Unit 3: Geometry, Proof and Systems of Equations (H7X3 77)

Applying Geometric skills to Complex Numbers

$\mathbb{C}$ is the set of **complex numbers**.

z denotes a complex number and is made up of two parts, a real part and an imaginary part.

$z = a + bi$ where $a = Re(z)$ and $b = Im(z)$ with $i^2 = -1 \Rightarrow i = \sqrt{-1}, i \in R$

$\bar{z} = a - bi$ is the complex conjugate of z , NB - $z\bar{z} = a^2 + b^2$

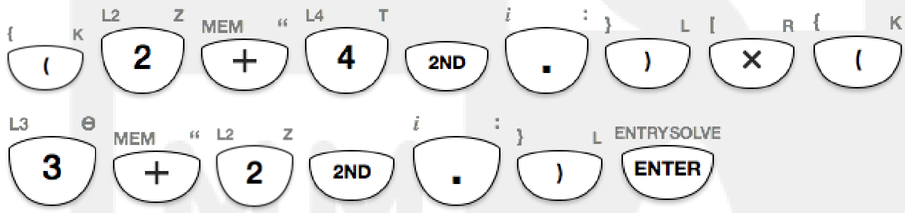
ADDITION, SUBTRACTION, MULTIPLICATION AND DIVISION

For $z_1 = 2 + 4i$ and $z_2 = 3 + 2i$

❶ $z_1 + z_2 = 2 + 4i + 3 + 2i = 5 + 6i$ Add real parts then add imaginary parts

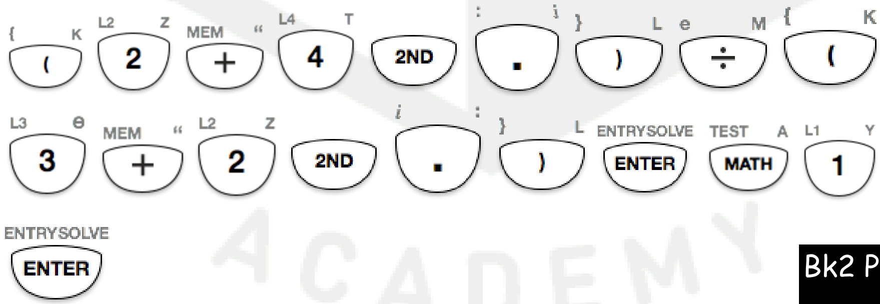
❷ $z_1 - z_2 = 2 + 4i - 3 - 2i = -1 + 2i$ Subtract real then imaginary parts

❸ $z_1 z_2 = (2 + 4i)(3 + 2i) = 6 + 4i + 12i + 8i^2 = 6 + 16i - 8 = -2 + 16i$



❹ $z_1 \div z_2 = \frac{z_1}{z_2} = \frac{2+4i}{3+2i} \times \frac{3-2i}{3-2i}$ multiply by the complex conjugate

$$= \frac{(2+4i)(3-2i)}{(3+2i)(3-2i)} = \frac{6+8i-8i^2}{9-4i^2} = \frac{14+8i}{13} = \frac{2}{13}(7 + 4i)$$



Bk2 P90 Ex1
Q2, 6, 7
P91 Ex2
Q1,3, 4

Powers of z

We use the Binomial Theorem for whole number powers

❺ Find z^4 if $z = 2 + i$

$$(2+i)^4 = 2^4 + 4(2)^3 i + 6(2)^2 i^2 + 4(2) i^3 + i^4 = 16 + 32i - 24 - 8i + 1 = -7 + 24i$$

❻ For $= 3 - 2i$, find z^{-2}

$$z^{-2} = \frac{1}{z^2} = \frac{1}{(3-2i)^2} = \frac{1}{9-12i-4} = \frac{1}{5-12i} \times \frac{5+12i}{5+12i} = \frac{13+12i}{169} = \frac{1}{169}(5+12i)$$

❼ For $= 5 + 12i$, find $z^{\frac{1}{2}}$

Let $a + bi = \sqrt{5+12i}$ where a and b are real $\Rightarrow (a+bi)^2 = 5 + 12i$

$$\Rightarrow a^2 + 2abi - b^2 = 5 + 12i$$

Equating real and imaginary parts we get: $a^2 - b^2 = 5$ and $2ab = 12$

Re-arranging $2ab = 12$ we get $a = \frac{6}{b}$ which we substitute into $a^2 - b^2 = 5$

