NATIONAL 5 MATHEMATICS

COURSE NOTES

Expressions and Formulae

I

MATHS

INDEX:

SURDS

Rational numbers can be written as fractions, irrational numbers cannot.
Real numbers are all rational and irrational numbers.

SURDS ARE IRRATIONAL ROOTS:

$\sqrt{2}$, $\sqrt{\frac{5}{9}}$, $\sqrt[3]{16}$ are surds

$\sqrt{25}$, $\sqrt{\frac{4}{9}}$, $\sqrt[3]{-8}$ are **not** surds as they are 5 , $\frac{2}{3}$ and -2 respectively.

RULES:

$$\sqrt{m \times n} = \sqrt{m} \times \sqrt{n}$$

$$\sqrt{\frac{m}{n}} = \frac{\sqrt{m}}{\sqrt{n}}$$

(1) $\sqrt{72}$

$= \sqrt{36} \times \sqrt{2}$

$= 6\sqrt{2}$

largest square number factor of 72

(2) $\sqrt{\frac{5}{9}}$

$= \frac{\sqrt{5}}{\sqrt{9}}$

$= \frac{\sqrt{5}}{3}$

REMOVING BRACKETS: root × root
non-root × non-root

$$2\sqrt{3}\,(\sqrt{3} - 3)$$

$$= 6 - 6\sqrt{3}$$

$2\sqrt{3} \times \sqrt{3} = 2 \times \sqrt{9} = 6$

$2\sqrt{3} \times 3 = 2 \times 3 \times \sqrt{3} = 6\sqrt{3}$

ADD/SUBTRACT:

$\sqrt{36} \times \sqrt{2}$ $\quad$ $\sqrt{16} \times \sqrt{3}$ $\quad$ $\sqrt{25} \times \sqrt{2}$

$$\sqrt{72} + \sqrt{48} - \sqrt{50}$$

$$= 6\sqrt{2} + 4\sqrt{3} - 5\sqrt{2}$$

$$= \sqrt{2} + 4\sqrt{3}$$

RATIONALISING DENOMINATORS

Remove the surd from the denominator of a fraction:
multiply the 'top' and 'bottom' by the surd.

(1) $\dfrac{4}{\sqrt{6}}$

$= \dfrac{4 \times \sqrt{6}}{\sqrt{6} \times \sqrt{6}}$

$= \dfrac{4\sqrt{6}}{6}$

$= \dfrac{2\sqrt{6}}{3}$

(2) $\sqrt{\dfrac{2}{3}}$

$= \dfrac{\sqrt{2}}{\sqrt{3}}$

$= \dfrac{\sqrt{2} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$

$= \dfrac{\sqrt{6}}{3}$

(3) $\dfrac{1}{2\sqrt{3}}$

$= \dfrac{1 \times \sqrt{3}}{2\sqrt{3} \times \sqrt{3}}$

$= \dfrac{\sqrt{3}}{2 \times \sqrt{9}}$

$= \dfrac{\sqrt{3}}{6}$

To remove a compound surd from the denominator:
multiply the 'top' and 'bottom' by the conjugate surd.

$\dfrac{1}{\sqrt{3} + 2}$

$= \dfrac{1 \quad (\sqrt{3} - 2)}{(\sqrt{3} + 2)(\sqrt{3} - 2)}$

$= \dfrac{\sqrt{3} - 2}{-1}$

$= \quad 2 - \sqrt{3}$

CONJUGATE SURDS:
product rational (no surds)

$(\sqrt{3} + 2)(\sqrt{3} - 2)$

$= \; 3 - 2\sqrt{3} + 2\sqrt{3} - 4$

$= \; -1$

INDICES

index or exponent

base

$a^n = a \times a \times a \times \ldots$ to n terms

RULES: same base

$a^m \times a^n = a^{m+n}$

$a^m \div a^n = a^{m-n}$

$$\frac{2^{-3} \times 2^9}{2^4} = \frac{2^{\overset{-3+9=6}{6}}}{2^4} = 2^{\overset{6-4=2}{2}} = 4$$

