QUADRATIC EQUATIONS

FUNCTION: pairs one number with another, its IMAGE.
can be defined by a formula

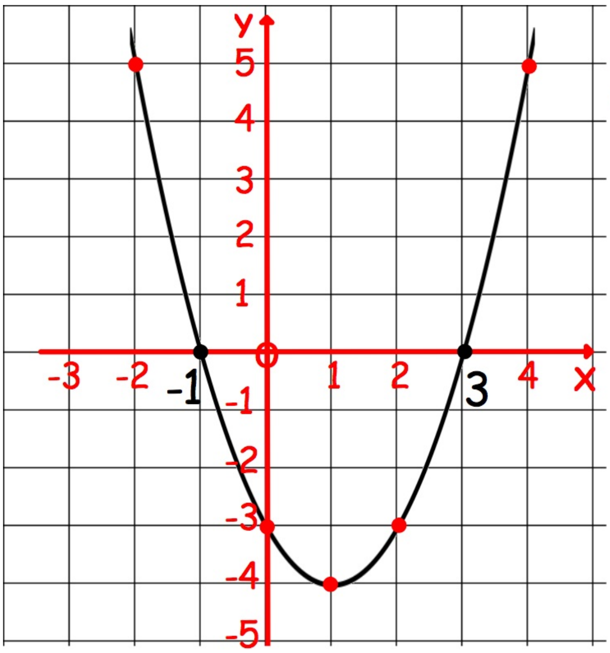
QUADRATIC FUNCTION: $f(x) = ax^2 + bx + c$

a, b and c are constants, $a \neq 0$

$f(x) = x^2 - 2x - 3$

$f(1) = 1^2 - 2 \times 1 - 3 = -4$

On the graph $y = f(x)$ this gives point $(1,-4)$



$y = x^2 - 2x - 3$

Graph is a PARABOLA,
a symmetrical curve.

The x-intercepts -1 and 3
are where $x^2 - 2x - 3 = 0$
The ZEROS are -1 and 3

Equation $x^2 - 2x - 3 = 0$ has ROOTS -1 and 3

$(-1)^2 - 2 \times (-1) - 3 = 0$

$3^2 - 2 \times 3 - 3 = 0$

SOLVING QUADRATIC EQUATIONS

The ROOTS of $ax^2 + bx + c = 0$, $a \neq 0$
are the values of x which satisfy the equation.

FACTORISATION

write in form $ax^2 + bx + c = 0$

factorise $(\quad)(\quad) = 0$

separate factors $(\quad) = 0$ or $(\quad) = 0$

and solve

(1) $12x - 8x^2 = 0$

$4x(3 - 2x) = 0$

$4x = 0$ or $3 - 2x = 0$

$2x = 3$

$\underline{\underline{x = 0 \text{ or } x = {}^3/_2}}$

(2) $x^2 - 6x + 8 = 0$

$(x - 2)(x - 4) = 0$

$x - 2 = 0$ or $x - 4 = 0$

$\underline{\underline{x = 2 \text{ or } x = 4}}$

(3) $4x^2 - 9 = 0$

$(2x + 3)(2x - 3) = 0$

$2x + 3 = 0$ or $2x - 3 = 0$

$2x = -3$ $\quad$ $2x = 3$

$\underline{\underline{x = -{}^3/_2 \text{ or } x = {}^3/_2}}$

(4) $$6x^2 + x - 2 = 0$$
$$(3x + 2)(2x - 1) = 0$$

$3x + 2 = 0$ or $2x - 1 = 0$

$3x = -2$ $\qquad$ $2x = 1$

$x = -^2/_3$ or $x = {}^1/_2$

(5) $$3x^2 - 6x - 24 = 0$$
$$x^2 - 2x - 8 = 0$$
$$(x + 2)(x - 4) = 0$$

can simplify by dividing by the common factor 3

$x + 2 = 0$ or $x - 4 = 0$

$x = -2$ or $x = 4$

equation requires rearrangement to $ax^2 + bx + c = 0$

(6) $$(t + 1)^2 = 2(t + 5)$$

$$t^2 + 2t + 1 = 2t + 10$$
$$t^2 - 9 = 0$$
$$(t + 3)(t - 3) = 0$$

$t = -3$ or $t = 3$

(7) $$\frac{2}{x} = \frac{x + 3}{5}$$

$$10 = x(x + 3)$$
$$10 = x^2 + 3x$$
$$0 = x^2 + 3x - 10$$

$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 2) = 0$$

$x = -5$ or $x = 2$

QUADRATIC FORMULA

The quadratic equation $ax^2 + bx + c = 0$

can be rearranged to $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The quadratic formula finds the ROOTS of the equation.

NOTE: If $b^2 - 4ac$ is 0, 1, 4, 9....
the equation can be solved by factorising.

