SCOTTISH
QUALIFICATIONS
AUTHORITY

2005 Mathematics

Advanced Higher

Finalised Marking Instructions

These Marking Instructions have been prepared by Examination Teams for use by SQA Appointed Markers when marking External Course Assessments.

General Marking Principles

These principles describe the approach taken when marking Advanced Higher Mathematics papers. For more detailed guidance please refer to the detailed Marking Instructions.

1 The main principle is to give credit for the skills demonstrated and the criteria met. Failure to have a correct method may not preclude a candidate gaining credit for their solution.

2 The answer to one part of a question, even if incorrect, is accepted as a basis for subsequent dependent parts of the question.

3 The following are not penalised:

- working subsequent to a correct answer (unless it provides firm evidence that the requirements of the question have not been met)
- legitimate variation in numerical values / algebraic expressions.

4 Full credit will only be given where the solution contains appropriate working. Where the correct answer might be obtained by inspection or mentally, credit may be given.

5 Sometimes the method to be used in a particular question is explicitly stated; no credit will be given where a candidate obtains the correct answer by an alternative method.

6 Where the method to be used in a particular question is not explicitly stated, full credit will be given for an alternative valid method.

In the detailed Marking Instructions which follow, marks are shown alongside the line for which they are awarded. There are two codes used, M and E. M indicates a method mark, so in question 1, 1M, 1, 1 means a method mark for the product rule (and then a mark for each of the terms). E is shorthand for error. In question 3, 2E1, means that a correct answer is awarded 2 marks but that 1 mark is deducted for each error.

Advanced Higher Mathematics 2005
Marking Instructions

1. (a) $f(x) = x^3 \tan 2x$

$f'(x) = 3x^2 \tan 2x + x^3\left(2\sec^2 2x\right)$ **1M, 1, 1**

(b)
$$y = \frac{1 + x^2}{1 + x}$$

$$\frac{dy}{dx} = \frac{2x(1 + x) - \left(1 + x^2\right).1}{(1 + x)^2}$$ **1M,1**

$$= \frac{x^2 + 2x - 1}{(1 + x)^2}$$ **1**

Alternative 1

$$y = \frac{1 + x^2}{1 + x} = x - 1 + \frac{2}{1 + x}$$ **M1,1**

$$\frac{dy}{dx} = 1 - \frac{2}{(1 + x)^2} \text{ or } 1 - 2(1 + x)^{-2}$$ **1**

Alternative 2

$$y = \frac{1 + x^2}{1 + x} = \left(1 + x^2\right)(1 + x)^{-1}$$

$$\frac{dy}{dx} = 2x(1 + x)^{-1} + \left(1 + x^2\right)(-1)(1 + x)^2$$ **M1,1**

$$\frac{dy}{dx} = \frac{2x}{(1 + x)} - \frac{1 + x^2}{(1 + x)^2}$$ **1**

2.
$$2y^2 - 2xy - 4y + x^2 = 0$$

$$4y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y - 4\frac{dy}{dx} + 2x = 0$$ **1M, 1**

For a horizontal tangent $\frac{dy}{dx} = 0$, so $-2y + 2x = 0$, i.e. $y = x$. **1**

This gives $2x^2 - 2x^2 - 4x + x^2 = 0 \Rightarrow x(x - 4) = 0$, i.e. $x = 0$ or 4. **1**

[Both x values are needed.]

3.
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$ **2E1**

$$e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \ldots$$ **1**

$$e^{x + x^2} = e^x e^{x^2}$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\right)\left(1 + x^2 + \frac{x^4}{2} + \ldots\right)$$ **M1**

$$= 1 + x + x^2 + \frac{x^2}{2} + \frac{x^3}{6} + x^3 + \frac{x^4}{2} + \frac{x^4}{2} + \frac{x^4}{24} + \ldots$$

$$= 1 + x + \tfrac{3}{2}x^2 + \tfrac{7}{6}x^3 + \tfrac{25}{24}x^4 + \ldots$$ **1,1**

[1 mark for first 3 terms, 1 mark for final two terms.]

Alternatives for $f(x) = e^{x + x^2}$:

1: Applying Maclaurin directly: 1 method mark and then 2E1.

2: Using $(x + x^2)$ in the expansion of e^x: 1 method mark and then 2E1.

