



Higher Mathematics

UNIT 3

Mathematics 3

HSN23000

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OUTCOME 1

Vectors

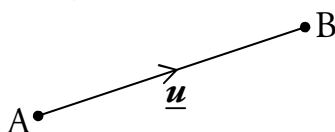
1 Vectors and Scalars

A **scalar** is a quantity with magnitude (size) only – for example, an amount of money or a length of time.

Sometimes size alone is not enough to describe a quantity – for example, directions to the nearest shop. For this we need to know a magnitude (i.e. how far), *and* a direction.

Quantities with both magnitude and direction are called **vectors**.

A vector is named either by using the points at the end of a directed line segment (e.g. \overrightarrow{AB} represents the vector starting at point A and ending at point B) or by using a bold letter (e.g. \mathbf{u}). You will see bold letters used in printed text, but in handwriting you should just underline the letter (e.g. \underline{u}).



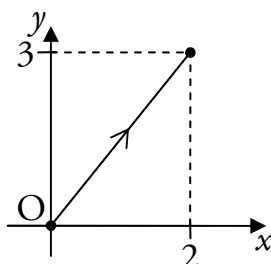
Throughout these notes, we will show vectors bold and underlined (e.g. \underline{u}).

2 Components

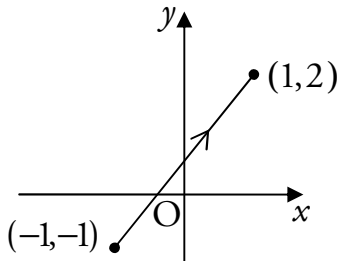
A vector may be represented by its **components**, which we write in a column. For example,

$\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ is a vector in two dimensions.

In this case the first component is 2 and this tells us to move 2 units in the x -direction. The second component tells us to move 3 units in the y -direction. So if the vector starts at the origin, it will look like:



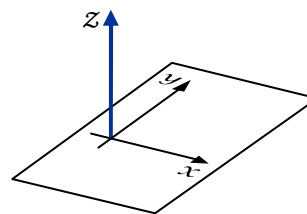
Note that we write the components in a column to avoid confusing them with coordinates. The following diagram also shows the vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$, but in this case it does not start at the origin.



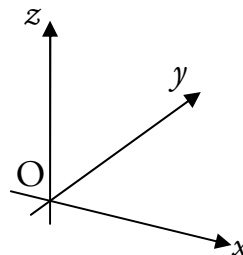
Vectors in Three Dimensions

In a vector with three components, the first two tell us how many units to move in the x - and y -directions, as before. The third component specifies how far to move in the z -direction.

When looking at a pair of (x, y) -axes, the z -axis points out of the page from the origin.



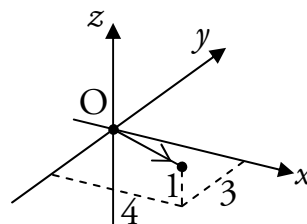
A set of 3D axes can be drawn on a page as shown to the right.



For example,

$$\begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$$

is a vector in three dimensions. This vector is shown in the diagram, starting from the origin.



Zero Vectors

Any vector with all its components zero is called a zero vector and can be

written as $\underline{\mathbf{0}}$, e.g. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \underline{\mathbf{0}}$.

3 Magnitude

The **magnitude** (or length) of a vector \underline{u} is written as $|\underline{u}|$. It can be calculated as follows.

$$\text{If } \overrightarrow{PQ} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ then } |\overrightarrow{PQ}| = \sqrt{a^2 + b^2}.$$

$$\text{If } \overrightarrow{PQ} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ then } |\overrightarrow{PQ}| = \sqrt{a^2 + b^2 + c^2}.$$

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 5 \\ -12 \end{pmatrix}$, find $|\underline{u}|$.

2. Find the length of $\underline{a} = \begin{pmatrix} -\sqrt{5} \\ 6 \\ 3 \end{pmatrix}$.

Unit Vectors

Any vector with a magnitude of one is called a **unit vector**. For example:

$$\begin{aligned} \text{if } \underline{u} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{\sqrt{3}}{2} \end{pmatrix} \text{ then } |\underline{u}| &= \sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \\ &= \sqrt{\frac{4}{4}} \\ &= 1 \text{ unit.} \end{aligned}$$

So \underline{u} is a unit vector.

Distance in Three Dimensions

The distance between the points A and B is $d_{AB} = |\overline{AB}|$ units.

So given $\overline{AB} = \begin{pmatrix} -1 \\ 2 \\ 5 \end{pmatrix}$, we find $d_{AB} = \sqrt{(-1)^2 + 2^2 + 5^2} = \sqrt{30}$.

In fact, there is a three dimensional version of the distance formula.

The distance d between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \text{ units.}$$

EXAMPLE

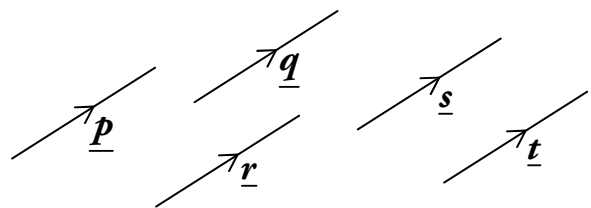
Find the distance between the points $(-1, 4, 1)$ and $(0, 5, -7)$.

4 Equal Vectors

Vectors with the same magnitude and direction are said to be equal.

For example, all the vectors shown to the right are equal.

If vectors are equal to each other, then all of their components are equal, i.e.

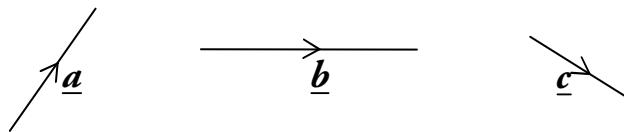


$$\text{if } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} d \\ e \\ f \end{pmatrix} \text{ then } a = d, b = e \text{ and } c = f.$$

Conversely, two vectors are only equal if all of their components are equal.

