Marking Scheme - Advanced Higher Prelim 2001

	Give one mark for each •	Illustrations for awarding each mark
1(a)	**ans:** $\frac{dy}{dx} = -\frac{1}{x^2+1}$ **4 marks** • know how to differentiate $\tan^{-1}$ • chain rule factor • manipulating algebra • answer in simplest form	 • $\frac{1}{1+\left(\frac{x+1}{x-1}\right)^2}$ • $-\frac{2}{(x-1)^2}$ • $\frac{(x-1)^2}{2x^2+2} \times -\frac{2}{(x-1)^2}$ • $-\frac{1}{x^2+1}$
1(b)	**ans:** $\frac{dy}{dx} = \tan x$ **3 marks** • know how to differentiate log • chain rule factor • answer in simplest form	 • $\frac{1}{\sec x}$ • $\sec x \tan x$ • $\tan x$
2.	**ans:** (2, -1, 1) **5 marks** • write system as an augmented matrix with 1 in top left-hand corner (optional) • first modified system • second modified system • using back-substitution to find z • using back-substitution to find x and y	 • $\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 3 \\ 2 & 3 & -4 & -3 \\ 3 & -1 & -1 & 6 \end{array}\right]$ • $\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -10 & -9 \\ 0 & -7 & -10 & -3 \end{array}\right]$ • $\left[\begin{array}{ccc\|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -10 & -9 \\ 0 & 0 & 60 & 60 \end{array}\right]$ • $z = 1$ • $y = -1$, $x = 2$

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3.	**ans:** proof by induction **5 marks**	
	• show true for n = 1	• $\begin{cases} LHS = \frac{d}{dx}(x) = 1;\ RHS = 1\times x^{1-1} = 1 \\ So\ true\ when\ n = 1 \end{cases}$
	• state inductive hypothesis	• Assume $\frac{d}{dx}\left(x^k\right) = k\,x^{k-1}$
	• consider the case for $n = k + 1$	• Consider $\frac{d}{dx}\left(x^{k+1}\right)$
	• carry out manipulation	• $\frac{d}{dx}\left(x \cdot x^k\right) = x^k + x \cdot kx^{k-1} = x^k + kx^k$ $= (k+1)x^k$
	• state conclusion	• So, if the formula is valid for n, it is valid for n+1. Since it is valid for n = 1, it is therefore true for all $n \geq 1$.
4.	**ans:** $ln\ 3$ **7 marks**	
	• rewrite integral in terms of x • correct limits	• and • $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2x}{x^2 + x}\,dx$
	• tidy up integral	• $\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{2}{x+1}\,dx$
	• integrate	• $2\,ln(x+1)\Big]_{1/\sqrt{3}}^{\sqrt{3}}$
	• evaluate limits	• $2\,ln\left(\sqrt{3}+1\right) - 2\,ln\left(\frac{1}{\sqrt{3}}+1\right)$
	• manipulate surds • final answer	• and • $2\,ln\left(\frac{\sqrt{3}+1}{\frac{1}{\sqrt{3}}+1} \times \frac{\frac{1}{\sqrt{3}}-1}{\frac{1}{\sqrt{3}}-1}\right)$ $= 2\,ln\left(-\frac{3}{2}\left(\frac{1}{\sqrt{3}} - \sqrt{3}\right)\right) = 2\,ln\,\sqrt{3} = ln\ 3$
5.	**ans:** 560 **3 marks**	
	• correct general term	• $\binom{7}{r}\left(x^3\right)^{7-r}\left(\frac{2}{x}\right)^r = \binom{7}{r} 2^r x^{21-4r}$
	• put power of x equal to 5 and solve for r • calculate coefficient	• $21 - 4r = 5;\ r = 4$ • $\binom{7}{4} 2^4 = 35 \times 16 = 560$