This gives $\left(\frac{6}{b}\right)^2 - b^2 = 5$ which we multiply through by b^2 to get $36 - b^4 = 5b^2$

Re-arranging we get $b^4 + 5b^2 - 36 = 0$ which factorises to $(b^2 - 4)(b^2 + 9) = 0$

Since $b^2 + 9 = 0$ has no solution then $b = \pm 2$ leading to $a = \pm 3$

So $\sqrt{5+12i} = 3 + 2i$ or $\sqrt{5+12i} = -3 - 2i$

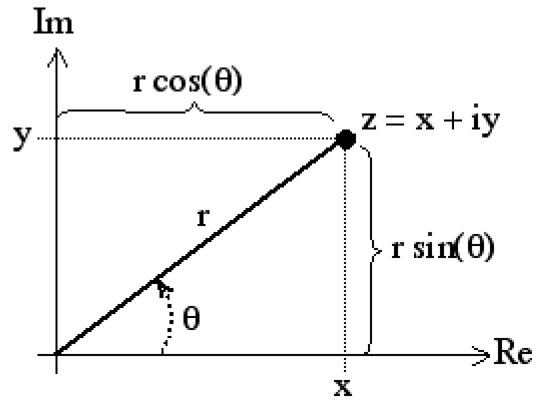
Bk2 P101 Ex6
Q3 (use Binomial)

P91/92 Ex2
Q5

DeMoivre's theorem offers us an alternative method to find powers of complex numbers but before we learn to use it, we must complete some prior learning.

ARGAND DIAGRAMS

The complex number $z = x + iy$ can be plotted on an Argand diagram as the point (x, y).



r is the length of the line from the origin to the point (x, y). $r = \sqrt{x^2 + y^2}$

This is called the modulus of z and is denoted $|z|$

The size of rotation is called the **argument** of z and is denoted by $\arg z$, $\theta = \arg z$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ with the domain restricted to $-\pi < \theta \leq \pi$ $(-180° < \theta \leq 180°)$

POLAR FORM:

Using right-angled triangle trigonometry we can write that $x = r\cos\theta$ and $x = r\sin\theta$

$\pi - \theta$	θ
$-\pi + \theta$	$-\theta$

So $z = x + iy$ becomes $z = r(\cos\theta + i\sin\theta)$

This is known as the **polar form of** z.

❽ Express $z = 3 + 4i$ in polar form.

$(3,4)$ is in the 1st quadrant $\qquad |z| = \sqrt{3^2 + 4^2} = 5$

$\tan\theta = \frac{4}{3} = 0.927$ so $\qquad z = 5(\cos 0.927 + i\sin 0.927)$

❾ Express $z = -3 - 4i$ in polar form.

$(-3, -4)$ is in the 3rd quadrant $\qquad |z| = \sqrt{(-3)^2 + (-4)^2} = 5$

$\tan\theta = \dfrac{-4}{-3} = 0.927 \Rightarrow \theta = -\pi + 0.927 = -2.215$

so $\quad z = 5(\cos(-2.215) + i\sin(-2.215)) = 5(\cos 2.215 - i\sin 2.215)$

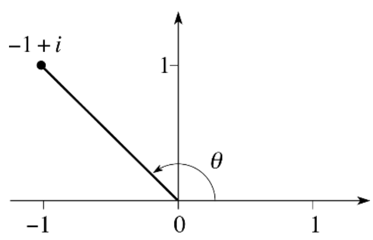
❶⓿ Express $z = -1 + i$ in polar form $r(\cos\theta° + i\sin\theta°)$.

$(-1,1)$ is in the 2nd quadrant

$|z| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

$\tan\theta = \dfrac{1}{1} = 45° \Rightarrow \theta = 180 - 45 = 135°$

so $\quad z = \sqrt{2}(\cos 135° + i\sin 135°)$



❶❶ Find the complex number with $|z| = 2$, and $\arg z = \frac{\pi}{3}$,

$\frac{\pi}{3} = \tan^{-1}\left(\frac{y}{x}\right) \Rightarrow \frac{y}{x} = \tan\frac{\pi}{3} = \sqrt{3}$ this gives $\Rightarrow \frac{y}{x} = \sqrt{3} \Rightarrow y = \sqrt{3}x$

