Prelim Examination 2002 / 2003
(Assessing Units 1 & 2)

MATHEMATICS

Advanced Higher Grade

Time allowed - 2 hours 30 minutes

Read Carefully

1. Full credit will be given only where the solution contains appropriate working.
2. **Calculators may be used in this paper.**
3. Answers obtained by readings from scale drawings will not receive any credit.
4. **This examination paper contains questions graded at all levels.**

All questions should be attempted

1. Evaluate $\int_{-1}^{0} \frac{dx}{\sqrt{3-2x-x^2}}$. (5)

2. Verify that $2i$ is a solution of $z^4 - 4z^3 + 17z^2 - 16z + 52 = 0$.

   Hence find all the solutions. (5)

3. Use Gaussian Elimination to solve the following system of equations.

$$\begin{aligned} x - y + 2z &= 7 \\ 3x + 2y + z &= -9 \\ 2x + y - 3z &= -20 \end{aligned}$$

(5)

4. (*a*) Write the binomial expansion of $(a+b)^4$. (2)

   (*b*) Find $\left(x + \frac{2}{x}\right)^4 - \left(x - \frac{2}{x}\right)^4$ in its simplest form. (5)

5. (*a*) Differentiate $f(x) = e^{\frac{x+1}{x-1}}$, $x > 1$. (4)

   (*b*) Find the equation of the tangent to the curve $2x^2 - 3xy + y^2 = 0$ at the point (1, 1). (4)

6. Let $z = 3 - i$ and let $\bar{z}$ be the complex conjugate of z.

   Solve $a\left(\frac{z}{\bar{z}}\right) + bz = 22 - 14i$, for $a, b \in \mathbf{R}$. (4)

7. If k is a positive integer and the coefficient of x^2 in the expansion of $(k - 4x)^6$ is 19440, find the value of k . (4)

8. The parametric equations $x = \dfrac{t-3}{2(1+2t)}$, $y = \dfrac{t}{2(1+2t)}$ represent a line, where $t \in \mathbf{R}$.

Find the Cartesian equation of the line, and show that the point $\left(\tfrac{11}{2}, 1\right)$ lies on the line. (5)

9. The first three terms of an arithmetic series are 8 + 16 + 24 + . . .

(*a*) Find, in terms of *n*, an expression for u_n, the n^{th} term, and S_n, the sum to *n* terms. (4)

(*b*) Hence find the sum of the natural numbers that are both multiples of 8 and smaller than 1000. (3)

10. Express $2 - 2\sqrt{3}i$ in polar form and hence find values for $\left(2 - 2\sqrt{3}\,i\right)^{3/2}$, writing your answers in the form $p + qi$, where $p, q \in \mathbf{R}$. (6)

11. By expressing $0\cdot 2\dot{3}$ as a geometric series, write $0\cdot 2\dot{3}$ in the form $\dfrac{a}{b}$, where a, b $\in \mathbf{N}$. (3)

12. Find the 4th roots of unity and show that the sum of these roots is zero. (5)

13. The integral I_n is given by

$$I_n = \int \sin^n x \, dx .$$

(*a*) By using the fact that $\sin^n x = \sin x \sin^{n-1} x$, prove the reduction formula

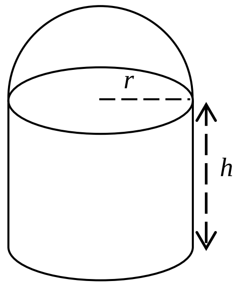
$$I_n = -\frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}$$

(5)

(*b*) Use the above result to find a reduction formula for $\int_0^{\pi/2} \sin^n x$. (2)

(*c*) Hence evaluate $\int_0^{\pi/2} \sin^8 x$. (3)

14. A child's drinking cup is made in the shape of a circular cylinder with a hemispherical top.



The cylinder has height h cm and radius r cm, and the cup has a total surface area of 80π cm^2.

(*a*) Find an expression for the height h in terms of the radius r. (2)
[surface area of a sphere $= 4\pi r^2$]

(*b*) Find the values of h and r for which the cup has a maximum volume.
Hence find the maximum volume. (8)

15. The function $f(x)$ is given by $f(x) = \dfrac{x^2 - 4}{x^2 + 8x}$.

(*a*) Write down the equations of the asymptotes of $f(x)$. (2)

(*b*) Prove that $f(x)$ has no stationary points. (3)

(*c*) Sketch the curve of $f(x)$, showing clearly all its features. (3)

(*d*) Using the sketch in part (c), construct the graph of $\dfrac{1}{f(x)}$, the curve of the reciprocal function. (4)

End of Question Paper