Find the roots of $3x^2 - 4x - 9 = 0$

$ax^2 + bx + c = 0$

$a = 3\ ,\ b = -4\ ,\ c = -9$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+4 \pm \sqrt{124}}{6}$$

$$= \frac{4 - \sqrt{124}}{6} \quad \text{or} \quad \frac{4 + \sqrt{124}}{6}$$

$$= -1.189... \quad \text{or} \quad 2.522...$$

$\underline{\underline{x = -1.2 \quad \text{or} \quad x = 2.5}}$

$b^2 - 4ac$

$= (-4)^2 - 4 \times 3 \times (-9)$

$= 124$

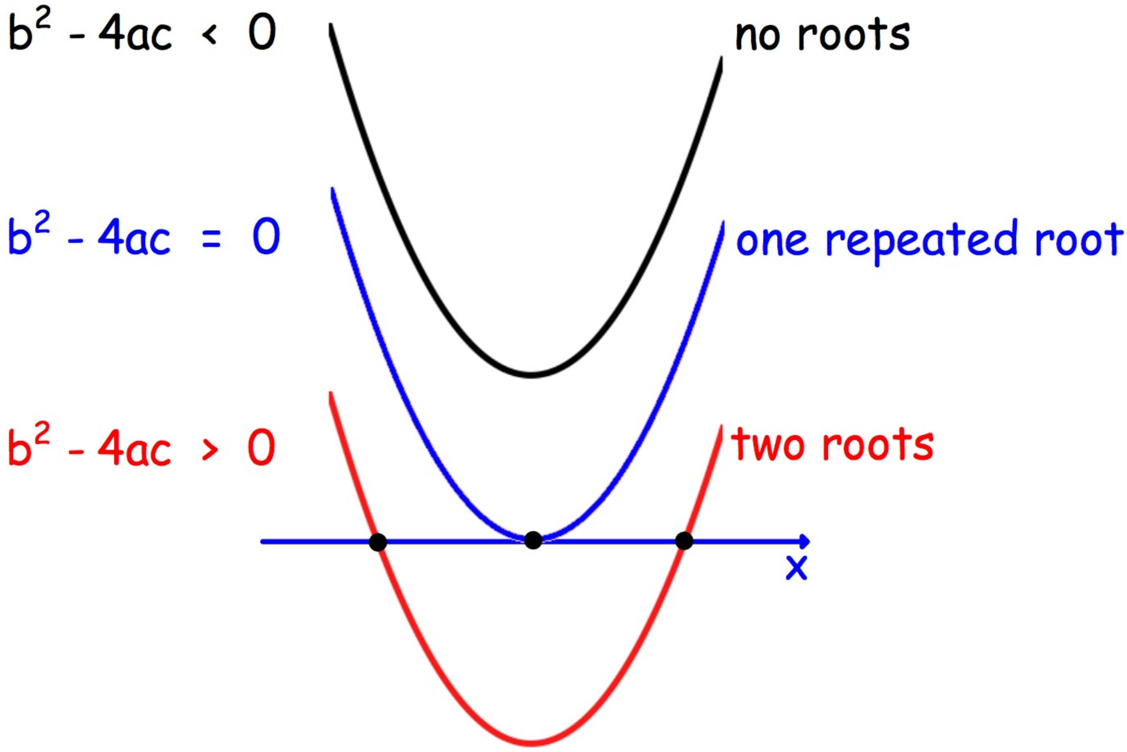
DISCRIMINANT

The ROOTS of the quadratic equation $ax^2 + bx + c = 0$

are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\pm\sqrt{\text{positive}}$	two solutions
$\pm\sqrt{\ \ 0\ \ }$	one solution
$\pm\sqrt{\text{negative}}$	no solution

The DISCRIMINANT $b^2 - 4ac$
is used to determine the NATURE of the roots.



NATURE OF THE ROOTS

$b^2 - 4ac > 0$ TWO REAL AND DISTINCT ROOTS

$b^2 - 4ac = 0$ TWO REAL AND EQUAL ROOTS

$b^2 - 4ac < 0$ NO REAL ROOTS

NOTE:

(i) condition for REAL ROOTS $b^2 - 4ac \geq 0$

(ii) if $b^2 - 4ac$ is a square number 0, 1, 4, 9.... then the roots are RATIONAL, otherwise IRRATIONAL. (SURD)

(1) Find the nature of the roots of $3x^2 - 4x - 9 = 0$

$a = 3$, $b = -4$, $c = -9$

$b^2 - 4ac = (-4)^2 - 4 \times 3 \times (-9) = 124$

$b^2 - 4ac > 0 \Rightarrow$ two real and distinct roots (and irrational)

(2) Find the nature of the roots of $2x^2 - x + 1 = 0$

$a = 2$, $b = -1$, $c = 1$

$b^2 - 4ac = (-1)^2 - 4 \times 2 \times 1 = -7$

$b^2 - 4ac < 0 \Rightarrow$ no real roots

UNKNOWNS

(1) Given $px^2 + 4x + p = 0$ has EQUAL roots, find p.