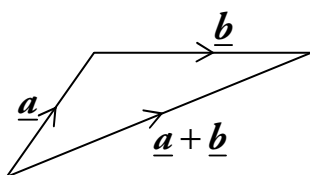
5 Addition and Subtraction of Vectors

Consider the following vectors:



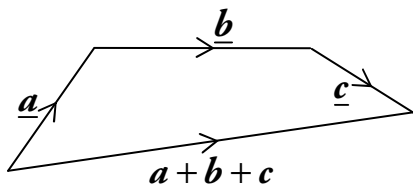
Addition

We can construct $\underline{a} + \underline{b}$ as follows:



$\underline{a} + \underline{b}$ means \underline{a} followed by \underline{b} .

Similarly, we can construct $\underline{a} + \underline{b} + \underline{c}$ as follows:

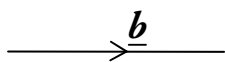


$\underline{a} + \underline{b} + \underline{c}$ means \underline{a} followed by \underline{b} followed by \underline{c} .

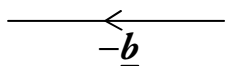
To add vectors, we position them nose-to-tail. Then the sum of the vectors is the vector between the first tail and the last nose.

Subtraction

Now consider $\underline{a} - \underline{b}$. This can be written as $\underline{a} + (-\underline{b})$, so if we first find $-\underline{b}$ we can use vector addition to obtain $\underline{a} - \underline{b}$.

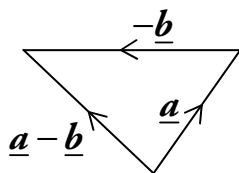


$-\underline{b}$ is just \underline{b} but in the opposite direction.



$-\underline{b}$ and \underline{b} have the same magnitude, i.e. $|\underline{b}| = |-\underline{b}|$.

Therefore we can construct $\underline{a} - \underline{b}$ as follows:



$\underline{a} - \underline{b}$ means \underline{a} followed by $-\underline{b}$.

Using Components

If we have the components of vectors, then things become much simpler.

The following rules can be used for addition and subtraction.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a+d \\ b+e \\ c+f \end{pmatrix}$$

add the components

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} - \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a-d \\ b-e \\ c-f \end{pmatrix}$$

subtract the components

EXAMPLES

1. Given $\underline{u} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, calculate $\underline{u} + \underline{v}$ and $\underline{u} - \underline{v}$.

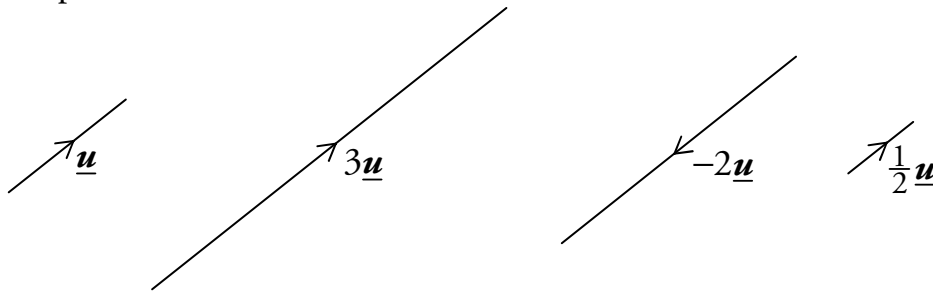
2. Given $\underline{p} = \begin{pmatrix} 4 \\ \frac{3}{2} \\ 3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} -1 \\ 3 \\ -\frac{6}{5} \end{pmatrix}$, calculate $\underline{p} - \underline{q}$ and $\underline{q} + \underline{p}$.

6 Multiplication by a Scalar

A vector \underline{u} which is multiplied by a scalar $k > 0$ will give the result $k\underline{u}$. This vector will be k times as long, i.e. its magnitude will be $k|\underline{u}|$.

Note that if $k < 0$ this means that the vector $k\underline{u}$ will be in the opposite direction to \underline{u} .

For example:



$$\text{If } \underline{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ then } k\underline{u} = \begin{pmatrix} ka \\ kb \\ kc \end{pmatrix}.$$

Each component is multiplied by the scalar.

EXAMPLES

1. Given $\underline{v} = \begin{pmatrix} 1 \\ 5 \\ -3 \end{pmatrix}$, find $3\underline{v}$.

2. Given $\underline{r} = \begin{pmatrix} -6 \\ 3 \\ 1 \end{pmatrix}$, find $-4\underline{r}$.

Negative Vectors

The negative of a vector is the vector multiplied by -1 .

If we write a vector as a directed line segment \overline{AB} , then $-\overline{AB} = \overline{BA}$:

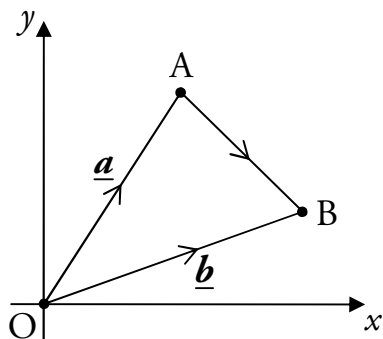
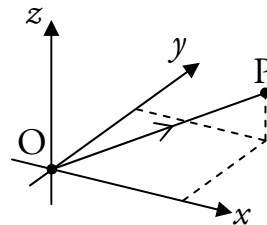


7 Position Vectors

\overline{OA} is called the **position vector** of point A relative to the origin O, and is written as \underline{a} .

\overline{OB} is called the position vector of point B, written \underline{b} .

Given $P(x, y, z)$, the position vector \overline{OP} or \underline{p} has components $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.



To move from point A to point B we can move back along the vector \underline{a} to the origin, and along vector \underline{b} to point B, i.e.

$$\begin{aligned} \overline{AB} &= \overline{AO} + \overline{OB} \\ &= -\overline{OA} + \overline{OB} \\ &= -\underline{a} + \underline{b} \\ &= \underline{b} - \underline{a}. \end{aligned}$$

For the vector joining any two points P and Q, $\overline{PQ} = \underline{q} - \underline{p}$.

EXAMPLE

R is the point $(2, -2, 3)$ and S is the point $(4, 6, -1)$. Find \overrightarrow{RS} .

Note

You don't need to write this line down in the exam.

8 Basis Vectors

A vector may also be defined in terms of the basis vectors \underline{i} , \underline{j} and \underline{k} .

These are three mutually perpendicular unit vectors (i.e. they are perpendicular to each other).