$a^0 = 1$

$$2^{-3} \times 2^3 = 2^{\overset{-3+3=0}{0}} = 1$$

$(a^m)^n = a^{mn}$

$(ab)^n = a^n b^n$

$$(2a^2b)^3 = 2^3 a^{\overset{3 \times 2=6}{6}} b^3 = 8a^6b^3$$

$\frac{1}{a^p} = a^{-p}$

$$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$$

Note: $\frac{4}{x^3} = 4x^{-3}$

$$\frac{1}{4x^3} = \frac{1}{4}x^{-3}$$

$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$

$$8^{\frac{4}{3}} = (\sqrt[3]{8})^4 = 2^4 = 16$$

$$8^{-\frac{4}{3}} = \frac{1}{8^{\frac{4}{3}}} = \frac{1}{16}$$

Brackets:

$-3 + 3 = 0$ $\quad$ $-2 + 3 = 1$

$$x^3(x^{-3} + x^{-2}) = x^0 + x^1 = 1 + x$$

$$2x^3(4x^{-3} + 3x^{-2}) = 8x^0 + 6x^1 = 8 + 6x$$

$-1 + -1 = -2$

$$(1 - x^{-1})(1 - x^{-1}) = 1 - x^{-1} - x^{-1} + x^{-2} = 1 - 2x^{-1} + x^{-2}$$

Surds:

(1) $$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

(2) $$8^{\frac{3}{2}} = (\sqrt[2]{8})^3$$

$$= \sqrt{8} \times \sqrt{8} \times \sqrt{8}$$

$$= 8 \times 2\sqrt{2}$$

$$= 16\sqrt{2}$$

SCIENTIFIC NOTATION (STANDARD FORM)

Used to write very large and very small numbers.

Form $a \times 10^n$

$1 \leq a < 10$ ie. between 1 and 10, excluding 10

n is an INTEGER ie. ...-3,-2,-1,0,1,2,3...

Place the decimal point after the first non-zero digit.
Count the number of places the decimal point moves.

$257000 = 2.57 \times 10^5$

2 5 7 0 0 0.

negative power - small number between 0 and 1,
the point moves to the right,

$0.0000257 = 2.57 \times 10^{-5}$

0·0 0 0 0 2 5 7

USING THE CALCULATOR

eg. 2.08×10^{-3}

$\boxed{2}$ $\boxed{.}$ $\boxed{0}$ $\boxed{8}$ $\boxed{\times 10^n}$ $\boxed{(-)}$ $\boxed{3}$

Examples: answers in scientific notation correct to 3 significant figures.

(1) 1 milligram of hydrogen contains 2.987×10^{20} molecules. Find the number of molecules in 5 grams.

$2.987 \times 10^{20} \quad \times \; 5000$ $\qquad$ $5\text{ g} = 5000\text{ mg}$

$= 1.4935 \times 10^{24}$

$= 1.49 \times 10^{24}$ molecules

(2) The total mass of argon in a flask is 4.15×10^{-2} grams. The mass of one atom of argon is 6.63×10^{-23} grams. Find the number of argon atoms in the flask.

$4.15 \times 10^{-2} \; \div \; 6.63 \times 10^{-23}$

$= 6.2594... \times 10^{20}$

$= 6.26 \times 10^{20}$ atoms

ALGEBRAIC EXPRESSIONS

REMOVING BRACKETS

SINGLE BRACKETS: $a \times (b + c) = a \times b + a \times c$

(1) $3p(2p + r)$

$= 6p^2 + 3pr$

(2) $2a(3a - b + 5)$

$= 6a^2 - 2ab + 10a$

Sign changes when multiplying by a negative term:

(3) $-3(2w - 3y)$

$= -6w + 9y$

(4) $-n(4n + 5m)$

$= -4n^2 - 5mn$

EXPRESSIONS: remove brackets then simplify

no sign change

(1) $2a + 3a(2 - 3a)$

$= 2a + 6a - 9a^2$

$= 8a - 9a^2$

sign change

(2) $5 - 3(2a - 3)$

$= 5 - 6a + 9$

$= 14 - 6a$

DOUBLE BRACKETS: "FOIL"

Pairs of terms between the two brackets multiplied

$$\overset{F\ \ O}{(\qquad)(\qquad)}\ \underset{I\ \ L}{}$$

F first pair

O outer pair

I inner pair

L last pair

(1) $(2x + 5)(2x - 3)$

$$(2x + 5)(2x - 3)$$

$$= \overset{F}{4x^2} \overset{O}{- 6x} \overset{I}{+ 10x} \overset{L}{- 15}$$

$$= 4x^2 + 4x - 15$$

(2) $(2x - 3)^2$

$$(2x - 3)(2x - 3)$$

$$= \overset{F}{4x^2} \overset{O}{- 6x} \overset{I}{- 6x} \overset{L}{+ 9}$$

$$= 4x^2 - 12x + 9$$

Same process for trinomials:

$(2x - 3)(3x^2 - 2x + 5)$

sign change

$= 2x(3x^2 - 2x + 5) \quad - 3(3x^2 - 2x + 5)$

$= 6x^3 - 4x^2 + 10x \quad - 9x^2 + 6x - 15$

$= 6x^3 - 4x^2 - 9x^2 + 10x + 6x - 15$

$= 6x^3 - 13x^2 + 16x - 15$

EXPRESSIONS: remove brackets then simplify.

$(x + 2)^2 - (x - 2)^2$

sign change

$= x^2 + 4x + 4 - (x^2 - 4x + 4)$

$= x^2 + 4x + 4 - x^2 + 4x - 4$

$= 8x$

FACTORISATION

COMMON FACTORS $a \times b + a \times c = a \times (b + c)$

To FULLY factorise a is the Highest Common Factor.

(1) $12w - 18y$

$= 6(2w - 3y)$

(2) $4n^2 + 5mn - n$

$= n(4n + 5m - 1)$

(3) $16p^2 + 8pr$

$= 8p(2p + r)$

(4) $1.4 \times 3.6 - 1.4 \times 1.6$

$= 1.4 \times (3.6 - 1.6)$

$= 1.4 \times 2$

$= 2.8$

DIFFERENCE OF TWO SQUARES

$a^2 - b^2 = (a + b)(a - b)$

(1) $16 - 9y^2$

$= 4^2 - (3y)^2$

$= (4 + 3y)(4 - 3y)$

(2) $(6.4)^2 - (3.6)^2$

$= (6.4 + 3.6)(6.4 - 3.6)$

$= 10 \times 2.8$

$= 28$

FULLY factorise:

common factor first

(3) $36n^2 - 9$

$= 9(4n^2 - 1)$

$= 9(2n + 1)(2n - 1)$

further factorisation

(4) $n^4 - 16$

$= (n^2 + 4)(n^2 - 4)$

$= (n^2 + 4)(n + 2)(n - 2)$

$(6n + 3)(6n - 3)$ is not FULLY factorised

$1x^2 + bx + c$ — a factor pair of c add to b

$2 + 4 = +6$ $\qquad$ $2 \times 4 = +8$

(1) $x^2 + 6x + 8 = (x + 2)(x + 4)$

$-2 + -4 = -6$ $\qquad$ $-2 \times -4 = +8$

(2) $x^2 - 6x + 8 = (x - 2)(x - 4)$

$-2 + 4 = +2$ $\qquad$ $-2 \times 4 = -8$

(3) $x^2 + 2x - 8 = (x - 2)(x + 4)$

$ax^2 + bx + c$ — a factor pair of ac add to b

$3x \times 1x$ $\qquad$ 1×8 $\qquad$ 2×4

(1) $3x^2 + 14x + 8$

$= \underline{\underline{(3x + 2)(x + 4)}}$

$3 \times 8 = 24$

$12 \times 2 = 24$

$3x^2 + 14x + 8$

$12 + 2 = 14$

so 12x and 2x

try combinations to make: 12x and 2x

$3x$ ╲╱ $+1$	$3x$ ╲╱ $+8$	$3x$ ╲╱ $+2$	$3x$ ╲╱ $+4$
$1x$ ╱╲ $+8$	$1x$ ╱╲ $+1$	$1x$ ╱╲ $+4$	$1x$ ╱╲ $+2$
$+1x$ $\quad$ $+24x$	$+8x$ $\quad$ $+3x$	$\underline{\underline{+2x \quad +12x}}$	$+4x$ $\quad$ $+6x$
		correct combination	

(2) $3x^2 - 14x + 8$

$= \underline{\underline{(3x - 2)(x - 4)}}$

$-12 \times -2 = +24$

$3 \times 8 = 24$

$3x^2 - 14x + 8$

$-12 + -2 = -14$

so -12x and -2x

try combinations to make: -12x and -2x

$$\begin{array}{cc} 3x & -1 \\ 1x & -8 \\ \hline -1x & -24x \end{array} \qquad \begin{array}{cc} 3x & -8 \\ 1x & -1 \\ \hline -8x & -3x \end{array} \qquad \begin{array}{cc} 3x & -2 \\ 1x & -4 \\ \hline -2x & -12x \\ \hline\hline \end{array} \qquad \begin{array}{cc} 3x & -4 \\ 1x & -2 \\ \hline -4x & -6x \end{array}$$

correct combination

(3) $3x^2 + 10x - 8$

$= \underline{\underline{(3x - 2)(x + 4)}}$

$+12 \times -2 = -24$

$3 \times -8 = -24$

$3x^2 + 10x - 8$

$+12 + -2 = +10$

so +12x and -2x

try combinations to make: +12x and -2x

$$\begin{array}{cc} 3x & -1 \\ 1x & +8 \\ \hline -1x & +24x \end{array} \qquad \begin{array}{cc} 3x & +8 \\ 1x & -1 \\ \hline +8x & -3x \end{array} \qquad \begin{array}{cc} 3x & -2 \\ 1x & +4 \\ \hline -2x & +12x \\ \hline\hline \end{array} \qquad \begin{array}{cc} 3x & +4 \\ 1x & -2 \\ \hline +4x & -6x \end{array}$$