$a = p \quad , \quad b = 4 \quad , \quad c = p$

$b^2 - 4ac = 4^2 - 4 \times p \times p = 16 - 4p^2$

for equal roots $\Rightarrow$

$$\begin{aligned} b^2 - 4ac &= 0 \\ 16 - 4p^2 &= 0 \\ 4 - p^2 &= 0 \\ p^2 &= 4 \\ \underline{\underline{p = -2 \text{ or } p = 2}} \end{aligned}$$

(2) Given $6x^2 + 12x + k = 0$ has REAL roots, find k.

$a = 6 \quad , \quad b = 12 \quad , \quad c = k$

$b^2 - 4ac = 12^2 - 4 \times 6 \times k = 144 - 24k$

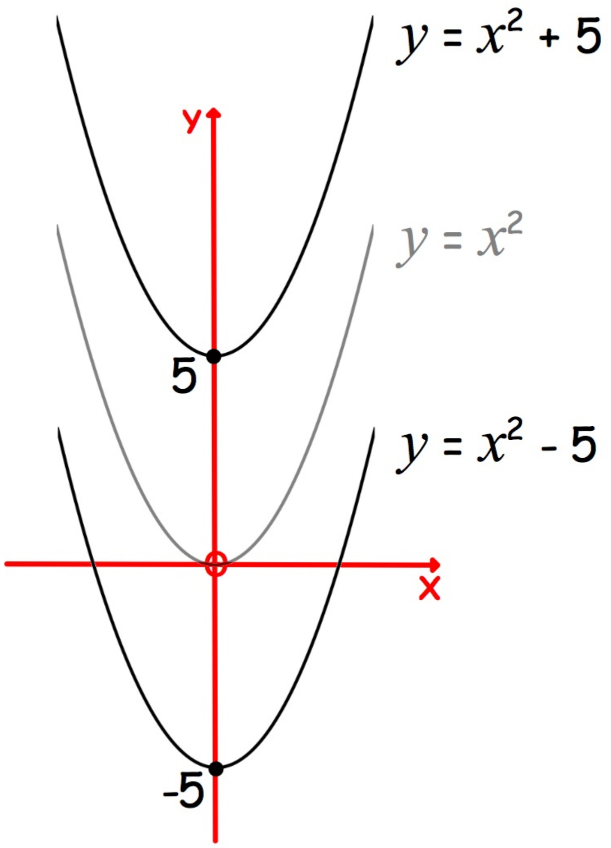
for real roots $\Rightarrow$

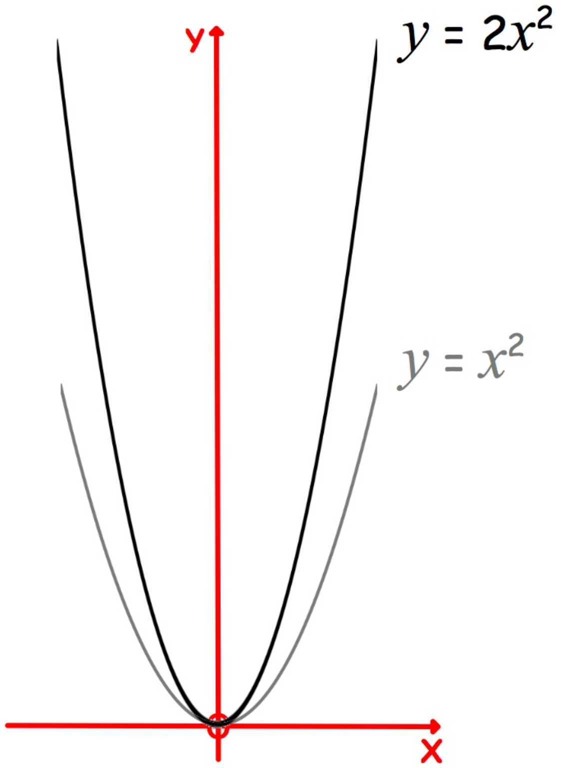
$$\begin{aligned} b^2 - 4ac &\geq 0 \\ 144 - 24k &\geq 0 \\ -24k &\geq -144 \\ \underline{\underline{k \leq 6}} \end{aligned}$$

