Unit 2: Applications in Algebra and Calculus (H7X1 77)

Factorials and the Binomial Theorem

Pascal's triangle:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

Using Pascal's Triangle we can quickly expand brackets of the type $(x+y)^n$

$$\text{e.g. } (x+y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$$
$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Note: the powers of the first term decrease from 4 to zero and vice versa

Reminder: ***any value to an even number power gives a positive answer and a negative to an odd number power gives a negative answer.***

Now try "THE BINOMIAL THEOREM 1" worksheet (1st column).

The co-efficients of the expansions can be calculated without using Pascal's Triangle using the Combination Function:

$$^nC_r = \frac{n!}{r!\,(n-r)!}$$

This is one of a group of functions which involve factorials – see below.

Factorial Function: $n! = n \times (n-1) \times (n-2) \times \ldots\ldots\ldots \times 3 \times 2 \times 1 \quad n \in W$

e.g. $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

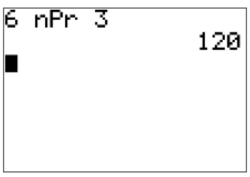
This can be used to calculate the number of different ways that 5 people can be sat on a bench – 5! = 120

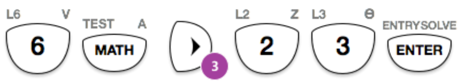
Permutation Function: $^nP_r = \frac{n!}{(n-r)!}$

How many ways can 1st, 2nd and 3rd place be awarded within a group of 6 people?

$$^6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = 120$$

We are choosing 3 from 6 then considering how many ways the 3 can be arranged.

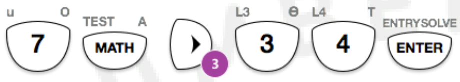


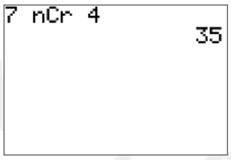


Combination Function: $^{n}C_r = \frac{n!}{r!(n-r)!}$

From a palette of 7 colours, pick 4 to mix. How many ways can this be done?

$$^{7}C_4 = \frac{7!}{4!\,(7-4)!} = 35$$





P5 Ex2A Q1, 3, 5

We need not consider order as the order we mix the paint doesn't matter.

Another way of writing $^{n}C_r$ is $\binom{n}{r}$ which is used in the notation for expanding a bracket when not using Pascal's Triangle. It is known as "The Binomial Theorem".

Binomial Theorem:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots \ldots + \binom{n}{n}x^{n-n}y^n$$

It appears in the AH Formula Sheet as:

Binomial theorem

$$(a+b)^n = \sum_{r=0}^{n}\binom{n}{r}a^{n-r}b^r \quad \text{where} \quad \binom{n}{r} = {}^{n}C_r = \frac{n!}{r!(n-r)!}$$

Σ is the Greek letter "sigma" and stands for sum of:

$$\sum_{r=1}^{5} \mathrm{n} = 1+2+3+4+5 \qquad \sum_{i=0}^{6} n^2 = 0^2+1^2+2^2+3^2+4^2+5^2+6^2$$

Examples: ❶ Expand $(2+3x)^5$ using the binomial theorem

$$(2+3x)^5 = \sum_{r=0}^{5}\binom{5}{r}(2)^{5-r}(3x)^r = \binom{5}{0}2^5 + \binom{5}{1}2^4(3x) + \binom{5}{2}2^3(3x)^2 + \binom{5}{3}2^2(3x)^3 + \binom{5}{4}2(3x)^4 + \binom{5}{5}(3x)^5$$

$$= 32 + 240x + 720x^2 + 1080x^3 + 810x^4 + 243x^5$$

❷ Expand $(x-2)^4$ using the binomial theorem

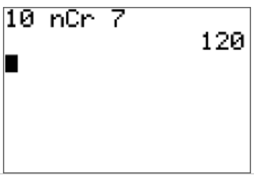
$$(x-2)^4 = \sum_{r=0}^{4}\binom{4}{r}(x)^{4-r}(-2)^r = \binom{4}{0}x^4 + \binom{4}{1}x^3(-2) + \binom{4}{2}x^2(-2)^2 + \binom{4}{3}x^1(-2)^3 + \binom{4}{4}(-2)^4$$

$$= x^4 - 8x^3 + 24x^2 - 32x + 16$$

Now try "THE BINOMIAL THEOREM 2" worksheet (1st column).

❸ Find the x^7 term in the expansion of $(1+x)^{10}$ – *7^{th} term needed*

$$^{10}C_7(1)^3(x)^7 = 120x^7$$



❹ Identify the y^3 term in the expansion of $\left(y-\frac{5}{y}\right)^7$ – *Harder!!*

We solve this by finding the ***GENERAL TERM*** *for this expansion and simplifying:*

$$\left(y-\frac{5}{y}\right)^7 = \sum_{r=0}^{7}\binom{7}{r}(y)^{7-r}\left(\frac{5}{y}\right)^r = \sum_{r=0}^{7}\binom{7}{r}(y)^{7-r}(5)^r(y^{-1})^r$$

$$= \sum_{r=0}^{7}\binom{7}{r}(y)^{7-r}(5)^r(y)^{-r} = \sum_{r=0}^{7}\binom{7}{r}(y)^{7-2r}(5)^r$$

