

**X100/701**

---

NATIONAL  
QUALIFICATIONS  
2005

FRIDAY, 20 MAY  
1.00 PM – 4.00 PM

MATHEMATICS  
ADVANCED HIGHER

**Read carefully**

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**



## Answer all the questions.

1. (a) Given  $f(x) = x^3 \tan 2x$ , where  $0 < x < \frac{\pi}{4}$ , obtain  $f'(x)$ . 3
- (b) For  $y = \frac{1+x^2}{1+x}$ , where  $x \neq -1$ , determine  $\frac{dy}{dx}$  in simplified form. 3
2. Given the equation  $2y^2 - 2xy - 4y + x^2 = 0$  of a curve, obtain the  $x$ -coordinate of each point at which the curve has a horizontal tangent. 4
3. Write down the Maclaurin expansion of  $e^x$  as far as the term in  $x^4$ . 2  
 Deduce the Maclaurin expansion of  $e^{x^2}$  as far as the term in  $x^4$ . 1  
 Hence, or otherwise, find the Maclaurin expansion of  $e^{x+x^2}$  as far as the term in  $x^4$ . 3
4. The sum,  $S(n)$ , of the first  $n$  terms of a sequence,  $u_1, u_2, u_3, \dots$  is given by  $S(n) = 8n - n^2$ ,  $n \geq 1$ .  
 Calculate the values of  $u_1, u_2, u_3$  and state what type of sequence it is. 3  
 Obtain a formula for  $u_n$  in terms of  $n$ , simplifying your answer. 2
5. Use the substitution  $u = 1 + x$  to evaluate  $\int_0^3 \frac{x}{\sqrt{1+x}} dx$ . 5
6. Use Gaussian elimination to solve the system of equations below when  $\lambda \neq 2$ :  

$$\begin{aligned} x + y + 2z &= 1 \\ 2x + \lambda y + z &= 0 \\ 3x + 3y + 9z &= 5. \end{aligned}$$
 4  
 Explain what happens when  $\lambda = 2$ . 2
7. Given the matrix  $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$ , show that  $A^2 + A = kI$  for some constant  $k$ , where  $I$  is the  $3 \times 3$  unit matrix. 4  
 Obtain the values of  $p$  and  $q$  for which  $A^{-1} = pA + qI$ . 2
8. The equations of two planes are  $x - 4y + 2z = 1$  and  $x - y - z = -5$ . By letting  $z = t$ , or otherwise, obtain parametric equations for the line of intersection of the planes. 4  
 Show that this line lies in the plane with equation  

$$x + 2y - 4z = -11. \quad 1$$

9. Given the equation  $z + 2i\bar{z} = 8 + 7i$ , express  $z$  in the form  $a + ib$ .

4

10. Prove by induction that, for all positive integers  $n$ ,

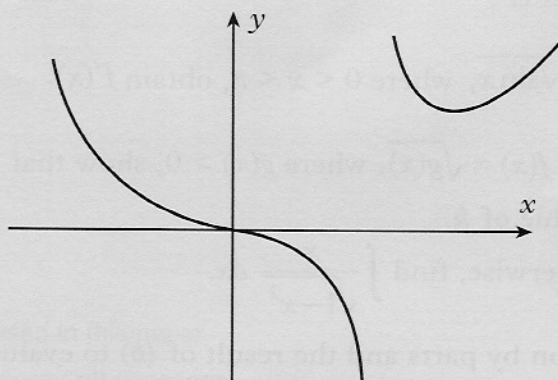
$$\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

5

State the value of  $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{r(r+1)(r+2)}$ .

1

11. The diagram shows part of the graph of  $y = \frac{x^3}{x-2}$ ,  $x \neq 2$ .



(a) Write down the equation of the vertical asymptote.

1

(b) Find the coordinates of the stationary points of the graph of  $y = \frac{x^3}{x-2}$ .

4

(c) Write down the coordinates of the stationary points of the graph of  $y = \left| \frac{x^3}{x-2} \right| + 1$ .

2

12. Let  $z = \cos \theta + i \sin \theta$ .

(a) Use the binomial expansion to express  $z^4$  in the form  $u + iv$ , where  $u$  and  $v$  are expressions involving  $\sin \theta$  and  $\cos \theta$ .

3

(b) Use de Moivre's theorem to write down a second expression for  $z^4$ .

1

(c) Using the results of (a) and (b), show that

$$\frac{\cos 4\theta}{\cos^2 \theta} = p \cos^2 \theta + q \sec^2 \theta + r, \text{ where } -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

stating the values of  $p$ ,  $q$  and  $r$ .

6

[Turn over for Questions 13, 14 and 15 on Page four

13. Express  $\frac{1}{x^3 + x}$  in partial fractions. 4

Obtain a formula for  $I(k)$ , where  $I(k) = \int_1^k \frac{1}{x^3 + x} dx$ , expressing it in the form  $\frac{a}{b}$ , where  $a$  and  $b$  depend on  $k$ . 4

Write down an expression for  $e^{I(k)}$  and obtain the value of  $\lim_{k \rightarrow \infty} e^{I(k)}$ . 2

14. Obtain the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 20 \sin x. \quad 7$$

Hence find the particular solution for which  $y = 0$  and  $\frac{dy}{dx} = 0$  when  $x = 0$ . 3

15. (a) Given  $f(x) = \sqrt{\sin x}$ , where  $0 < x < \pi$ , obtain  $f'(x)$ . 1

(b) If, in general,  $f(x) = \sqrt{g(x)}$ , where  $g(x) > 0$ , show that  $f'(x) = \frac{g'(x)}{2\sqrt{g(x)}}$ , stating the value of  $k$ . 2

Hence, or otherwise, find  $\int \frac{x}{\sqrt{1-x^2}} dx$ . 3

- (c) Use integration by parts and the result of (b) to evaluate

$$\int_0^{1/2} \sin^{-1} x \, dx. \quad 4$$

[END OF QUESTION PAPER]