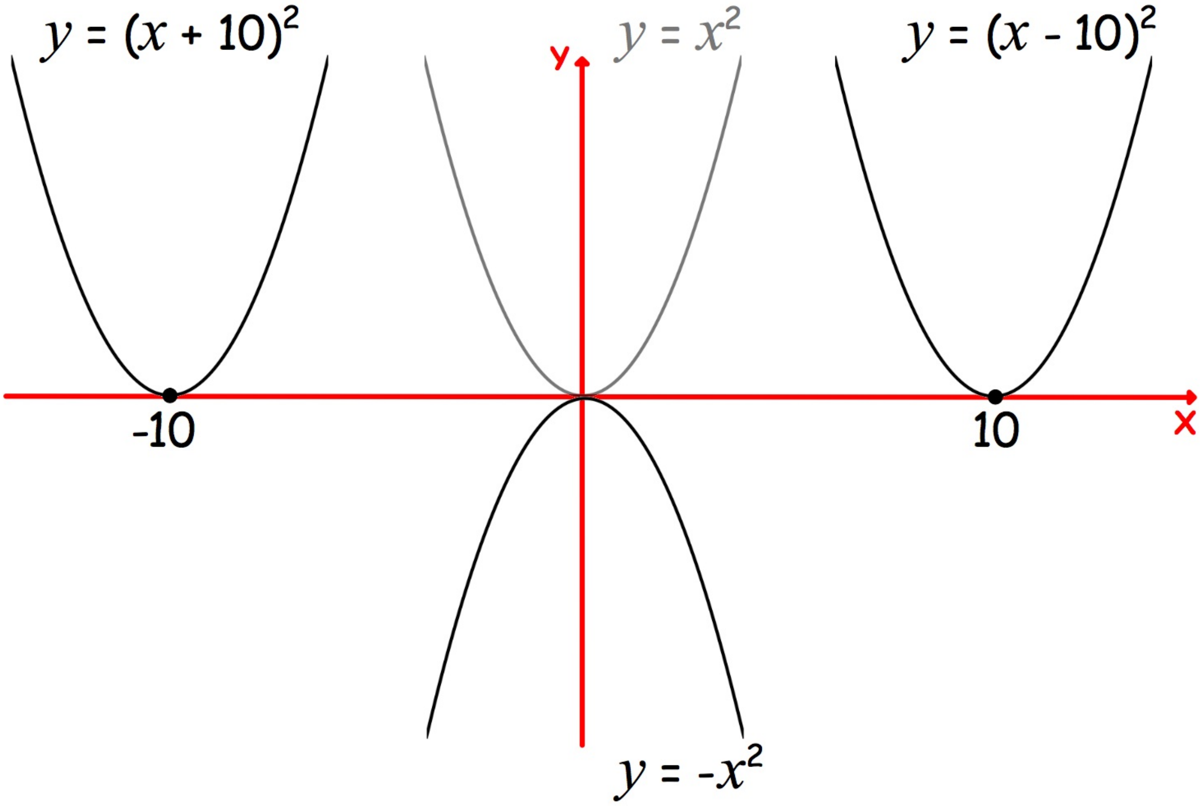
GRAPH TRANSFORMATIONS

Transform graph $y = x^2$

$y = ax^2$	stretch a units vertically
$y = -1x^2$	reflect in the x-axis
$y = (x + b)^2$	shift -b horizontally
$y = x^2 + c$	shift c units vertically

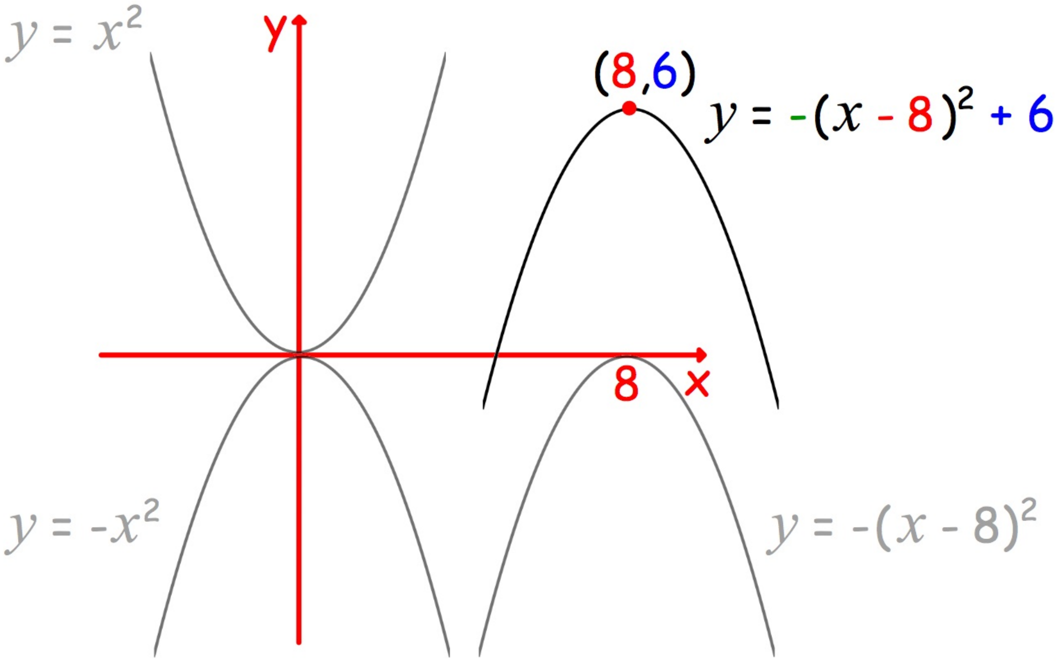






COMMBINING TRANSFORMATIONS

$y = 6 - (x - 8)^2$



SKETCH GRAPHS

(i) Y-INTERCEPT: substitute $x = 0$

(ii) X-INTERCEPTS (if any): substitute $y = 0$

solve $(x - a)(x - b) = 0$

(iii) TURNING POINT:

$$y = \pm 1(x + b)^2 + c$$

axis of symmetry $x = -b$

TP $(-b, c)$

+1

minimum

-1

maximum

OR

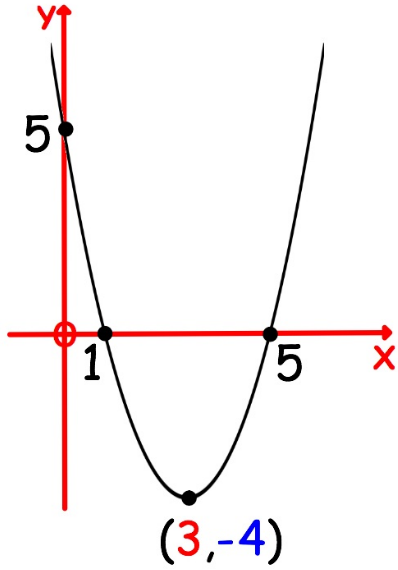
(iii) TURNING POINT:

$$y = (x - a)(x - b)$$

axis of symmetry $x = \dfrac{a + b}{2}$

substitute for x in equation

(1) $y = x^2 - 6x + 5$



(i) $y = 0^2 - 6 \times 0 + 5 = 5$

(ii) $x^2 - 6x + 5 = 0$

$(x - 1)(x - 5) = 0$

$x = 1$ or $x = 5$

$\frac{1+5}{2}$

(iii) axis of symmetry: $x = 3$

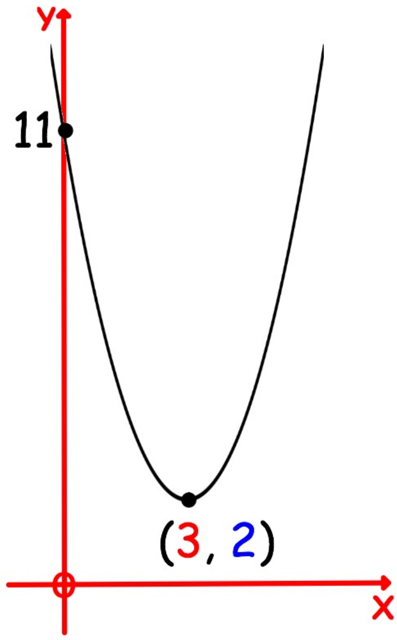
$y = x^2 - 6x + 5$

$= 3^2 - 6 \times 3 + 5$

$= -4$

minimum TP: (3,-4)

(2) $y = (x - 3)^2 + 2$



(i) $y = (0 - 3)^2 + 2$

$= 9 + 2$

$= 11$

(ii) no zeros

make $(0)^2$

(iii) axis of symmetry: $x = 3$

$y = (x - 3)^2 + 2$

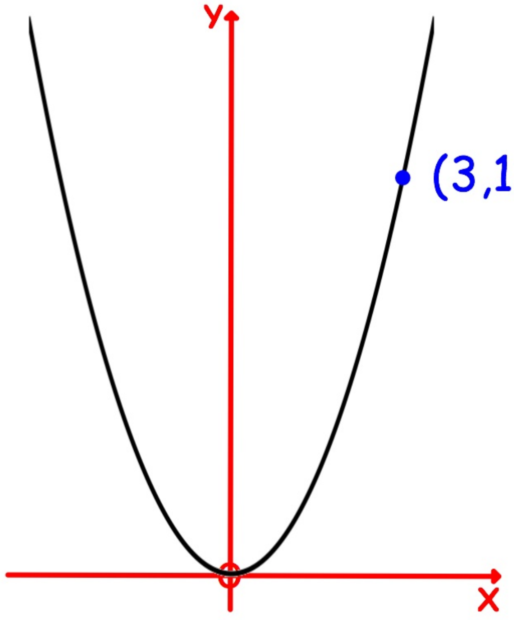
$y = (3 - 3)^2 + 2$

$= 0^2 + 2$

minimum TP: (3, 2)

