Unit 1: Methods in Algebra and Calculus (H7X2 77)

Differentiation 1 – New rules and Functions

In Higher Maths we learned how to differentiate functions involving a bracket without having to break it. This was called the 'Chain Rule' where we multiplied the derivative of outside the bracket by the derivative of inside the bracket.

Examples: ❶ Differentiate $y = (3x^2 + 5)^4$

$$\frac{dy}{dx} = 4 \times (3x^2 + 5)^3 \times 6x = 24x(3x^2 + 5)^3$$

❷ Differentiate $y = cos^3x$

Re-write using a bracket - $y = cos^3x = (\cos x)^3$

$$\frac{dy}{dx} = 3(\cos x)^2 \times - \sin x = -3(\cos x)^2 \sin x$$

❸ Differentiate $y = cos^2 3x$

Three functions! - $cos^2 3x = (\cos 3x)^2 = (\cos(3x))^2$

Hence: $$\frac{dy}{dx} = 2(\cos 3x)^1 \times \frac{d}{dx}[\cos(3x)]$$

$$\frac{dy}{dx} = 2(\cos 3x) \times - \sin 3x \times 3$$

$$\frac{dy}{dx} = -6 \cos 3x \sin 3x = -3 \sin 6x$$

P32 Ex3A
1st Column

P33 Ex3B
1st Column

Product Rule:

The product rule is used to differentiate a product of functions (without having to break the brackets) e.g. $y = (2x + 3)(x^2 - 2x)^4$.

Let $f(x) = uv$ where u and v are separate functions of x

Then: $$f'(x) = u'v + uv'$$

❹ Differentiate $f(x) = 5x(3x^2+2)^5$

Let $u = 5x$ *and* $v = (3x^2+2)^5$

$u' = 5$ *and* $v' = 5(3x^2+2)^4 \times 6x = 30x(3x^2+2)^4$

$f'(x) = u'v + uv'$

$f'(x) = 5 \times (3x^2+2)^5 + 5x \times 30x(3x^2+2)^4$

$f'(x) = 5(3x^2+2)^5 + 150x^2(3x^2+2)^4$

$f'(x) = 5(3x^2+2)^4[(3x^2+2) + 30x^2]$ ***NB always factorise your answer***

$f'(x) = 5(3x^2+2)^4(33x^2+2)$

❺ Differentiate $f(x) = 3x^3 \cos 2x$

Let $u = 3x^3$ *and* $v = \cos(2x)$

$u' = 9x^2$ *and* $v' = -\sin(2x) \times 2 = -2\sin 2x$

$f'(x) = u'v + uv'$

$f'(x) = 9x^2 \times \cos 2x + 3x^3 \times -2\sin 2x$

$f'(x) = 9x^2 \cos 2x - 6x^3 \sin 2x$

$f'(x) = 3x^2[3\cos 2x - 2x \sin 2x]$

P35 Ex4A
1st Column

P36 Ex4B
1st column

NB: For 3 functions - $f(x) = uvw$ *then:* $f'(x) = u'vw + uv'w + uvw'$

❻ Find the equation of the tangent to the curve $y = x^2(x-1)^5$ at the point (2, 4)

Let $u = x^2$ *and* $v = (x-1)^5$

$u' = 2x$ *and* $v' = 5(x-1)^4 \times 1 = 5(x-1)^4$

$$\frac{dy}{dx} = u'v + uv'$$

$$\frac{dy}{dx} = 2x \times (x-1)^5 + x^2 \times 5(x-1)^4 = x(x-1)^4[2(x-1) + 5x]$$

at $x = 2$ $\quad m = x(x-1)^4(7x-2) = 2(2-1)^4(7(2)-2) = 24$

$$y - 4 = 24(x-2)$$

$$y = 24x - 44$$

Quotient Rule:

The quotient rule is used to differentiate an algebraic fraction e.g. $y = \frac{3x+4}{x^2-2}$

Let $f(x) = \frac{u}{v}$ where u and v are separate functions of x

Then: $$f'(x) = \frac{u'v - uv'}{v^2}$$

❼ Differentiate $f(x) = \frac{2x}{x+3}$

Let $u = 2x$ *and* $v = x + 3$

$u' = 2$ *and* $v' = 1$

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{2(x+3) - 2x}{(x+3)^2} = \frac{6}{(x+3)^2}$$

❽ Differentiate $f(x) = \frac{4x+3}{\sqrt{2x-1}}$

Let $u = 4x + 3$ *and* $v = (2x-1)^{\frac{1}{2}}$

$u' = 4$ *and* $v' = \frac{1}{2}(2x-1)^{-\frac{1}{2}} \times 2 = (2x-1)^{-\frac{1}{2}}$

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{4(2x-1)^{\frac{1}{2}} - (4x+3)(2x-1)^{-\frac{1}{2}}}{(2x-1)}$$

$$= \frac{(2x-1)^{-\frac{1}{2}}[4(2x-1)^1 - (4x+3)]}{(2x-1)}$$

$$= \frac{(2x-1)^{-\frac{1}{2}}[8x - 4 - 4x - 3]}{(2x-1)^1}$$

$$= \frac{4x-7}{(2x-1)^{\frac{3}{2}}}$$

P37 Ex5A
Columns 1&2

P37 Ex5B
1st Column

P38 Ex6
Mixed Exercise

Three new functions are defined in the AH course:

Secant	Cosecant	Cotangent
$\sec x = \frac{hypotenuse}{adjacent} = \frac{1}{\cos x}$	$\operatorname{cosec} x = \frac{hypotenuse}{opposite} = \frac{1}{\sin x}$	$\cot x = \frac{adjacent}{opposite} = \frac{1}{\tan x}$

These give us some new trigonometric identities to learn:

$$\cot x = \frac{\cos x}{\sin x} \qquad 1 + tan^2x = sec^2x \qquad 1 + cot^2x = cosec^2x$$

These can be proved using the substitutions $\tan x = \frac{\sin x}{\cos x}$ &/or $sin^2x + cos^2x = 1$ if you are brave enough?

Similar to Higher, the derivatives of these new functions, along with those of $\tan x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\ln x$ and e^x, are given in the AH formula sheet:

Higher

Standard derivatives	
$f(x)$	$f'(x)$
$\sin ax$	$a \cos ax$
$\cos ax$	$-a \sin ax$

Advanced Higher

Standard derivatives	
$f(x)$	$f'(x)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\ln x$	$\frac{1}{x}$
e^x	e^x
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$

We can be asked to find the derivative of functions involving these as well as a combination of them i.e. using the chain, product or quotient rules:

❾ Differentiate $f(x) = \sin(\tan x)$

$$f'(x) = -\cos(\tan x) \times \sec^2 x = -\sec^2 x \cos(\tan x)$$

10 Differentiate $f(x) = e^{2x}(\cot x)$

Let $u = e^{2x}$ *and* $v = \cot x$

$u' = 2e^{2x}$ *and* $v' = -\text{cosec}^2 x$

$f'(x) = u'v + uv'$

$f'(x) = 2e^{2x} \times \cot x + e^{2x} \times (-\text{cosec}^2 x)$

$f'(x) = 2e^{2x}(\cot x) - e^{2x}(\text{cosec}^2 x)$

$f'(x) = e^{2x}[2\cot x - \text{cosec}^2 x]$

11 Show that the derivative of $y = \frac{\ln(x^2)}{\tan x}$ is $\frac{2}{x\tan x} - \frac{\ln(x^2)}{\tan^2 x} - \ln(x^2)$

Let $u = \ln(x^2)$ *and* $v = \tan x$

$u' = \frac{1}{x^2} \times 2x = \frac{2}{x}$ *and* $v' = \sec^2 x$

$$f'(x) = \frac{u'v - uv'}{v^2} = \frac{\frac{2}{x} \times \tan x - \ln(x^2) \times \sec^2 x}{\tan^2 x}$$

$$f'(x) = \frac{2}{x\tan x} - \frac{\sec^2 x \ln(x^2)}{\tan^2 x}$$

$$f'(x) = \frac{2}{x\tan x} - \frac{\sec^2 x \ln(x^2)}{\tan^2 x}$$ *using* $1 + tan^2 x = sec^2 x$

$$f'(x) = \frac{2}{x\tan x} - \frac{(1 + tan^2 x)\ln(x^2)}{\tan^2 x} = \frac{2}{x\tan x} - \frac{\ln(x^2)}{\tan^2 x} - \frac{\ln(x^2)\, tan^2 x}{\tan^2 x}$$

$$f'(x) = \frac{2}{x\tan x} - \frac{\ln(x^2)}{\tan^2 x} - \ln(x^2)$$

P40 Ex7
All

P43 Ex8A
2 Columns

Book 2
P33 Ex3A
Q1, 3, 5

P44 Ex8B
Q1

Rectilinear Motion:

Body moving in a straight line, normally assumed to be the x-axis.

Displacement from the origin is a function of time and is denoted:

$$x = f(t) \quad \text{or} \quad s = f(t)$$

Velocity is the rate of displacement with time and is given by: $v = \frac{dx}{dt}$

Acceleration is the rate of change of velocity with time and is given by:

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

❶❷ A body travels along a straight line such that $x = t^3 - 6t^2 + 9t + 1$ where x represents its displacement in centimetres from the origin t seconds after observation begins.

a) When is the velocity zero?

b) When is the acceleration zero?

c) When is the velocity of the body decreasing?

a) $v = 0 \Rightarrow \frac{dx}{dt} = 0$ so $3t^2 - 12t + 9 = 0 \Rightarrow 3(t-3)(t-1) = 0$

At the 1st and 3rd second, the velocity is zero.

b) $a = 0 \Rightarrow \frac{dv}{dt} = 0$ so $6t - 12 = 0 \Rightarrow t = 2$

At the 2nd second, the acceleration is zero.

c) v decreasing so $a < 0 \Rightarrow 6t - 12 < 0 \Rightarrow t < 2$

The velocity is decreasing up to the 2nd second.

P51 Ex1
1st Column Q1&2
Q3, 5, 7, 11 (optional)