These basis vectors can be written in component form as

$$\underline{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \underline{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \underline{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Any vector can be written in **basis form** using \underline{i} , \underline{j} and \underline{k} . For example:

$$\underline{v} = \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 3 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 6 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 2\underline{i} - 3\underline{j} + 6\underline{k}.$$

There is no need for the working above if the following is used:

$$a\underline{i} + b\underline{j} + c\underline{k} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

9 Collinearity

In Straight Lines (Unit 1 Outcome 1), we learned that points are collinear if they lie on the same straight line.

The points A, B and C in 3D space are collinear if \overline{AB} is parallel to \overline{BC} , with B a common point.

Note that we *cannot* find gradients in three dimensions – instead we use the following.

Non-zero vectors are **parallel** if they are scalar multiples of the same vector.

For example:

$$\underline{u} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}, \quad \underline{v} = \begin{pmatrix} 6 \\ 3 \\ 12 \end{pmatrix} = 3 \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 3\underline{u}.$$

So \underline{u} and \underline{v} are parallel.

$$\underline{p} = \begin{pmatrix} 15 \\ 9 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}, \quad \underline{q} = \begin{pmatrix} 20 \\ 12 \\ -8 \end{pmatrix} = 4 \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}.$$

So \underline{p} and \underline{q} are parallel.

EXAMPLE

A is the point $(1, -2, 5)$, B $(8, -5, 9)$ and C $(22, -11, 17)$.

Show that A, B and C are collinear.

10 Dividing Lines in a Ratio

There is a simple process for finding the coordinates of a point which divides a line segment in a given ratio.

EXAMPLE

1. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

The point T divides PR in the ratio $2 : 3$. Find the coordinates of T.

Step 1

Make a sketch of the line, showing the ratio in which the point divides the line segment.

Step 2

Using the sketch, equate the ratio of the two line segments with the given ratio.

Step 3

Cross multiply, then change directed line segments to position vectors.

Step 4

Rearrange to give the position vector of the unknown point.

Step 5

From the position vector, state the coordinates of the unknown point.

Using the Section Formula

The previous method can be condensed into a formula as shown below.

If the point P divides the line AB in the ratio $m : n$, then

$$\underline{p} = \frac{n\underline{a} + m\underline{b}}{n + m},$$

where \underline{a} , \underline{b} and \underline{p} are the position vectors of A, B and P respectively.

This is referred to as the **section formula**.

It is not necessary to know this, since the approach explained above will always work.

EXAMPLE

2. P is the point $(-2, 4, -1)$ and R is the point $(8, -1, 19)$.

The point T divides PR in the ratio $2 : 3$. Find the coordinates of T.

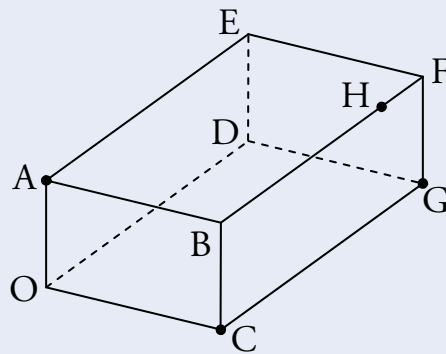
Note

If you are confident with arithmetic, this step can be done mentally.

Further Examples

EXAMPLES

3. The cuboid OABCDEFG is shown in the diagram below.



The point A has coordinates $(0,0,5)$, $C(8,0,0)$ and $G(8,12,0)$. The point H divides BF in the ratio $4:1$. Find the coordinates of H.

Note

$$\frac{BH}{BF} = \frac{4}{5}, \text{ so } \vec{BH} = \frac{4}{5}\vec{BF}.$$

4. The points $P(6,1,-3)$, $Q(8,-3,1)$ and $R(9,-5,3)$ are collinear. Find the ratio in which Q divides PR.

Note

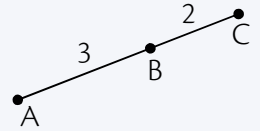
The ratio is $2:1$ since

$$\frac{PQ}{QR} = \frac{2}{1}.$$

5. The points $A(7, -4, -4)$, $B(13, 5, -7)$ and C are collinear. Given that B divides AC in the ratio $3 : 2$, find the coordinates of C .

Note

A sketch may help you to see this:



11 The Scalar Product

So far we have added and subtracted vectors and multiplied a vector by a scalar. Now we will consider the scalar product, which is a form of vector multiplication.

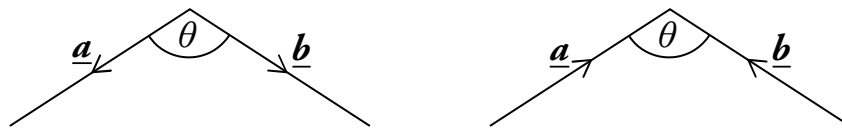
The **scalar product** is denoted by $\underline{a} \cdot \underline{b}$ (sometimes it is called the dot product) and can be calculated using the formula:

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta,$$

where θ is the angle between the two vectors \underline{a} and \underline{b} .

This is given in the exam.

The definition above assumes that the vectors \underline{a} and \underline{b} are positioned so that they both point away from the angle, or both point into the angle.



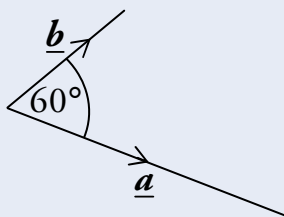
However, if one vector is pointing away from the angle, while the other points into the angle,



we find that $\underline{a} \cdot \underline{b} = -|\underline{a}||\underline{b}|\cos\theta$.

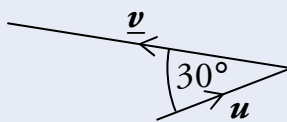
EXAMPLES

- Two vectors, \underline{a} and \underline{b} have magnitudes 7 and 3 units respectively and are at an angle of 60° to each other as shown below.



What is the value of $\underline{a} \cdot \underline{b}$?

- The vector \underline{u} has magnitude k and \underline{v} is twice as long as \underline{u} . The angle between \underline{u} and \underline{v} is 30° , as shown below.



Find an expression for $\underline{u} \cdot \underline{v}$ in terms of k .

Remember

When one vector points in and one points out,
 $\underline{u} \cdot \underline{v} = -|\underline{u}||\underline{v}|\cos\theta$.