EQUATION FROM THE GRAPH

(1) Form $y = kx^2$



$$y = kx^2$$

$$18 = k \times 3 \times 3$$

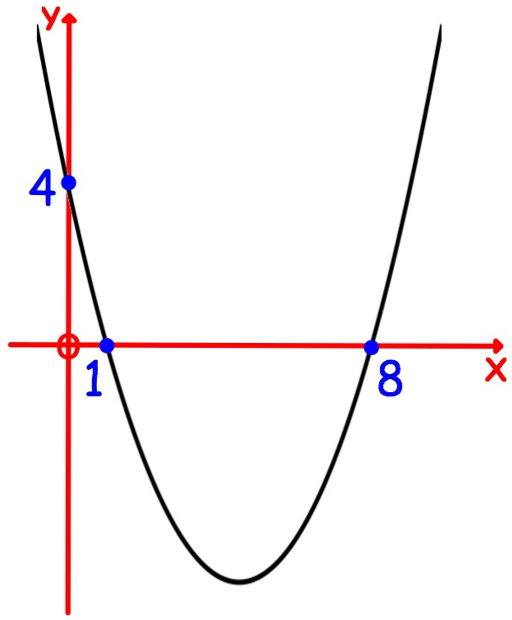
$$18 = 9k$$

$$k = 2$$

$$\underline{\underline{y = 2x^2}}$$

for (3,18) substitute $x = 3$ and $y = 18$

(2) Form $y = k(x - a)(x - b)$



$$y = k(x - 1)(x - 8)$$

$$4 = k(0 - 1)(0 - 8)$$

$$4 = 8k$$

$$k = {}^{4}/_{8}$$

$$k = {}^{1}/_{2}$$

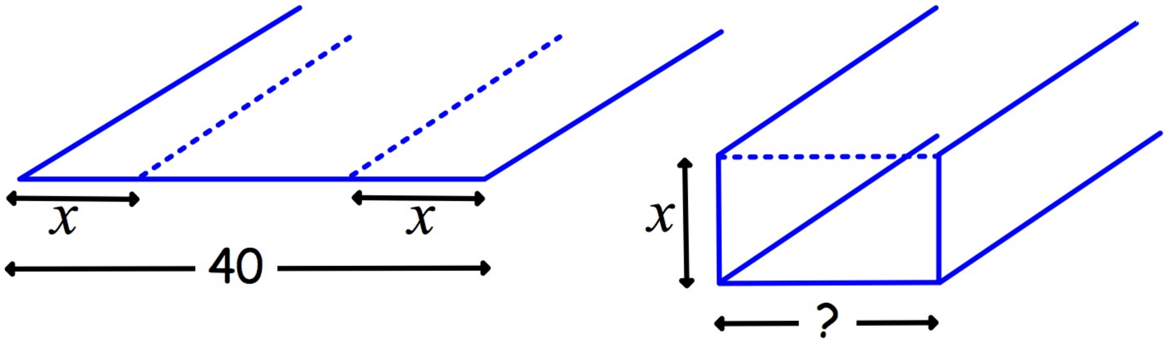
$$\underline{\underline{y = {}^{1}/_{2}\,(x - 1)(x - 8)}}$$

for (0,4) substitute $x = 0$ and $y = 4$

MAXIMUM / MINIMUM PROBLEMS

Find the turning point of the graph.

A rectangular sheet of metal 40cm wide is folded x cm from each end to form a gutter.



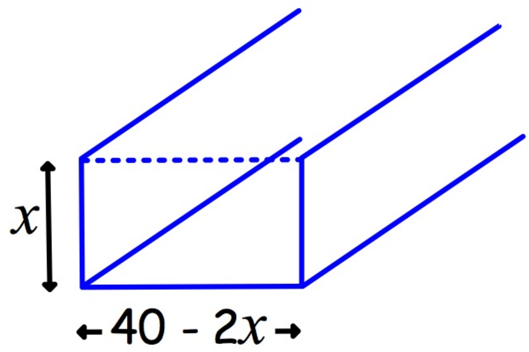
(a) Show the cross-sectional area is $A(x) = 40x - 2x^2$

(b) Find the value of x which will maximise the water flow.