$|z| = \sqrt{x^2 + y^2} = 2$ squaring both sides gives $x^2 + y^2 = 4$

Substituting $y = \sqrt{3}x$ into $x^2 + y^2 = 4$ we get $x^2 + 3x^2 = 4$

$4x^2 = 4 \Rightarrow x = \pm 1$ since $\frac{\pi}{3}$ in 1st quadrant, we get $x = 1$ and $y = \sqrt{3}$

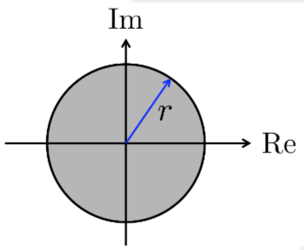
$z = 1 + \sqrt{3}i$

Bk2 P94 Ex3
Q1-7
1st column

Sets of points on the Complex Plane (Locus)

Part of the AH course requires us to "interpret geometrically certain equations or inequalities in the complex plane, i.e. find the loci defined by (in) equalities".
We need to look at how a set of a points move in the complex plane when there are restrictions on its modulus and/or argument.

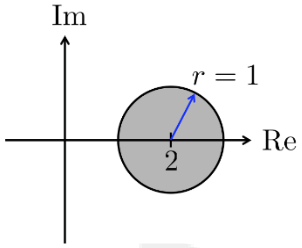
If the restriction is on its **modulus** then we get a **circle**:



$|z| = r \Rightarrow$ points lie on circumference

$|z| < r \Rightarrow$ points lie inside the circle

$|z| > r \Rightarrow$ points lie outside the circle

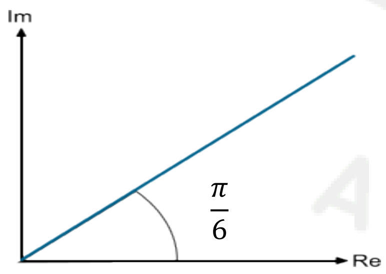


$|z - 2| = 1 \Rightarrow |x + iy - 2| = 1 \Rightarrow |x - 2 + iy| = 3$

$\Rightarrow (x - 2)^2 + y^2 = 1$

Circle with centre (2, 0) and radius 1

If the restriction is on its **argument** then we get a **straight line**:



$\arg z = \frac{\pi}{6} \Rightarrow \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{6}$

$\Rightarrow \frac{y}{x} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \Rightarrow y = \frac{1}{\sqrt{3}}x$

Locus is a straight line through the origin with gradient $\frac{1}{\sqrt{3}}$

For inequalities, shade above or below the line as appropriate.

❶❷ Describe the loci in the complex plane given by $|z-1| = |z+5|$

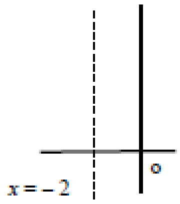
$$|x+iy-1| = |x+iy+5| \Rightarrow |(x-1)+iy| = |(x+5)+iy|$$

$$\Rightarrow (x-1)^2 + y^2 = (x+5)^2 + y^2$$

$$\Rightarrow x^2 - 2x + 1 = x^2 + 10x + 25$$

$$\Rightarrow -12x = 24 \Rightarrow x = -2$$

Vertical line $x = -2$



NB: (-2) is the middle between 1 and -5, all the points along this line are equidistant from (1, 0) and (-5, 0)

Bk2 P96 Ex4
Q1 (1st column)

DeMOIVRE'S THEOREM

For	$z = r(\cos\theta + i\sin\theta)$	then	$z^n = r^n(\cos n\theta + i\sin n\theta)$

This gives us an alternative method to find z^n for complex numbers, including fractional indices.

❶❸ Find z^7 if $z = 1 + \sqrt{3}i$

Convert to polar form: $r = |z| = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\tan\theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3}$ since $(1, \sqrt{3})$ in 1st quadrant

Apply DeMoivre: $z^7 = (2)^7\left(\cos 7(\frac{\pi}{3}) + i\sin 7(\frac{\pi}{3})\right)$

$$z^7 = 128\left(\cos(\tfrac{7\pi}{3}) + i\sin(\tfrac{7\pi}{3})\right)$$

$\frac{7\pi}{3}$ is outwith the domain $-\pi < \theta < \pi$ so subtract 2π repeatedly until θ is within the domain

$$z^7 = 128\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 128\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) = 64 + 64\sqrt{3}i$$

Some Useful Points

If z_1 and z_2 are two complex numbers expressed in polar form then:

- $|z_1 z_2| = |z_1| \times |z_2|$
- $\left|\frac{z_1}{z_2}\right| = |z_1| \div |z_2|$
- $Arg(z_1 z_2) = Arg\ z_1 + Arg\ z_2$
- $Arg\left(\frac{z_1}{z_2}\right) = Arg\ z_1 - Arg\ z_2$

As with DeMoivre, if the new argument does not lie in the range $(-\pi, \pi]$ or $(-180°, 180°]$ then we have to add or subtract $2\pi/360°$ repeatedly until it does.