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4. $$u_1 = S_1 = 8 - 1 = 7$$

$$u_2 = S_2 - S_1 = 12 - 7 = 5 \qquad \mathbf{1}$$

$$u_3 = S_3 - S_2 = 15 - 12 = 3 \qquad \mathbf{1}$$

It is an arithmetic series. **1**

Method 1: $a = 7; d = -2.$ **1**

$u_n = 7 + (n - 1)(-2) = 9 - 2n.$ **1**

Method 2: $u_n = S(n) - S(n-1) = 8n - n^2 - \big(8(n-1) - (n-1)^2\big)$ **1**

$= 8n - n^2 - 8n + 8 + n^2 - 2n + 1 = 9 - 2n$ **1**

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5. $$u = 1 + x \Rightarrow dx = du \qquad \mathbf{1}$$

When $x = 0, u = 1$ and when $x = 3, u = 4$ **1**

$$\therefore \int_0^3 \frac{x}{\sqrt{1+x}}\,dx = \int_1^4 \frac{u-1}{u^{1/2}}\,du \qquad \mathbf{1}$$

$$= \int_1^4 \left[u^{1/2} - u^{-1/2}\right] du$$

$$= \left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right]_1^4 \qquad \mathbf{1}$$

$$= \left[\frac{2 \times 8}{3} - 2 \times 2\right] - \left[\frac{2}{3} - 2\right]$$

$$= \tfrac{14}{3} - 2 = 2\tfrac{2}{3} \text{ (2.67 is acceptable).} \qquad \mathbf{1}$$

Alternative for the final 2 marks:

$$\int_1^4 \frac{u-1}{u^{1/2}}\,du = \int_1^4 (u-1)u^{-1/2}du$$

$$= \left[(u-1)\int u^{-1/2}du - \int 1.\frac{u^{1/2}}{\frac{1}{2}}\,du\right]_1^4$$

$$= \left[(u-1)2u^{1/2} - \tfrac{4}{3}u^{3/2}\right]_1^4 \qquad \mathbf{1}$$

$$= \left[12 - \tfrac{32}{3}\right] - \left[0 - \tfrac{4}{3}\right] = 2\tfrac{2}{3} \qquad \mathbf{1}$$

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6. $$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & \lambda & 1 & 0 \\ 3 & 3 & 9 & 5 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & \lambda - 2 & -3 & -2 \\ 0 & 0 & 3 & 2 \end{array}\right) \qquad \mathbf{2E1}$$

$z = \tfrac{2}{3};$ **1**

$(\lambda - 2)y - 2 = -2 \Rightarrow y = 0; x = 1 - 0 - \tfrac{4}{3} = -\tfrac{1}{3}.$ **1**

When $\lambda = 2$, the second and third rows of the second matrix are the same, **1**(†)

so there is an infinite number of solutions. **1**

(†) Use of 'redundant' is worth a mark.

Interpretation in geometrical terms can be given both the marks.

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7. $A = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}\begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix}$ **2E1**

$$A^2 + A = \begin{pmatrix} 2 & -4 & -2 \\ -1 & 2 & -1 \\ 1 & 2 & 5 \end{pmatrix} + \begin{pmatrix} 0 & 4 & 2 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{pmatrix}$$ **1**

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} = 2I$$ **1**

$$A^{-1}(A^2 + A) = 2A^{-1}$$ **1**

$$2A^{-1} = A + I$$

$$A^{-1} = \tfrac{1}{2}A + \tfrac{1}{2}I$$ **1**

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8. Using $z = t$, then **1**

$$x - 4y = 1 - 2t$$ **1M**

$$x - y = t - 5$$

Subtracting:

$$3y = 3t - 6 \Rightarrow y = t - 2$$ **1**

$$x = y + t - 5 \Rightarrow x = 2t - 7$$ **1**

The line of intersection is given by: $x = 2t - 7, y = t - 2, z = t$.

$$x + 2y - 4z = 2t - 7 + 2(t - 2) - 4t$$

$$= 2t - 7 + 2t - 4 - 4t = -11$$ **1**

Alternative for first 4 marks:

The normals to the planes are $\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and the vector product of these will give the direction of the line of intersection.

$$(\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) \wedge (\mathbf{i} - \mathbf{j} - \mathbf{k}) = 6\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$$ **1M,1**