correct combination

ALTERNATIVE

$12 \times 2 = 24$

$3 \times 8 = 24$

$3x^2 + 14x + 8$

$12 + 2 = 14$

so 12x and 2x

(1) $3x^2 + 14x + 8$

$= 3x^2 + 12x + 2x + 8$ replace $+14x$ by $+12x + 2x$

$= (3x^2 + 12x) + (2x + 8)$ bracket first and last pairs

$= 3x(x + 4) + 2(x + 4)$ fully factorise each bracket

$= \underline{\underline{(3x + 2)(x + 4)}}$ $(x + 4)$ is a common factor

(2) $3x^2 - 14x + 8$

$= 3x^2 - 12x - 2x + 8$

$= 3x(x - 4) - 2(x - 4)$

$= \underline{\underline{(3x - 2)(x - 4)}}$

sign change

$-12 \times -2 = +24$

$3 \times 8 = 24$

$3x^2 - 14x + 8$

$-12 + -2 = -14$

so -12x and -2x

-2x first avoids sign change

(3) $3x^2 + 10x - 8$

$= 3x^2 - 2x + 12x - 8$

$= x(3x - 2) + 4(3x - 2)$

$= \underline{\underline{(3x - 2)(x + 4)}}$

$+12 \times -2 = -24$

$3 \times -8 = -24$

$3x^2 + 10x - 8$

$+12 + -2 = +10$

so +12x and -2x

COMPLETING THE SQUARE

form $(x + a)^2 + b$

Squaring brackets $\quad (x \pm a)^2 = x^2 \pm 2ax + a^2$

$$(x + 3)^2 = x^2 + 6x + 9$$

÷ 2 $\quad$ ÷ 2 and square

rearranging $\quad x^2 \pm 2ax = (x \pm a)^2 - a^2$

$$x^2 + 6x = (x + 3)^2 - 9$$

÷ 2 $\quad$ square

(1) $\quad x^2 - 8x + 10$

÷ 2 and square

$-8 \div 2 = -4$

$(-4)^2 = 16$

$= (x - 4)^2 - 16 + 10$

$= (x - 4)^2 - 6$

(2) $\quad x^2 + 3x - 1$

÷ 2 and square

$3 \div 2 = {}^3/_2$

$({}^3/_2)^2 = {}^9/_4$

$= (x + {}^3/_2)^2 - {}^9/_4 - {}^4/_4$

$= (x + {}^3/_2)^2 - {}^{13}/_4$

ALGEBRAIC FRACTIONS

SIMPLIFYING:

(i) fully factorise numerator and denominator

(ii) cancel common factors

(1) $\dfrac{4x^2 - 6x}{4x^2 - 9}$

$= \dfrac{2x(2x - 3)}{(2x + 3)(2x - 3)}$

$= \dfrac{2x}{2x + 3}$

(2) $\dfrac{2t^2 - 9t + 9}{(t - 3)^3}$

$= \dfrac{(2t - 3)(t - 3)}{(t - 3)^2(t - 3)}$

$= \dfrac{2t - 3}{(t - 3)^2}$

(3) $\dfrac{4x^2y}{4x^2y + 2xy^2}$

$= \dfrac{2xy \times 2x}{2xy(2x + y)}$

$= \dfrac{2x}{2x + y}$

MULTIPLY/DIVIDE: same rules as arithmetic

$$\frac{a}{b} \times \frac{c}{d} = \frac{a \times c}{b \times d}$$

(1) $\dfrac{m}{2} \times \dfrac{6}{m^2}$

$= \dfrac{6m}{2m^2}$

$= \dfrac{2m \times 3}{2m \times m}$

$= \dfrac{3}{m}$

(2) $\dfrac{4}{x} \div \dfrac{y}{2x}$

$\div \dfrac{c}{d} \longrightarrow \times \dfrac{d}{c}$

reciprocal

$= \dfrac{4}{x} \times \dfrac{2x}{y}$

$= \dfrac{8x}{xy}$

$= \dfrac{8}{y}$

ADD/SUBTRACT: requires a common denominator

(1) $\frac{m}{2} + \frac{m-1}{3}$

$= \frac{3m}{6} + \frac{2(m-1)}{6}$

$= \frac{3m + 2m - 2}{6}$

$= \frac{5m-2}{6}$

(2) $\frac{y}{x^2} + \frac{1}{2x}$

$= \frac{2y}{2x^2} + \frac{1x}{2x^2}$

$= \frac{2y + x}{2x^2}$

(3) $\frac{3}{t-3} - \frac{3}{t+3}$

$= \frac{3(t+3)}{(t-3)(t+3)} - \frac{3(t-3)}{(t-3)(t+3)}$

$= \frac{3(t+3) - 3(t-3)}{(t-3)(t+3)}$

$= \frac{3t+9 - 3t+9}{(t-3)(t+3)}$

$= \frac{18}{(t-3)(t+3)}$

(4) $\frac{1}{x} + \frac{3}{x(x-3)}$

$= \frac{1(x-3)}{x(x-3)} + \frac{3}{x(x-3)}$

$= \frac{x - 3 + 3}{x(x-3)}$

$= \frac{x}{x(x-3)}$

$= \frac{1}{x-3}$