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6(a)	**ans:** $\frac{6}{x^2+9}+\frac{2}{x+3}$ **4 marks** • know how to find partial fractions • know how to find A, B and C • finds A • finds B and C	 • $\frac{Ax+B}{x^2+9}+\frac{C}{x+3}$ • $2x^2+6x+36=(x+3)(Ax+B)+C(x^2+9)$ • $A=0$ • $B=6$ and $C=2$
6(b)	**ans:** 3·37 units^2 **5 marks** • knows to express integral in partial fractions • and • integrates terms correctly • evaluates limits • final answer	 • $\int_{-2}^{0}\left(\frac{6}{x^2+9}+\frac{2}{x+3}\right)dx$ • and • $2tan^{-1}\frac{x}{3}+2ln\lvert x+3\rvert$ • $2tan^{-1}0+2ln3-\left(2tan^{-1}\left(-\frac{2}{3}\right)+2ln1\right)$ • 3·37 units^2
7.	**ans:** $x(t)=3t+1$ **6 marks** • knows formula for $\frac{d^2y}{dx^2}$ in parametric form • finds $\frac{d}{dt}\left(\frac{dy}{dx}\right)$ • substitutes information into formula • finds $\frac{dx}{dt}$ in simplest form • integrates $\frac{dx}{dt}$ to find x • finds constant of integration	 • $\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$ • $3t^2+3$ • $t^2+1=\frac{3t^2+3}{\frac{dx}{dt}}$ • 3 • $x(t)=\int 3dt=3t+c$ • $x(1)=4$; $c=1$
8.	**ans:** 8 units **6 marks** • knows to find max. and min. turning points • knows to use implicit differentiation • differentiates correctly • finds x-coordinate of relevant turning point • finds corresponding y-coordinates • finds max. distance	 • • • $\frac{dy}{dx}=\frac{2x(4-x^2)}{y}$ • x = -2, 0 or 2 and chooses x =2 from diagram • y = -4 or 4 • 8

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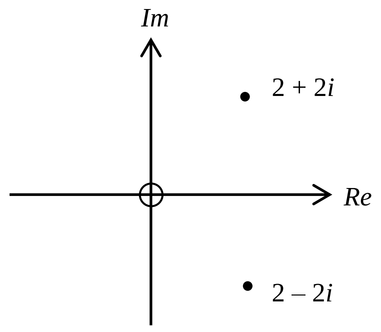
9(a)	**ans:** $2 + 2i,\ 2 - 2i$ **4 marks** • set up system of equations • use substitution to obtain quadratic • use quadratic formula to solve quadratic • correct answer	 • $x + y = 4;\ \ xy = 8$ • $x^2 - 4x + 8 = 0$ • $x = \dfrac{4 \pm \sqrt{16 - 4(1)(8)}}{2}$ • $x = 2 + 2i$ or $x = 2 - 2i$
9(b)	**ans:** Diagram **2 marks** • Argand diagram correctly labelled • both points plotted and labelled	
10.	**ans:** Proof **5 marks** • first application of integration by parts • second application of integration by parts • knowing to use integ. by parts again • third application of integration by parts • answer in required form	 • $x^3 \sin x - \int 3x^2 \sin x\, dx$ • and • $x^3 \sin x - \left[-3x^2 \cos x + \int 6x \cos x\, dx\right]$ $= x^3 \sin x + 3x^2 \cos x - \int 6x \cos x\, dx$ • $x^3 \sin x + 3x^2 \cos x - 6x \sin x - 6\cos x + C$ • $3\left(x^2 - 2\right)\cos x + \left(x^3 - 6x\right)\sin x + C$
11(a)	**ans:** $3\left(1 - \dfrac{1}{3^n}\right)$ **4 marks** • correct ratio • using correct formula • substituting correctly into formula • answer in simplest form	 • $r = \frac{1}{3}$ • $S_n = \dfrac{a\left(1 - r^n\right)}{1 - r}$ • $\dfrac{2\left(1 - \left(\frac{1}{3}\right)^n\right)}{1 - \frac{1}{3}} = \dfrac{2\left(1 - \frac{1}{3^n}\right)}{\frac{2}{3}}$ • $3\left(1 - \dfrac{1}{3^n}\right)$