Now obtain a point (e.g. let $z = 0$ and solve the equations $x - 4y = 1$, $x - y = -5$ gives $(-7, -2, 0)$). **1**

Then write down the symmetric equation:

$$\frac{x + 7}{6} = \frac{y + 2}{3} = \frac{z}{3} = \lambda$$

which leads to $x = 6\lambda - 7, y = 3\lambda - 2, z = 3\lambda$. **1**

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9. Let $z = a + ib$ so $\bar{z} = a - ib$ **1**

$$z + 2i\bar{z} = 8 + 7i$$

$$a + ib + 2ia + 2b = 8 + 7i$$ **1**

$$a + 2b = 8$$

$$2a + b = 7$$ **M1**

$$3a = 6$$

$$a = 2;\ b = 3$$

$$z = 2 + 3i.$$ **1**

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10. When $n = 1$, LHS $= \dfrac{1}{1 \times 2 \times 3} = \dfrac{1}{6}$, RHS $= \dfrac{1}{4} - \dfrac{1}{2 \times 2 \times 3} = \dfrac{1}{4} - \dfrac{1}{12} = \dfrac{1}{6}$.

Thus true for $n = 1$. **1**

Assume true for $n = k$ and consider $n = k + 1$.

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)(k+2)(k+3)} \qquad \mathbf{1}$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \qquad \mathbf{1}$$

$$= \frac{1}{4} + \frac{-(k+3)+2}{2(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{k+1}{2(k+1)(k+2)(k+3)} \qquad \mathbf{1}$$

$$= \frac{1}{4} - \frac{1}{2\big((k+1)+1\big)\big((k+1)+2\big)}$$

Hence, since true for $n = 1$, true for all positive n. **1**

$$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}. \qquad \mathbf{1}$$

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11. (a) $x = 2$. [2 alone is not enough.] **1**

(b)

$$y = \frac{x^3}{x-2}$$

$$\frac{dy}{dx} = \frac{3x^2(x-2) - x^3}{(x-2)^2} = \frac{x^2(2x-6)}{(x-2)^2} = 0 \qquad \mathbf{2E1}$$

when $x = 0$ and when $x = 3$. **1**

Stationary points are $(0, 0)$ and $(3, 27)$. **1**

(c)

$$y = \left|\frac{x^3}{x-2}\right| + 1$$

Stationary points are $(0, 1)$ and $(3, 28)$. **1, 1**

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12. (a) $z^4 = (\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 4\cos^3\theta(i\sin\theta) + 6\cos^2\theta(i^2\sin^2\theta) + 4\cos\theta(i^3\sin^3\theta) + i^4\sin^4\theta$ **M1**

$= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta$ **1**

$= \left(\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta\right) + i\left(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta\right)$ **1**

(b) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ **1**

(c) Equating the real parts from (a) and (b): **M1**

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ **1**

$\dfrac{\cos 4\theta}{\cos^2\theta} = \cos^2\theta - 6\sin^2\theta + \sin^2\theta\dfrac{\sin^2\theta}{\cos^2\theta}$ **1**

$= \cos^2\theta - 6\left(1 - \cos^2\theta\right) + \left(1 - \cos^2\theta\right)\dfrac{1 - \cos^2\theta}{\cos^2\theta}$ **2E1**

$= 7\cos^2\theta - 6 + \left(\sec^2\theta - 2 + \cos^2\theta\right)$

$= 8\cos^2\theta + \sec^2\theta - 8$

$p = 8, q = 1, r = -8.$ **1**

13.

$\dfrac{1}{x^3 + x} = \dfrac{A}{x} + \dfrac{Bx + C}{x^2 + 1}$ **1**

$1 = A\left(x^2 + 1\right) + (Bx + C)x$

$x = 0 \Rightarrow 1 = A \Rightarrow A = 1$ **1**

$x = 1 \Rightarrow 1 = 2 + B + C$

$x = -1 \Rightarrow 1 = 2 + B - C$

$\Rightarrow C = 0, B = -1$ **1,1**

$\dfrac{1}{x^3 + x} = \dfrac{1}{x} - \dfrac{x}{x^2 + 1}$

$I(k) = \displaystyle\int_1^k \frac{1}{x^3 + x}\,dx = \int_1^k \left(\frac{1}{x} - \frac{x}{x^2 + 1}\right) dx$

