y'(t) < 0$	$x'(t) > 0$ $y'(t) < 0$

$Acceleration = |a| = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2}$

$\tan\theta = \frac{y''(t)}{x''(t)}$ *as above for direction*

❶❹ The motion of a particle is modelled by the equations $x = 5t$ and $y = 5\sqrt{3}t - 5t^2$. x is the horizontal displacement, y is the vertical displacement and t is the time. Find the position and speed of the particle as well as its direction of motion after 1 second.

At $t = 1$ $\quad x = 5 \times 1 = 5 \quad y = 5\sqrt{3} \times 1 - 5(1)^2 = 5\sqrt{3} - 5$

Position is $\left(5, 5\sqrt{3} - 5\right)$

$\frac{dx}{dt} = 5 \quad \frac{dy}{dt} = 5\sqrt{3} - 10t$

At $t = 1$ $\quad \frac{dx}{dt} = 5 \quad \frac{dy}{dt} = 5\sqrt{3} - 10(1) = 5\sqrt{3} - 10$

$$Speed = |v| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(5)^2 + \left(5\sqrt{3} - 10\right)^2} = 5.2ms^{-1}$$

Direction: $\tan\theta = \frac{y'(1)}{x'(1)} = \frac{5\sqrt{3}-10}{5} = -0.2679$

$\theta = 165°$ or $\theta = 345°$

Since $x'(t) > 0$ and $y'(t) < 0$ then $\theta = 345°$ is the direction

Bk2 P50 Ex1
Odd numbers