The Component Form of the Scalar Product

The scalar product can also be calculated as follows:

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad \text{where } \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \text{ and } \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

This is given in the exam.

EXAMPLES

3. Find $\underline{p} \cdot \underline{q}$, given that $\underline{p} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ and $\underline{q} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$.

4. If A is the point (2, 3, 9), B(1, 4, -2) and C(-1, 3, -6), calculate $\overline{AB} \cdot \overline{AC}$.

12 The Angle Between Vectors

The formulae for the scalar product can be rearranged to give the following equations, both of which can be used to calculate θ , the angle between two vectors.

$$\cos \theta = \frac{\underline{a} \cdot \underline{b}}{|\underline{a}| |\underline{b}|} \quad \text{or} \quad \cos \theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\underline{a}| |\underline{b}|}.$$

Look back to the formulae for finding the scalar product, given on the previous pages. Notice that the first equation is simply a rearranged form of the one which can be used to find the scalar product. Also notice that the second simply substitutes $\underline{a} \cdot \underline{b}$ for the component form of the scalar product.

These formulae are *not* given in the exam but can both be easily derived from the formulae on the previous pages (which *are* given in the exam).

EXAMPLES

1. Calculate the angle θ between vectors $\underline{p} = 3\underline{i} + 4\underline{j} - 2\underline{k}$ and

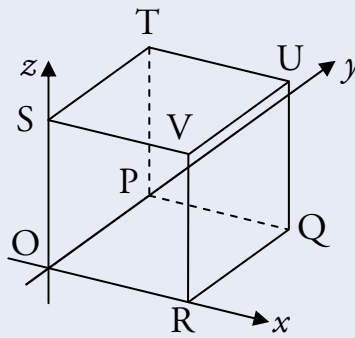


$$\underline{q} = 4\underline{i} + \underline{j} + 3\underline{k}.$$



2. K is the point $(1, -7, 2)$, $L(-3, 3, 4)$ and $M(2, 5, 1)$. Find \widehat{KLM} .

3. The diagram below shows the cube OPQRSTUV.



The point R has coordinates $(4,0,0)$.

- Write down the coordinates of T and U.
- Find the components of \overrightarrow{RT} and \overrightarrow{RU} .
- Calculate the size of angle TRU.



13 Perpendicular Vectors

If \underline{a} and \underline{b} are perpendicular then $\underline{a} \cdot \underline{b} = 0$.

$$\begin{aligned}\text{This is because } \underline{a} \cdot \underline{b} &= |\underline{a}| |\underline{b}| \cos \theta \\ &= |\underline{a}| |\underline{b}| \cos 90^\circ \quad (\theta = 90^\circ \text{ since perpendicular}) \\ &= 0 \quad (\text{since } \cos 90^\circ = 0).\end{aligned}$$

Conversely, if $\underline{a} \cdot \underline{b} = 0$ then \underline{a} and \underline{b} are perpendicular.

EXAMPLES

- Two vectors are defined as $\underline{a} = 4\underline{i} + 2\underline{j} - 5\underline{k}$ and $\underline{b} = 2\underline{i} + \underline{j} + 2\underline{k}$.
Show that \underline{a} and \underline{b} are perpendicular.

- $\overline{PQ} = \begin{pmatrix} 4 \\ a \\ 7 \end{pmatrix}$ and $\overline{RS} = \begin{pmatrix} 2 \\ -3 \\ a \end{pmatrix}$ where a is a constant.

Given that \overline{PQ} and \overline{RS} are perpendicular, find the value of a .

14 Properties of the Scalar Product

Some useful properties of the scalar product are as follows:

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad (\text{Expanding brackets})$$

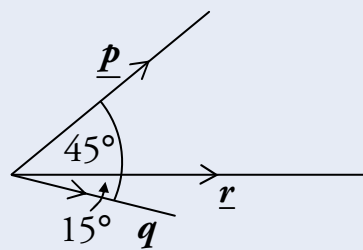
$$\underline{a} \cdot \underline{a} = |\underline{a}|^2.$$

Note that these are not given in the exam, so you need to remember them.

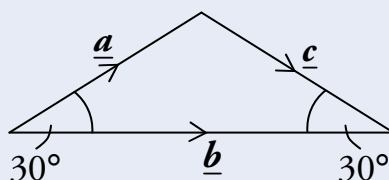
EXAMPLES

1. In the diagram, $|\underline{p}| = 3$, $|\underline{r}| = 4$ and $|\underline{q}| = 2$.

Calculate $\underline{p} \cdot (\underline{q} + \underline{r})$.



2. In the diagram below, $|\underline{a}| = |\underline{c}| = 2$ and $|\underline{b}| = 2\sqrt{3}$.



Calculate $\underline{a} \cdot (\underline{a} + \underline{b} + \underline{c})$.

Remember

$\underline{a} \cdot \underline{c} = -|\underline{a}||\underline{c}|\cos\theta_2$
since \underline{a} points into θ_2
and \underline{c} points away.

OUTCOME 2

Further Calculus

1 Differentiating $\sin x$ and $\cos x$

In order to differentiate expressions involving trigonometric functions, we use the following rules:

$$\frac{d}{dx}(\sin x) = \cos x, \quad \frac{d}{dx}(\cos x) = -\sin x.$$

These rules only work when x is an angle measured in radians. A form of these rules is given in the exam.

EXAMPLES

1. Differentiate $y = 3 \sin x$ with respect to x .

2. A function f is defined by $f(x) = \sin x - 2 \cos x$ for $x \in \mathbb{R}$.

Find $f'\left(\frac{\pi}{3}\right)$.

3. Find the equation of the tangent to the curve $y = \sin x$ when $x = \frac{\pi}{6}$.

2 Integrating $\sin x$ and $\cos x$

We know the derivatives of $\sin x$ and $\cos x$, so it follows that the integrals are:

$$\int \cos x \, dx = \sin x + c, \quad \int \sin x \, dx = -\cos x + c.$$

Again, these results only hold if x is measured in radians.

EXAMPLES

1. Find $\int (5 \sin x + 2 \cos x) \, dx$.

2. Find $\int_0^{\frac{\pi}{4}} (4 \cos x + 2 \sin x) \, dx$.

Note

It is good practice to rationalise the denominator.