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11(b)	**ans:** $n = 5$ **3 marks** • use formula correctly • manipulate formula • answer	 • $\frac{242}{81} = 3\left(1 - \frac{1}{3^n}\right) \Rightarrow \frac{242}{243} = 1 - \frac{1}{3^n}$ • $3^n = 243$ • $n = 5$ (using logs or trial and error)
12(a)	**ans:** $\frac{dx}{dt} = 1000 + 0\cdot 1x$ **2 marks** • amount of money going into account each year • interest @ 10%	 • 1000 • $0\cdot 1x$
12(b)	**ans:** $t = 10\,ln\frac{1000 + 0\cdot 1x}{1200}$ **7 marks** • know to use method of separating variables • separates variables correctly • integrates LHS correctly • integrates RHS correctly (incl. constant of integration) • correct initial conditions • finds correct value of C • finds required solution	 • and • $\int \frac{dx}{1000+0\cdot 1x} = \int dt$ • and • $10\,ln(1000 + 0\cdot 1x) = t + C$ • $x = 2000$ at $t = 0$ • $C = 10\,ln 1200$ • $t = 10\,ln\frac{1000 + 0\cdot 1x}{1200}$
12(c)	**ans:** 23 years **2 marks** • substitute in value for x • answer	 • $t = 10\,ln\frac{1000+0\cdot 1\times 100000}{1200} = 10\,ln\frac{11000}{1200}$ • $22\cdot 16$ years $\approx$ 23 years
13(a)	**ans:** $x = 1$ **1 mark** • states equation of vertical asymptote	 • $x = 1$
13(b)	**ans:** $y = x - 1$ **3 marks** • knows to divide • restating function • correctly stating equation of asymptote	 • and • $\frac{x^2 - 2x + 2}{x - 1} = (x - 1) + \frac{1}{x - 1}$ • $y = x - 1$

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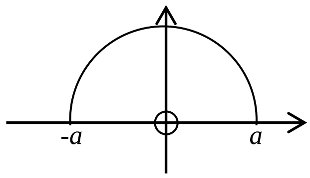
13(c)	**ans:** Max at (0, -2), Min at (2, 2) **5 marks** • knows to find $\frac{dy}{dx}$ • knows to put $\frac{dy}{dx}$ =0 • finds x-coordinates • finds y-coordinates • determines nature of each by second derivative or nature table	• $\frac{dy}{dx} = 1 - \frac{1}{(x-1)^2}$ • $1 - \frac{1}{(x-1)^2} = 0$ • $x = 0$ or $x = 2$ • (0, -2), (2, 2) • $\frac{d^2y}{dx^2} = \frac{2}{(x-1)^3}$; *Max at* $(0,-2)$, *Min at* $(2,2)$
13(d)	**ans:** sketch **4 marks** • sketch showing all relevant points • correctly shows how curve approaches asymptotes • knows to reflect all parts of graph from below the *x*-axis to above the *x*-axis • reflects correctly	See sketch at end of marking scheme
14(a)	**ans:** $\frac{4}{3}\pi a^3$ **8 marks** • draws sketch showing semi-circle above *x*-axis • Roots of semi-circle at –*a* and *a* • knows how to find volume of revolution • limits of integration as –*a* and *a* • applies formula correctly • integrates correctly • evaluates limits • correct answer	• and •  • and • $V = \int_{-a}^{a} \pi y^2 dx$ • $V = \int_{-a}^{a} \pi \left(a^2 - x^2\right) dx$ • $\pi\left[a^2x - \frac{x^3}{3}\right]_{-a}^{a}$ • $\pi\left[a^2(a) - \frac{a^3}{3}\right] - \pi\left[a^2(-a) - \frac{(-a)^3}{3}\right]$ • $\frac{4}{3}\pi a^3$
14(b)	**ans:** 523·6 units^3 **2 marks** • knows to put $a = 5$ • finds volume	• $\frac{4}{3}\pi\left(5^3\right)$ • 523·6 units^3

Total 100 Marks

Sketch for question 13(d)