❶❹ Simplify: $2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \times 5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

- Multiply modulus's and add arguments $2 \times 5\left(\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)\right)$
- Simplify: $10\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$

❶❺ Simplify: $9(\cos 45° + i\sin 45°) \div 3(\cos 60° + i\sin 60°)$

- Divide modulus's and subtract arguments

$$9 \div 3(\cos(45° - 60°) + i\sin(45° - 60°))$$

- Simplify: $3(\cos(-15°) + i\sin(-15°)) = 3(\cos(15°) - i\sin(15°))$

Bk2 P101 Ex6
Q1, 2, 3a, 3c, 4a, c, e, g, i, k

Multiple Angle Formulae

At Higher level, we were given the double angle formulae:

$$\sin 2A = 2\sin A\cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A$$

At AH level, we can be asked to develop further multiple angle formulae by equating the DeMoivre expansion with the Binomial expansion of the complex number:

$$z = \cos\theta + i\sin\theta \quad \text{i.e.} \quad r = |z| = 1$$

❶❻ $(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$ using DeMoivre

$(\cos\theta + i\sin\theta)^2 = \cos^2\theta + i2\cos\theta\sin\theta - \sin^2\theta$ using Binomial

Equating real parts we get: $\cos 2\theta = \cos^2\theta - \sin^2\theta$

Equating imaginary pars we get: $\sin 2\theta = 2\sin\theta\cos\theta$

❶❼ $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$ using DeMoivre

$(\cos\theta + i\sin\theta)^5 = \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

using Binomial

Equating real parts we get: $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

Equating imaginary pars we get: $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

This can be developed further using: $\sin^2\theta + \cos^2\theta = 1$

If we substitute $\cos^2\theta = 1 - \sin^2\theta$ into the imaginary part above, we get:

$$\sin 5\theta = 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$\sin 5\theta = 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta - 10\sin^3\theta + 10\sin^5\theta + \sin^5\theta$$

$$\sin 5\theta = 5\sin\theta - 10\sin^3\theta + 5\sin^5\theta - 10\sin^3\theta + 11\sin^5\theta$$

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

These can be re-arranged to make

$\sin^n\theta$ / $\cos^n\theta$ the subject

Bk2 P102 Ex6
Q5, 6, 7(a)

Roots of a Complex Number

DeMoivre can also be used to find fractional powers of z in polar form:

$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{1}{n}\theta + i\sin\frac{1}{n}\theta\right)$	or	$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$

NB the n^{th} root of z has n solutions and will have arguments in the range $(-n\pi, n\pi]$

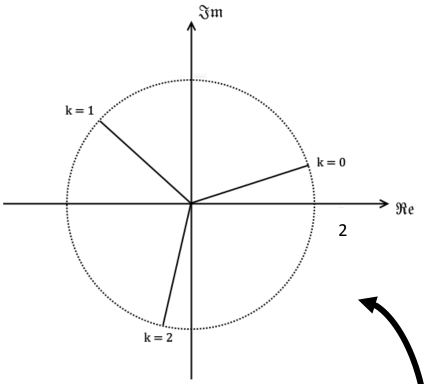
❶❽ Solve the equation $z^3 = 4 + i4\sqrt{3}$ i.e. find $\sqrt[3]{(4 + i4\sqrt{3})}$

$|z^3| = \sqrt{4^2 + (4\sqrt{3})^2} = 8$ $arg(z^3) = \tan^{-1}\frac{4\sqrt{3}}{4} = \frac{\pi}{3}$

$z^8 = 8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Solutions are of the form:

$z = r^{\frac{1}{3}}\left(\cos\frac{1}{3}\left(\frac{\pi}{3} + 2k\pi\right) + i\sin\frac{1}{3}\left(\frac{\pi}{3} + 2k\pi\right)\right)$ where $k = 0,1,2$