3. Find the value of $\int_0^4 \frac{1}{2} \sin x \, dx$.

Remember

We must use radians when integrating or differentiating trig. functions.

3 The Chain Rule

We will now look at how to differentiate composite functions, such as $f(g(x))$. If the functions f and g are defined on suitable domains, then

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \times g'(x).$$

Stated simply: differentiate the outer function, the bracket stays the same, then multiply by the derivative of the bracket.

This is called the **chain rule**. You will need to remember it for the exam.

EXAMPLE

If $y = \cos\left(5x + \frac{\pi}{6}\right)$, find $\frac{dy}{dx}$.

Note

The “ $\times 5$ ” comes from $\frac{d}{dx}\left(5x + \frac{\pi}{6}\right)$.

4 Special Cases of the Chain Rule

We will now look at how the chain rule can be applied to particular types of expression.

Powers of a Function

For expressions of the form $[f(x)]^n$, where n is a constant, we can use a simpler version of the chain rule:

$$\frac{d}{dx}[(f(x))^n] = n[f(x)]^{n-1} \times f'(x).$$

Stated simply: the power (n) multiplies to the front, the bracket stays the same, the power lowers by one (giving $n-1$) and everything is multiplied by the derivative of the bracket ($f'(x)$).

EXAMPLES

1. A function f is defined on a suitable domain by $f(x) = \sqrt{2x^2 + 3x}$.
Find $f'(x)$.

2. Differentiate $y = 2\sin^4 x$ with respect to x .

Powers of a Linear Function

The rule for differentiating an expression of the form $(ax + b)^n$, where a , b and n are constants, is as follows:

$$\frac{d}{dx}[(ax + b)^n] = an(ax + b)^{n-1}.$$

EXAMPLES

3. Differentiate $y = (5x + 2)^3$ with respect to x .

4. If $y = \frac{1}{(2x+6)^3}$, find $\frac{dy}{dx}$.

5. A function f is defined by $f(x) = \sqrt[3]{(3x-2)^4}$ for $x \in \mathbb{R}$. Find $f'(x)$.

Trigonometric Functions

The following rules can be used to differentiate trigonometric functions.

$$\frac{d}{dx}[\sin(ax+b)] = a \cos(ax+b) \quad \frac{d}{dx}[\cos(ax+b)] = -a \sin(ax+b)$$

These are given in the exam.

EXAMPLE

6. Differentiate $y = \sin(9x + \pi)$ with respect to x .

5 A Special Integral

The method for integrating an expression of the form $(ax + b)^n$ is:

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c \quad \text{where } a \neq 0 \text{ and } n \neq -1.$$

Stated simply: raise the power (n) by one, divide by the new power and also divide by the derivative of the bracket ($a(n+1)$), add c .

EXAMPLES

1. Find $\int (x + 4)^7 dx$.

2. Find $\int (2x + 3)^2 dx$.

3. Find $\int \frac{1}{\sqrt[3]{5x+9}} dx$ where $x \neq -\frac{9}{5}$.



4. Evaluate $\int_0^3 \sqrt{3x+4} \, dx$ where $x \geq -\frac{4}{3}$.

Note

Changing powers back into roots here makes it easier to evaluate the two brackets.

Remember

To evaluate $\sqrt{4^3}$, it is easier to work out $\sqrt{4}$ first.

Warning

Make sure you don't confuse differentiation and integration – this could lose you a lot of marks in the exam.

Remember the following rules for differentiating and integrating expressions of the form $(ax + b)^n$:

$$\frac{d}{dx}[(ax + b)^n] = an(ax + b)^{n-1},$$

$$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c.$$

These rules will *not* be given in the exam.

Using Differentiation to Integrate

Recall that integration is just the process of undoing differentiation. So if we differentiate $f(x)$ to get $g(x)$ then we know that $\int g(x) dx = f(x) + c$.

EXAMPLES

5. (a) Differentiate $y = \frac{5}{(3x-1)^4}$ with respect to x .

(b) Hence, or otherwise, find $\int \frac{1}{(3x-1)^5} dx$.

Note

We could also have used the special integral to obtain this answer.

6. (a) Differentiate $y = \frac{1}{(x^3-1)^5}$ with respect to x .

(b) Hence, find $\int \frac{x^2}{(x^3-1)^6} dx$.

Note

In this case, the special integral cannot be used.

6 Integrating $\sin(ax + b)$ and $\cos(ax + b)$

Since we know the derivatives of $\sin(ax + b)$ and $\cos(ax + b)$, it follows that their integrals are:

$$\int \cos(ax + b) \, dx = \frac{1}{a} \sin(ax + b) + c,$$


$$\int \sin(ax + b) \, dx = -\frac{1}{a} \cos(ax + b) + c.$$

These are given in the exam.

EXAMPLES

1. Find $\int \sin(4x + 1) \, dx$.

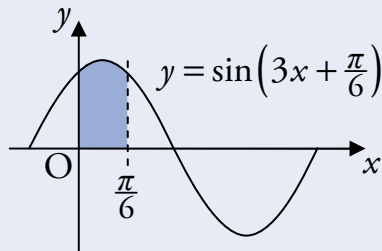
2. Find $\int \cos\left(\frac{3}{2}x + \frac{\pi}{5}\right) \, dx$.

 3. Find the value of $\int_0^1 \cos(2x - 5) \, dx$.

Remember

We must use radians when integrating or differentiating trig. functions.

4. Find the area enclosed by the graph of $y = \sin\left(3x + \frac{\pi}{6}\right)$, the x -axis and the lines $x = 0$ and $x = \frac{\pi}{6}$.



5. Find $\int 2 \cos\left(\frac{1}{2}x - 3\right) dx$.

6. Find $\int \left(5 \cos(2x) + \sin(x - \sqrt{3})\right) dx$.

7. (a) Differentiate $\frac{1}{\cos x}$ with respect to x .

(b) Hence find $\int \frac{\tan x}{\cos x} dx$.

OUTCOME 3

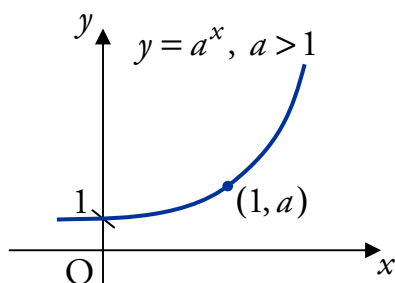
Exponentials and Logarithms

1 Exponentials

We have already met exponential functions in Unit 1 Outcome 2.