(a) $A = lb$

$A = x(40 - 2x)$

$A(x) = 40x - 2x^2$



(b) $40x - 2x^2 = 0$

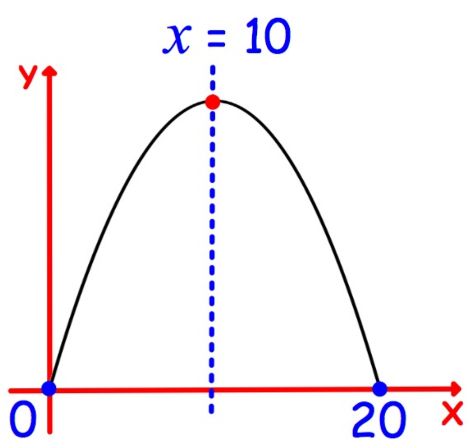
$2x(20 - x) = 0$

$2x = 0$ or $20 - x = 0$

$x = 0$ or $x = 20$

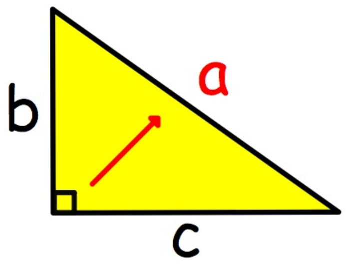
$(0 + 20) \div 2 = 10$

$\underline{\underline{x = 10}}$



PYTHAGORAS' THEOREM

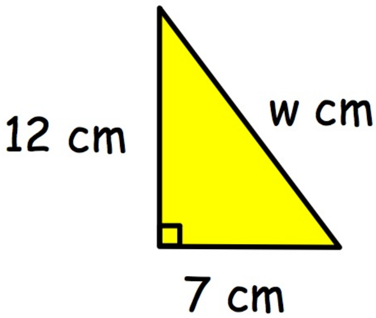
For right-angled triangles only:



$$a^2 = b^2 + c^2$$

HYPOTENUSE (largest side) is opposite the 90° angle.

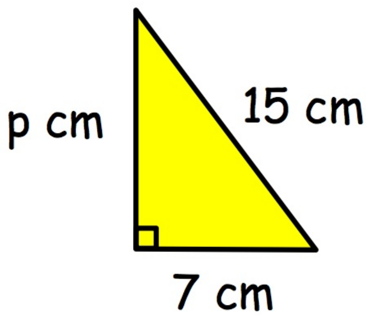
BIGGEST SIDE



$$\begin{aligned} w^2 &= 12^2 + 7^2 \\ &= 144 + 49 \\ &= 193 \end{aligned}$$

$$\begin{aligned} w &= \sqrt{193} \\ &= 13.892\ldots \\ w &\approx 13.9 \end{aligned}$$

SMALLER SIDE



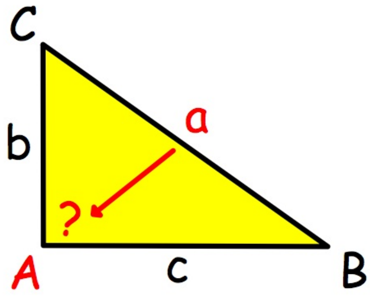
$$\begin{aligned} p^2 &= 15^2 - 7^2 \\ &= 225 - 49 \\ &= 176 \end{aligned}$$

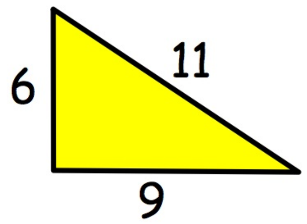
$$\begin{aligned} p &= \sqrt{176} \\ &= 13.266\ldots \\ p &\approx 13.3 \end{aligned}$$

CONVERSE OF PYTH. THM.

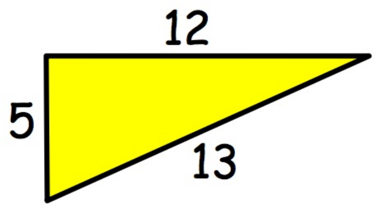
if $a^2 = b^2 + c^2$
then ΔABC is right-angled

(right-angled at A ie. $\angle BAC = 90°$)



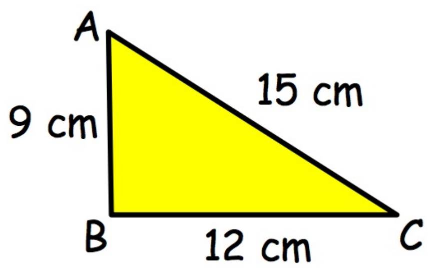


$6^2 + 9^2 \neq 11^2$
Δ is not right-angled



$5^2 + 12^2 = 13^2$
Δ is right-angled

Show that triangle ABC is right-angled.



$9^2 + 12^2 = 225$
$15^2 = 225$

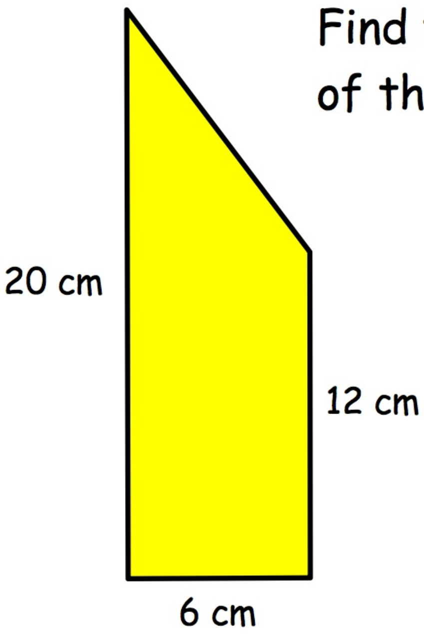
since $AB^2 + BC^2 = AC^2$

by the Converse of Pyth. Thm.
Δ ABC is right-angled at B.