$k = 0$ gives $z = 2\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$

$k = 1$ gives $z = 2\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right)$

$k = 2$ gives $z = 2\left(\cos\frac{13\pi}{9} + i\sin\frac{13\pi}{9}\right)$

Outwith $(-n\pi, n\pi]$ so subtract 2π

$z = 2\left(\cos -\frac{5\pi}{9} + i\sin -\frac{5\pi}{9}\right) = 2\left(\cos\frac{5\pi}{9} - i\sin\frac{5\pi}{9}\right)$

We can be asked to plot the solutions on an Argand diagram

❶❾ Solve the equation: $z^5 = 1$

$|z^5| = 1$ $arg(z^0) = 0$ $z^5 = 1(\cos 0 + i\sin 0)$

Solutions are of the form:

$z = 1^{\frac{1}{5}}\left(\cos\frac{1}{5}(0 + 2k\pi) + i\sin\frac{1}{5}(0 + 2k\pi)\right)$ where $k = 0,1,2,3,4$

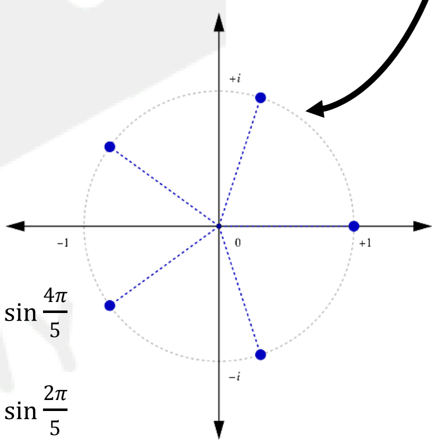
$k = 0$ gives $z = (\cos 0 + i\sin 0) = 1$

$k = 1$ gives $z = \left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$k = 2$ gives $z = \left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$k = 3$ gives $z = \left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right) = \cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}$

$k = 4$ gives $z = \left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right) = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$



Bk2 P106 Ex7 Q1 (1st col), 2a, 2c

These solutions are often referred to as the **5th roots of unity**

Finding Complex Roots of Polynomial Equations

A polynomial of degree n has n complex roots.

❷⓿ If $f(z) = z^4 - 6z^3 + 18z^2 - 30z + 25$, show that $z = 1 + 2i$ is a root of the equation $f(z) = 0$ and hence find the other roots of the polynomial.

$$f(1+2i) = (1+2i)^4 - 6(1+2i)^3 + 18(1+2i)^2 - 30(1+2i) + 25$$

Using the graphics calculator:

$$f(1+2i) = (-7-24i) - 6(-11-2i) + 18(-3+4i) - 30(1+2i) + 25$$

$$f(1+2i) = -7 - 24i + 66 + 12i - 54 + 72i - 30 - 60i + 25 = 0$$

So $z = 1 + 2i$ is a root of $f(z)$ since $f(1+2i) = 0$

If $1 + 2i$ is a root then its conjugate $1 - 2i$ is also a root.

This means that $z - 1 + 2i$ and $z - 1 - 2i$ are factors.

$$\begin{aligned}(z-1-2i)(z-1+2i) &= z^2 - z + 2iz - 1 + 1 - 2i - 2iz + 2i - 4i^2 \\ &= z^2 - 2z + 5\end{aligned}$$

$z^2 - 2z + 5$ is the first quadratic factor. To find the other quadratic factor we use polynomial division:

			z^2	$-4z$	$+5$
$z^2 \quad -2z \quad +5$	z^4	$-6z^3$	$+18z^2$	$-30z$	$+25$
	z^4	$-2z^3$	$+5z^2$		
		$-4z^3$	$+13z^2$	$-30z$	$+25$
		$-4z^3$	$+8z^2$	$-20z$	
			$5z^2$	$-10z$	$+25$
			$5z^2$	$-10z$	$+25$
					0

To find roots from the 2nd quadratic factor, $z^2 - 4z + 5$, equate to zero and use the quadratic formula:

$$z = \frac{4 \pm \sqrt{-4}}{2} = \frac{4 \pm 2i}{2} = 2 \pm i$$

Bk2 P108 Ex8

Q2e, 3e, 6c, 7c

Roots are: $1 + 2i$, $1 - 2i$, $2 + i$, and $2 - i$

NB: * There is the possibility that some of the roots are REAL!!

* You could be asked to plot these roots on an Argand diagram.