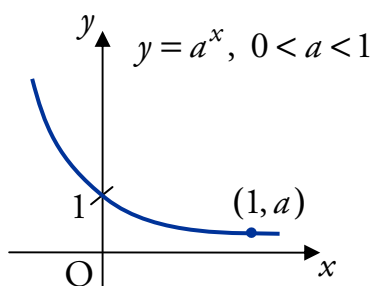
A function of the form $f(x) = a^x$ where $a, x \in \mathbb{R}$ and $a > 0$ is known as an **exponential function** to the base a .

If $a > 1$ then the graph looks like this:



This is sometimes called a **growth** function.

If $0 < a < 1$ then the graph looks like this:



This is sometimes called a **decay** function.

Remember that the graph of an exponential function $f(x) = a^x$ always passes through (0, 1) and (1, a) since

$$f(0) = a^0 = 1, \quad f(1) = a^1 = a.$$

EXAMPLES



1. The otter population on an island increases by 16% per year. How many full years will it take for the population to double?

On a calculator:

1	.	1	6	=	
1	.	1	6	ANS	=
				=	
				⋮	



2. The efficiency of a machine decreases by 5% each year. When the efficiency drops below 75%, the machine needs to be serviced. After how many years will the machine need serviced?

2 Logarithms

Having previously defined what a logarithm is (see Unit 1 Outcome 2) we want to look in more detail at the properties of these important functions.

The relationship between logarithms and exponentials is expressed as:

$$y = \log_a x \Leftrightarrow x = a^y \quad \text{where } a, x > 0.$$

Here, y is the power of a which gives x .

EXAMPLES

1. Write $5^3 = 125$ in logarithmic form.

2. Evaluate $\log_4 16$.

3 Laws of Logarithms

There are three laws of logarithms which you must know.

Rule 1

$$\log_a x + \log_a y = \log_a (xy) \quad \text{where } a, x, y > 0.$$

If two logarithmic terms with the same base number (a above) are being added together, then the terms can be combined by multiplying the arguments (x and y above).

EXAMPLE

1. Simplify $\log_5 2 + \log_5 4$.

Rule 2

$$\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right) \quad \text{where } a, x, y > 0.$$

If a logarithmic term is being subtracted from another logarithmic term with the same base number (a above), then the terms can be combined by dividing the arguments (x and y in this case).

Note that the argument which is being taken away (y above) appears on the bottom of the fraction when the two terms are combined.

EXAMPLE

2. Evaluate $\log_4 6 - \log_4 3$.

Rule 3

$$\log_a x^n = n \log_a x \quad \text{where } a, x > 0.$$

The power of the argument (n above) can come to the front of the term as a multiplier, and vice-versa.

EXAMPLE

3. Express $2 \log_7 3$ in the form $\log_7 a$.

Squash, Split and Fly

You may find the following names are a simpler way to remember the laws of logarithms.

- $\log_a x + \log_a y = \log_a (xy)$ – the arguments are **squashed** together by multiplying.
- $\log_a x - \log_a y = \log_a \left(\frac{x}{y} \right)$ – the arguments are **split** into a fraction.
- $\log_a x^n = n \log_a x$ – the power of an argument can **fly** to the front of the log term and vice-versa.

Note

When working with logarithms, you should remember:

$$\log_a 1 = 0 \quad \text{since } a^0 = 1, \quad \log_a a = 1 \quad \text{since } a^1 = a.$$

EXAMPLE

4. Evaluate $\log_7 7 + \log_3 3$.

Combining several log terms

When adding and subtracting several log terms in the form $\log_a b$, there is a simple way to combine all the terms in one step.

$$\log_a \left(\frac{\text{arguments of **positive** log terms}}{\text{arguments of **negative** log terms}} \right)$$

- Multiply the arguments of the positive log terms in the numerator.
- Multiply the arguments of the negative log terms in the denominator.

EXAMPLES

5. Evaluate $\log_{12} 10 + \log_{12} 6 - \log_{12} 5$

$$\log_{12} \left(\frac{+ \log_{12} 10 \quad + \log_{12} 6}{- \log_{12} 5} \right)$$

6. Evaluate $\log_6 4 + 2 \log_6 3$.

$$\begin{aligned} \text{OR} \quad & \log_6 4 + 2 \log_6 3 \\ & = \log_6 2^2 + 2 \log_6 3 \\ & = 2 \log_6 2 + 2 \log_6 3 \\ & = 2(\log_6 2 + \log_6 3) \\ & = 2(\log_6 (2 \times 3)) \\ & = 2 \log_6 6 \\ & = 2 \quad (\text{since } \log_6 6 = 1). \end{aligned}$$

4 Exponentials and Logarithms to the Base e

The constant e is an important number in Mathematics, and occurs frequently in models of real-life situations. Its value is roughly 2.718281828 (to 9 d.p.), and is defined as:

$$\left(1 + \frac{1}{n}\right)^n \text{ as } n \rightarrow \infty.$$

If you try large values of n on your calculator, you will get close to the value of e . Like π , e is an irrational number.

Throughout this section, we will use e in expressions of the form:

- e^x , which is called an exponential to the base e ,
- $\log_e x$, which is called a logarithm to the base e . This is also known as the **natural logarithm** of x , and is often written as $\ln x$ (i.e. $\ln x \equiv \log_e x$).

EXAMPLES



1. Calculate the value of $\log_e 8$.

On a calculator:



2. Solve $\log_e x = 9$.

On a calculator:

3. Simplify $4 \log_e (2e) - 3 \log_e (3e)$ expressing your answer in the form $a + \log_e b - \log_e c$ where a , b and c are whole numbers.

$$\text{OR } 4 \log_e (2e) - 3 \log_e (3e)$$

$$= \log_e (2e)^4 - \log_e (3e)^3$$

$$= \log_e \left(\frac{(2e)^4}{(3e)^3} \right)$$

$$= \log_e \left(\frac{16e^4}{27e^3} \right)$$

$$= \log_e \left(\frac{16e}{27} \right)$$

$$= \log_e e + \log_e 16 - \log_e 27$$

$$= 1 + \log_e 16 - \log_e 27.$$

Remember

$$(ab)^n = a^n b^n.$$

5 Exponential and Logarithmic Equations

Many mathematical models of real-life situations use exponentials and logarithms. It is important to become familiar with using the laws of logarithms to help solve equations.