$= \displaystyle\int_1^k \frac{1}{x}\,dx - \frac{1}{2}\int_1^k \frac{2x}{x^2 + 1}\,dx$

$= [\ln x]_1^k - \dfrac{1}{2}\left[\ln\left(x^2 + 1\right)\right]_1^k$ **2E1**

$= [\ln k - 0] - \dfrac{1}{2}\left[\ln\left(k^2 + 1\right) - \ln 2\right]$ **1**

$= \ln k - \ln\sqrt{k^2 + 1} + \tfrac{1}{2}\ln 2$

$= \ln\dfrac{k\sqrt{2}}{\sqrt{k^2 + 1}}$ **1**

$e^{I(k)} = \dfrac{k\sqrt{2}}{\sqrt{k^2 + 1}}$ **1**

$= \dfrac{\sqrt{2}}{\sqrt{1 + k^{-2}}} \to \sqrt{2}$ **1**

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14. Let $y = e^{mx}$, then the auxiliary equations is

$$m^2 - 3m + 2 = 0 \quad \mathbf{1}$$
$$(m - 1)(m - 2) = 0$$
$$m = 1 \text{ or } m = 2 \quad \mathbf{1}$$

The Complementary Function is $y = Ae^x + Be^{2x}$. **1**

For the Particular Integral, try $y = a \sin x + b \cos x$. **1**

$$\frac{dy}{dx} = a \cos x - b \sin x$$
$$\frac{d^2y}{dx^2} = -a \sin x - b \cos x$$

Substituting:

$$(-a \sin x - b \cos x) - 3(a \cos x - b \sin x) + 2(a \sin x + b \cos x) = 20 \sin x \quad \mathbf{1}$$
$$(-a + 3b + 2a) \sin x + (-b - 3a + 2b) \cos x = 20 \sin x$$
$$a + 3b = 20; \qquad -3a + b = 0$$
$$a = 2; \; b = 6. \quad \mathbf{1}$$

The general solution is

$$y = Ae^x + Be^{2x} + 2 \sin x + 6 \cos x \quad \mathbf{1}$$

$$\frac{dy}{dx} = Ae^x + 2Be^{2x} + 2 \cos x - 6 \sin x$$

$y = 0$ when $x = 0$ so $A + B + 6 = 0$. **1**

$\frac{dy}{dx} = 0$ when $x = 0$ so $A + 2B + 2 = 0$. **1**

$$B = 4; \qquad A = -10 \quad \mathbf{1}$$

The particular solution is

$$y = -10e^x + 4e^{2x} + 2 \sin x + 6 \cos x.$$

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15. (a) $$f(x) = (\sin x)^{1/2} \Rightarrow f'(x) = \frac{1}{2}\frac{\cos x}{(\sin x)^{1/2}}$$ **1**

(b) $f(x) = \sqrt{g(x)} = [g(x)]^{1/2}$. Thus

$$f'(x) = \frac{1}{2}g'(x)[g(x)]^{-1/2} = \frac{g'(x)}{2\sqrt{g(x)}} \text{ i.e. } k = 2$$ **1, 1**

$$\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{-2x}{2\sqrt{1-x^2}}\,dx$$ **1M,1**

$$= -\sqrt{1-x^2} + c$$ **1**

Alternative methods for working out $\int \frac{x}{\sqrt{1-x^2}}\,dx$ include use of substitutions.
For example $u = 1 - x^2$; $u^2 = 1 - x^2$; $x = \sin\theta$. Marks should be awarded:

For setting up an integral. **1**
For carrying out the integration. **1**
For returning to an answer in x. **1**

(c) $$\int_0^{1/2} \sin^{-1} x\,dx = \left[\sin^{-1} x \int 1.dx - \int x\frac{d}{dx}(\sin^{-1} x)\,dx\right]_0^{1/2}$$ **1**

$$= \left[x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}dx\right]_0^{1/2}$$ **1**

$$= \left[x\sin^{-1} x + \sqrt{1-x^2}\right]_0^{1/2}$$ **1**

$$= \frac{1}{2}\sin^{-1}\left(\frac{1}{2}\right) + \sqrt{\frac{3}{4}} - (0 + 1)$$

$$= \frac{1}{2}\frac{\pi}{6} + \frac{\sqrt{3}}{2} - 1 = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$ **1**

[END OF MARKING INSTRUCTIONS]