For the y^3 *term:* $7-2r=3$ *giving* $r=2$ *so we need the 2nd term:*

$${}^7C_2(y)^5\left(-\frac{5}{y}\right)^2 = 21y^5\left(\frac{25}{y^2}\right) = 525y^3$$

❺ Identify the term independent of x in the expansion of $\left(2x+\frac{1}{x}\right)^6$

Find the ***GENERAL TERM*** *for this expansion and simplify:*

$$\left(2x+\frac{1}{x}\right)^6 = \sum_{r=0}^{6}\binom{6}{r}(2x)^{6-r}\left(\frac{1}{x}\right)^r = \sum_{r=0}^{6}\binom{6}{r}(2)^{6-r}(x)^{6-r}(x^{-1})^r$$

$$= \sum_{r=0}^{6}\binom{6}{r}(2)^{6-r}(x)^{6-r}(x)^{-r} = \sum_{r=0}^{6}\binom{6}{r}(2)^r(x)^{6-2r}$$

For the x^0 *term:* $6-2r=0$ *giving* $r=3$ *so we need the 3rd term:*

Binomial Theorem – General Term worksheet

$${}^6C_3(2x)^3\left(\frac{1}{x}\right)^3 = 20\times 8x^3\times\left(\frac{1}{x^3}\right) = 160$$

P9/10 Ex3A Q1, 3, 4, 5

❻ Expand $(x+1)^2(2+x)^3$ using the binomial theorem.

Expand each bracket separately:

$$(x+1)^2 = x^2+2x+1$$

$$(2+x)^3 = \binom{3}{0}2^3+\binom{3}{1}2^2x+\binom{3}{2}2x^2+\binom{3}{3}x^3 = 8+12x+6x^2+x^3$$

$$(x+1)^2(2+x)^3 = (x^2+2x+1)(8+12x+6x^2+x^3)$$

$$= 8x^2+12x^3+6x^4+x^5+16x+24x^2+12x^3+2x^4+8+12x+6x^2+x^3$$

$$= 8+28x+38x^2+25x^3+8x^4+x^5$$

❼ Find the co-efficient of x^5 in the expansion of $(1+x)^4(1-2x)^3$

The x^5 terms are found with the following combinations:

x^2 in the 1st expansion, $\binom{4}{2}x^2$, with $(-2x)^3$ in the 2nd expansion, $\binom{3}{0}(-2x)^3$

x^3 in the 1st expansion, $\binom{4}{1}x^2$, with $(-2x)^2$ in the 2nd expansion, $\binom{3}{1}(-2x)^3$

x^4 in the 1st expansion, $\binom{4}{0}x^2$, with $(-2x)$ in the 2nd expansion, $\binom{3}{2}(-2x)$

$$\binom{4}{2}x^2\binom{3}{0}(-2x)^3+\binom{4}{1}x^2\binom{3}{1}(-2x)^3+\binom{4}{0}x^2\binom{3}{2}(-2x)$$

$$= 6x^2.-8x^3+4x^3.12x^2+x^4.-6x = 6x^5$$

❽ Find the greatest term in the expansion of $(1+3x)^{18}$ when $x=\frac{3}{4}$

$$\left(1+\frac{3}{4}\right)^{18} = 1+\binom{18}{1}\left(\frac{3}{4}\right)+\binom{18}{2}\left(\frac{3}{4}\right)^2+\binom{18}{3}\left(\frac{3}{4}\right)^3+\binom{18}{4}\left(\frac{3}{4}\right)^4+\ldots$$

$$= 1+13.5+86.1+344.3+979.2+2056.3+3341.5+4137+3650.6+\ldots$$

Numbers are starting to decrease now so the 9th term is the greatest:

$$\binom{18}{8}(1)^9\left(\frac{3}{4}\right)^9$$

P11/12 Ex3B
Q1-11 (odd numbers)

❾ Calculate 0.9^7 correct to 2 decimal places.

$$0.9^7 = (1-0.1)^7 = 1+\binom{7}{1}(-0.1)+\binom{7}{2}(-0.1)^2+\binom{7}{3}(-0.1)^4+\ldots$$

$$= 1-0.7+0.21-0.035+0.0035-..$$

(Numbers are becoming insignificant as they are beyond the 2 d.p. asked for)

$$= 0.475 = 0.48$$

P13 Ex4
Q1 & 3

<u>Problem solving with Combination Function:</u>

We could be asked to work backwards e.g. find the value of n for which $\binom{n}{2}=55$

$$\binom{n}{2} = \frac{n!}{2!\,(n-2)!} = \frac{n(n-1)(n-2)\times\ldots\times 3\times 2\times 1}{2\times 1\times(n-2)(n-3)\times\ldots\times 3\times 2\times 1} = \frac{n(n-1)}{2} = 55$$

Rearranging we get: $n^2-n-110=(n-11)(n+10)=0$

$n\neq -11$ so $\underline{n=10}$

<u>Commonly used relationships</u>: $\binom{n}{r}=\binom{n}{n-r}$ – *symmetry in Pascal's Triangle*

$\binom{n}{r-1}+\binom{n}{r}=\binom{n+1}{r}$ – *e.g.* ${}^6C_3+{}^6C_4$

Sum of 2 consecutive coefficients giving coefficient below $= {}^7C_4$

P7/8 Ex2B
Q2, 4, 6, 7