EXAMPLES

1. Solve $\log_a 13 + \log_a x = \log_a 273$ for $x > 0$.

2. Solve $\log_{11}(4x + 3) - \log_{11}(2x - 3) = 1$ for $x > \frac{3}{2}$.

3. Solve $\log_a(2p + 1) + \log_a(3p - 10) = \log_a(11p)$ for $p > 4$.

Dealing with Constants

Sometimes it may be necessary to write constants as logs, in order to solve equations.

EXAMPLE

4. Solve $\log_2 7 = \log_2 x + 3$ for $x > 0$.

Solving Equations with Unknown Exponents

If an unknown value (e.g. x) is the power of a term (e.g. e^x or 10^x), and its value is to be calculated, then we must take logs on both sides of the equation to allow it to be solved.

The same solution will be reached using any base, but calculators can be used for evaluating logs either in base e or base 10.

EXAMPLES



5. Solve $e^x = 7$.

OR Taking \log_{10} of both sides

$$\log_{10} e^x = \log_{10} 7$$

$$x \log_{10} e = \log_{10} 7$$

$$x = \frac{\log_{10} 7}{\log_{10} e}$$

$$x = 1.946 \text{ (to 3 d.p.)}$$



6. Solve $5^{3x+1} = 40$.

Note

\log_{10} could have been used instead of \log_e .

Exponential Growth and Decay

Recall from Section 1 that exponential functions are sometimes known as growth or decay functions. These often occur in models of real-life situations.

For instance, radioactive decay can be modelled using an exponential function. An important measurement is the **half-life** of a radioactive substance, which is the time taken for the mass of the radioactive substance to halve.

7. The mass G grams of a radioactive sample after time t years is given by the formula $G = 100e^{-3t}$.

- (a) What was the initial mass of radioactive substance in the sample?
(b) Find the half-life of the radioactive substance.



8. The world population, in billions, t years after 1950 is given by

$$P = 2.54e^{0.0178t}.$$

- (a) What was the world population in 1950?
- (b) Find, to the nearest year, the time taken for the world population to double.

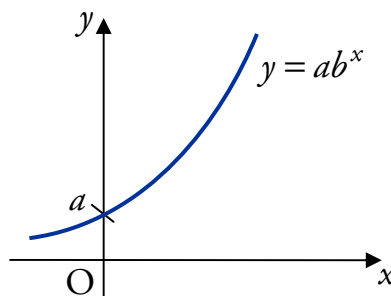


6 Graphing with Logarithmic Axes

It is common in applications to find an exponential relationship between variables; for instance, the relationship between the world population and time in the previous example. Given some data (e.g. from an experiment) we would like to find an explicit equation for the relationship.

Relationships of the form $y = ab^x$

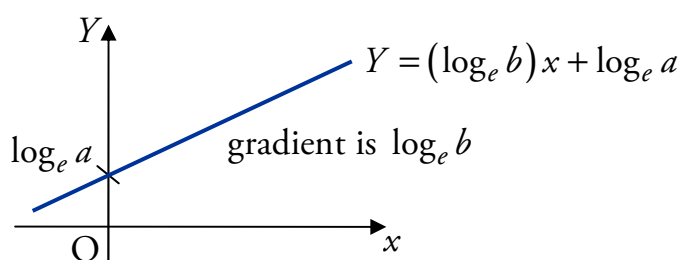
Suppose we have an exponential graph $y = ab^x$, where $a, b > 0$.



Taking logarithms we find that

$$\begin{aligned}\log_e y &= \log_e (ab^x) \\ &= \log_e a + \log_e b^x \\ &= \log_e a + x \log_e b.\end{aligned}$$

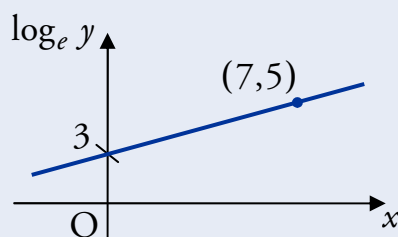
We can scale the y -axis so that $Y = \log_e y$; the Y -axis is called a **logarithmic axis**. Now our relationship is of the form $Y = (\log_e b)x + \log_e a$, which is a straight line in the (x, Y) -plane.



Since this is just a straight line, we can use known points to find the gradient $\log_e b$ and the Y -axis intercept $\log_e a$. From these we can easily find the values of a and b , and hence specify the equation $y = ab^x$.

EXAMPLES

- The relationship between two variables, x and y , is of the form $y = ab^x$. An experiment to test this relationship produced the data shown in the graph, where $\log_e y$ is plotted against x .



Find the values of a and b .

2. The results from an experiment were noted as follows:

x	1.30	2.00	2.30	2.80
$\log_e y$	2.04	2.56	2.78	3.14

The relationship between these data can be written in the form $y = ab^x$.



Find the values of a and b , and state the formula for y in terms of x .

Note

Depending on the points used, slightly different values for a and b may be obtained.

Equations in the form $y = ax^b$

Another common relationship is $y = ax^b$, where $a, x > 0$. In this case, the relationship can be represented by a straight line if we change *both* axes to logarithmic ones.

EXAMPLE

3. The results from an experiment were noted as follows:

$\log_{10} x$	1.70	2.29	2.70	2.85
$\log_{10} y$	1.33	1.67	1.92	2.01

The relationship between these data can be written in the form $y = ax^b$.

Find the values of a and b , and state the formula for y in terms of x .

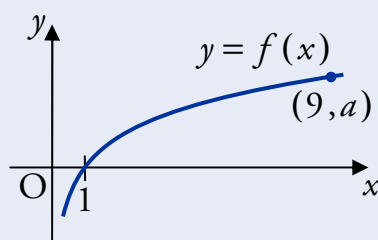


7 Graph Transformations

Graph transformations were covered in Unit 1 Outcome 2 – Functions and Graphs, but we will now look in more detail at applying transformations to graphs of exponential and logarithmic functions.

