# Unit 3: Geometry, Proof and Systems of Equations (H7X3 77)

## Applying Algebraic skills to Number Theory

## The Euclidean Algorithm

The **greatest common divisor** (gcd) of two integers is the same as the hcf (highest common factor)

The notation $(a, b)$ is used to denote the gcd of $a$ and $b$.

The hcf of 2 (or more) integers was found initially by writing out lists of factors:

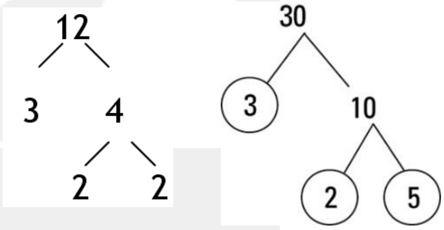
Factors of 12 – 1, 2, 3, 4, 6, 12

Factors of 30 – 1, 2, 3, 5, 6, 15, 30

Common factors were underlined and the highest offered as the answer - $(12,30) = 6$

Another method was to use prime factor trees:

The common prime factors (3 and 2) are then multiplied to give the hcf - $3 \times 2 = 6$




The Euclidean Algorithm gives us another method for this:

Write the larger number as a product of the smaller one – $a = q \times b + r$

$$30 = 2 \times 12 + 6$$

Now express $b$ in terms of $r$: $12 = 2 \times 6 + 0$

When $r = 0$ then the gcd is the last non-zero remainder: $gcd(12,30) = 6$

❶ Find the gcd of 424 and 132

$$424 = 3 \times 132 + 28$$
$$132 = 4 \times 28 + 20$$
$$28 = 1 \times 20 + 8$$
$$20 = 2 \times 8 + 4$$
$$8 = 2 \times 4 + 0$$

Last non-zero remainder is 4 so $gcd(424,132) = 4$

This works since the gcd must also be a factor of all the remainders:

$$(424,132) = (132,28) = (28,20) = (20,8) = (8,4) = 4$$

❷ Find the gcd of 280 and 117

$$\begin{aligned} 280 &= 2 \times 117 + 46 \\ 117 &= 2 \times 46 + 25 \\ 46 &= 1 \times 25 + 21 \\ 25 &= 1 \times 21 + 4 \\ 21 &= 5 \times 4 + 1 \\ 4 &= 4 \times 1 + 0 \end{aligned}$$

Last non-zero remainder is 1 so $gcd(280{,}117) = 1$

| |
|---|
| If $gcd(a, b) = 1$ then integers $a$ and $b$ are said to be **co-prime** or **relatively prime**. |

The graphic calculator has a "gcd" function built in:


NB – LCM of $a$ and $b$ = $\dfrac{ab}{(a,b)}$ Not part of the course!!

For 3 integers, find the gcd of the smallest two first then the gcd of the answer with the largest integer.

❸ Find the gcd of 259, 222 and 185

$$\begin{aligned} 222 &= 1 \times 185 + 37 \\ 185 &= 5 \times 37 + 0 \end{aligned}$$

$$gcd(222{,}185) = 37$$

$$259 = 7 \times 37 + 0$$

Bk3 P145 Ex4
Q1-3(1st column)

$$gcd(259{,}222{,}185) = 37$$

**DIOPHANTINE EQUATIONS**

Euclid's Algorithm can also be used to find integer solutions to equations of the form:

$$ax + by = kd \qquad \text{where } d = (a, b)$$

These types of equations are known as Diophantine Equations

**NB – the AH course does not go beyond $k = 1$ i.e. $ax + by = d$**

❹ Use the Euclidean algorithm to find integers $x$ and $y$ such that

$$149x + 139y = 1$$

**Use the algorithm as before but re-write each calculation making the remainder, $r$, the subject:**

$149 = 1 \times 139 + 10 \qquad \Rightarrow 10 = 149 - 1 \times 139$

$139 = 13 \times 10 + 9 \qquad \Rightarrow 9 = 139 - 13 \times 10$

$10 = 1 \times 9 + 1 \qquad \Rightarrow 1 = 10 - 1 \times 9$

$9 = 9 \times 1 + 0 \qquad \Rightarrow gcd(149,139) = 1$

**Use these new equations in reverse order and substitute for each remainder:**

$1 = 10 - 1 \times 9$ Substitute $9 = 139 - 13 \times 10$

$1 = 10 - 1 \times (139 - 13 \times 10) = 14 \times 10 - 1 \times 139$

Substitute $10 = 149 - 1 \times 139$

$1 = 14 \times (149 - 1 \times 139) - 1 \times 139 = 14 \times 149 - 15 \times 139$