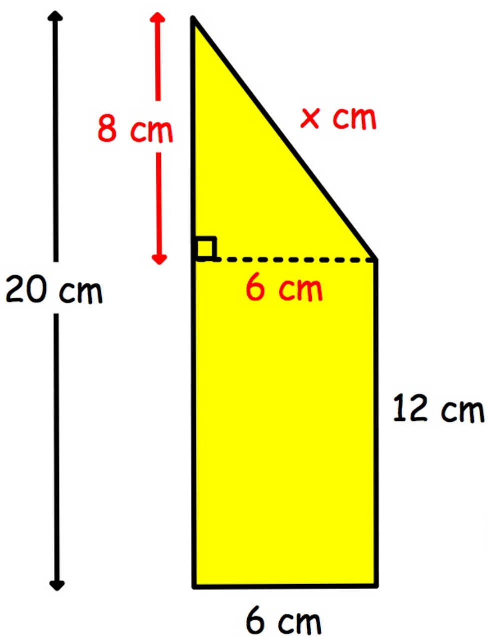
APPLYING PYTH. THM.

Identify the right angled triangle.

Find the perimeter of the shape.



height of triangle 20 cm - 12 cm = 8 cm



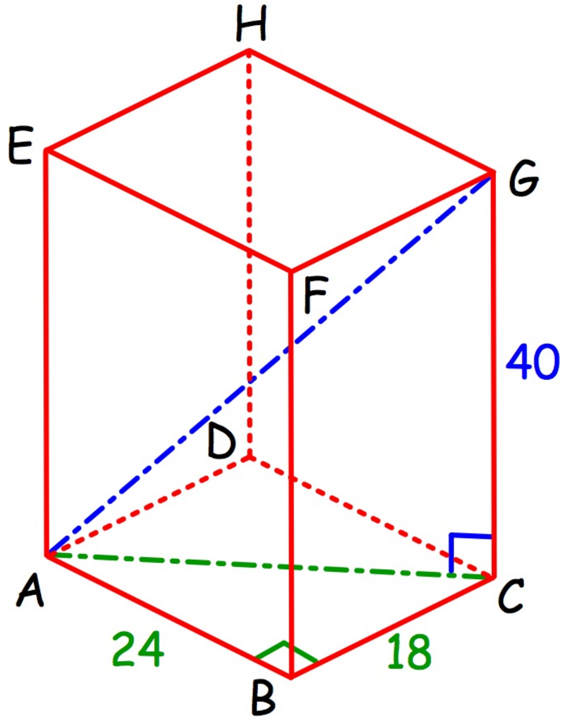
$$x^2 = 8^2 + 6^2$$
$$= 64 + 36$$
$$= 100$$
$$x = 10$$

P = 20cm + 6cm + 12cm + 10cm

= <u>48 cm</u>

3D

Find the length of space diagonal AG.



$\underline{\Delta ABC}$ face diagonal

$$AC^2 = 24^2 + 18^2$$
$$= 576 + 324$$
$$= 900$$
$$AC = \sqrt{900}$$
$$= 30$$

$\underline{\Delta ACG}$ space diagonal

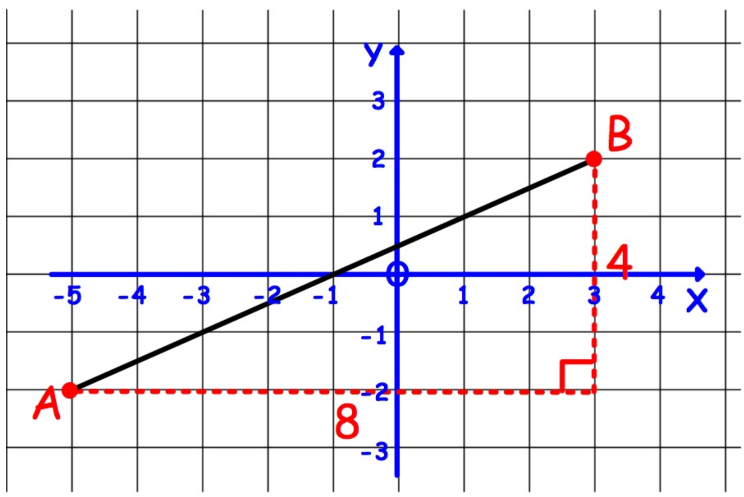
$$AG^2 = 40^2 + 30^2$$
$$= 1600 + 900$$
$$= 2500$$
$$AG = \sqrt{2500}$$
$$= 50$$

DISTANCE BETWEEN TWO POINTS

Plot the points.

Construct a right-angled triangle around them.

A (-5,-2) and B (3,2)



$$AB^2 = 8^2 + 4^2$$
$$= 64 + 16$$
$$= 80$$
$$AB = \sqrt{80}$$
$$= 8.944...$$
$$AB \approx 8.9 \text{ units}$$