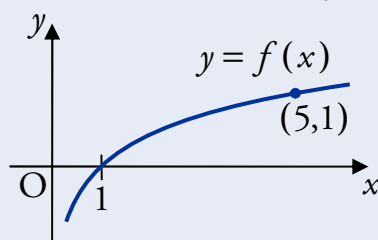
EXAMPLES

1. Shown below is the graph of $y = f(x)$ where $f(x) = \log_3 x$.



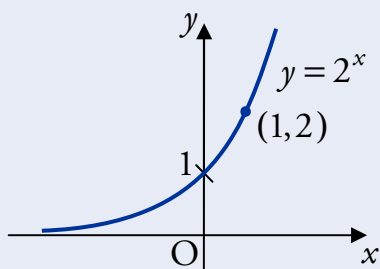
- (a) State the value of a .
(b) Sketch the graph of $y = f(x + 2) + 1$.

2. Shown below is part of the graph of $y = \log_5 x$.



Sketch the graph of $y = \log_5 \left(\frac{1}{x} \right)$.

3. The diagram shows the graph of $y = 2^x$.



On separate diagrams, sketch the graphs of:

- (a) $y = 2^{-x}$;
- (b) $y = 2^{2-x}$.

OUTCOME 4

Wave Functions

1 Expressing $p\cos x + q\sin x$ in the form $k\cos(x - a)$

An expression of the form $p\cos x + q\sin x$ can be written in the form $k\cos(x - a)$ where

$$k = \sqrt{p^2 + q^2} \quad \text{and} \quad \tan a = \frac{k \sin a}{k \cos a}.$$

The following example shows how to achieve this.

EXAMPLES

1. Write $5\cos x^\circ + 12\sin x^\circ$ in the form $k\cos(x^\circ - a^\circ)$ where $0 \leq a \leq 360$.

Step 1

Expand $k\cos(x - a)$ using the compound angle formula.

Step 2

Rearrange to compare with $p\cos x + q\sin x$.

Step 3

Compare the coefficients of $\cos x$ and $\sin x$ with $p\cos x + q\sin x$.

Step 4

Mark the quadrants on a CAST diagram, according to the signs of $k\cos a$ and $k\sin a$.

Step 5

Find k and a using the formulae above (a lies in the quadrant marked twice in **Step 4**).

Step 6

State $p\cos x + q\sin x$ in the form $k\cos(x - a)$ using these values.



2. Write $5 \cos x - 3 \sin x$ in the form $k \cos(x - a)$ where $0 \leq a \leq 2\pi$.

Note

Make sure your calculator is in radian mode.

2 Expressing $p \cos x + q \sin x$ in other forms

An expression in the form $p \cos x + q \sin x$ can also be written in any of the following forms using a similar method:

$$k \cos(x + a), \quad k \sin(x - a), \quad k \sin(x + a).$$

EXAMPLES

1. Write $4 \cos x^\circ + 3 \sin x^\circ$ in the form $k \sin(x^\circ + a^\circ)$ where $0 \leq a \leq 360$.

2. Write $\cos x - \sqrt{3} \sin x$ in the form $k \cos(x + a)$ where $0 \leq a \leq 2\pi$.

3 Multiple Angles

We can use the same method with expressions involving the same multiple angle, i.e. $p \cos(nx) + q \sin(nx)$, where n is a constant.

EXAMPLE

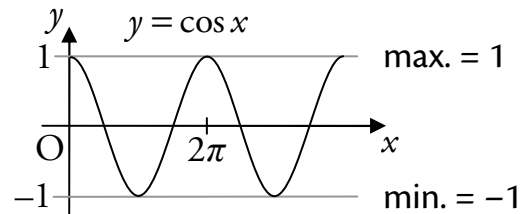
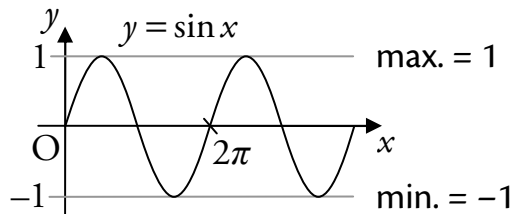


Write $5 \cos 2x^\circ + 12 \sin 2x^\circ$ in the form $k \sin(2x^\circ + a^\circ)$ where $0 \leq a \leq 360$.

4 Maximum and Minimum Values

To work out the maximum or minimum values of $p \cos x + q \sin x$, we can rewrite it as a single trigonometric function, e.g. $k \cos(x - a)$.

Recall that the maximum value of the sine and cosine functions is 1, and their minimum is -1 .



EXAMPLE

Write $4 \sin x + \cos x$ in the form $k \cos(x - a)$ where $0 \leq a < 2\pi$ and state:

- (i) the maximum value and the value of $0 \leq x < 2\pi$ at which it occurs
- (ii) the minimum value and the value of $0 \leq x < 2\pi$ at which it occurs.



5 Solving Equations

The method of writing two trigonometric terms as one can be used to help solve equations involving both a $\sin(nx)$ and a $\cos(nx)$ term.

EXAMPLES



1. Solve $5\cos x^\circ + \sin x^\circ = 2$ where $0 \leq x \leq 360$.



2. Solve $2 \cos 2x + 3 \sin 2x = 1$ where $0 \leq x \leq 2\pi$.

6 Sketching Graphs of $y = p\cos x + q\sin x$

Expressing $p\cos x + q\sin x$ in the form $k\cos(x - a)$ enables us to sketch the graph of $y = p\cos x + q\sin x$.

EXAMPLES

- (a) Write $7\cos x^\circ + 6\sin x^\circ$ in the form $k\cos(x^\circ - a^\circ)$, $0 \leq a \leq 360$.
(b) Hence sketch the graph of $y = 7\cos x^\circ + 6\sin x^\circ$ for $0 \leq x \leq 360$.



2. Sketch the graph of $y = \sin x^\circ + \sqrt{3} \cos x^\circ$ for $0 \leq x \leq 360$.

3. (a) Write $5 \sin x^\circ - \sqrt{11} \cos x^\circ$ in the form $k \sin(x^\circ - a^\circ)$, $0 \leq a \leq 360$.



(b) Hence sketch the graph of $y = 5 \sin x^\circ - \sqrt{11} \cos x^\circ + 2$, $0 \leq x \leq 360$.