i.e. $14 \times 149 - 15 \times 139 = 1$ so $x = 14$ and $y = -15$

❺ Use the Euclidean algorithm to show that $(29,113) = 1$ where $(a, b)$ denotes the highest common factor of $a$ and $b$.
Hence find integers $x$ and $y$ such that $29x + 113y = 1$

$113 = 3 \times 29 + 26 \qquad \Rightarrow 26 = 113 - 3 \times 29$

$29 = 1 \times 26 + 3 \qquad \Rightarrow 3 = 29 - 1 \times 26$

$26 = 8 \times 3 + 2 \qquad \Rightarrow 2 = 26 - 8 \times 3$

$3 = 1 \times 2 + 1 \qquad \Rightarrow 1 = 3 - 1 \times 2$

$2 = 1 \times 2 + 0 \qquad \Rightarrow gcd(29,113) = 1$ i.e. co-prime

Working backwards and substituting:

$1 = 3 - 1 \times 2$ Substitute $2 = 26 - 8 \times 3$

$1 = 3 - 1 \times (26 - 8 \times 3) = 9 \times 3 - 1 \times 26$

Substitute $3 = 29 - 1 \times 26$

$1 = 9 \times (29 - 1 \times 26) - 1 \times 26 = 9 \times 29 - 10 \times 26$

Substitute $26 = 113 - 3 \times 29$

$1 = 9 \times 29 - 10 \times (113 - 3 \times 29) = 12 \times 29 - 10 \times 113$

i.e. $12 \times 29 - 10 \times 113 = 1$ so $x = 12$ and $y = -10$

Bk3 P147 Ex5
Q1, 3, 5

**CHANGING NUMBER BASES**

In Maths we use base 10 ($base_{10}$) and you may be aware that computers use binary or base 2 ($base_2$) and hexadecimal ($base_{16}$) is used in computer programming.

For base 10 we have column headings:

| $10^3$(=1000) | $10^2$(=100) | $10^1$(=10) | $10^0$(=1) |
|---|---|---|---|
| 1 | 3 | 5 | 7 |

For base 2 we have column headings:

| $2^3$(=8) | $2^2(=\mathbf{4})$ | $2^1(=\mathbf{2})$ | $2^0$(=1) |
|---|---|---|---|
| 1 | 0 | 0 | 1 |

=8+1=9

and so on . . .

❻ Change $132_5$ to base 10

$1 \times 5^2 + 3 \times 5^1 + 2 \times 5^0 = 25 + 15 + 2 = 42$

❼ Change $436_7$ to base 10

$4 \times 7^2 + 3 \times 7^1 + 6 \times 7^0 = 4 \times 49 + 3 \times 7 + 6 = 223$

The Euclidean algorithm can be used to change numbers in base 10 to other base numbers.

❽ Change $1136_{10}$ to base 6

*Divide the number by the new base value leaving the answer as a quotient and a remainder:*

$$1136 \div 6 = 189r2$$

*Repeat this with each quotient until it equals zero:*

$$189 \div 6 = 31r3$$
$$31 \div 6 = 5r1$$
$$5 \div 6 = 0r5$$

*Now read the denominators from bottom to top:*

$$5132_6$$

❾ Convert $2213_4$ to base 5

*Convert to base 10 first then to the other base value:*

$$2213_4 = 2 \times 4^3 + 2 \times 4^2 + 1 \times 4^1 + 3 \times 4^0 = 167$$

$$167 \div 5 = 33r2$$
$$33 \div 5 = 6r3$$
$$6 \div 5 = 1r1$$
$$1 \div 5 = 0r1$$

Hence $2213_4 = 1132_5$

**Bk3 P151 Ex7 Q1, 2**