St Columba's High School
Advanced Higher Maths
Functions

Contents

1 Revision and Set Notation

A function is a relationship that links the members of one set with the members of another set. In the maths covered in school so far, it is usually a relationship that maps x values to a y value. The values of x for which the function is defined is called the *domain* of the function and the set of values which it can map to is called the *range*.

A function maps every x-value to a **unique** y-value. It is sometimes written as $y = f(x)$ where f maps x to y. It is possible for $f(a) = f(b)$ when $a \neq b$.

In higher, you learned how to differentiate, integrate and sketch graphs of functions. You should also be able to identify the domain and range for given functions.

Set Notation

Abbreviations are often used to represent the sets of numbers for which functions are defined. Some of these are described below:

$\mathbb{N}$ The set of natural numbers $\{1, 2, 3, 4, \ldots\}$

$\mathbb{Z}$ The set of integers $\{-2, -1, 0, 1, 2, \ldots\}$

$\mathbb{Q}$ The set of rational numbers

$\mathbb{R}$ The set of real numbers

Example 1. Write down the domain and range for the function $y = \cos x$.

Process

The function $y = \cos x$ has domain $x \in \mathbb{R}$. The range of the function is $-1 \leq f(x) \leq 1$. This can also be written as $f \in \{-1, 1\}$.

Example 2. Write down the domain and range for the function $y = \frac{1}{x-2}$.

Process

As dividing by zero is impossible, in this case, $x \neq 2$. Hence, the domain is $x \in \mathbb{R}, x \neq 2$. This can also be written as $\{\mathbb{R} - 2\}$. The range is all possible values of $\frac{1}{x-2}$. This must be every real number except 0. Hence the range is $f(x) \in \mathbb{R}, f(x) \neq 0$ or $\{\mathbb{R} - 0\}$.

2 Asymptotes and Continuity

2.1 Continuity

Functions may be continuous or non-continuous. For continuous functions, the graph of the function is a continuous curve or line with no gaps. However, some functions are undefined for certain values of x and this leads to gaps in the graphs of the function.

A function is continuous if there is no break in the curve across the domain of the function. More formally, continuity is defined as:

> A function is continuous if, for every point a on the domain,
>
> $$\lim_{x \to a} f(x) = f(a)$$

Similarly, a function, $f(x)$, is discontinuous if there is a break in the graph for a value or values of x. Hence, a function is discontinuous if there exists a point or points on the function at which the function is undefined.

> A function is discontinuous if, for some point a on the domain,
>
> $$\lim_{x \to a} f(x) \neq f(a)$$

Example 3. Identify any points of discontinuity of the function

$$f(x) = \frac{-2x}{x2 - 9}$$

Process

Rewriting the function by factorising the denominator gives:

$$f(x) = \frac{-2x}{(x-3)(x+3)}$$

The function is not defined for $x = \pm 3$. Hence the graph of the function has discontinuities at $x = \pm 3$.

Example 4. Identify any points of discontinuity for the multistep function

$$f(x) = \begin{cases} x^2, & \text{if } x < 0 \\ 4x+1, & \text{if } x \geq 0 \end{cases}$$

Process

There are no points for which the function is undefined. Hence, to identify any discontinuities, consider any points where the graph would not "match up" between the two specified regions.

When $x = 0$, $f(x) = 4 \times 0 + 1 = 1$.

However, for values of x less than 0, $f(x) = x^2$. As

$$x \to 0^-, f(x) \to 0^+$$

.

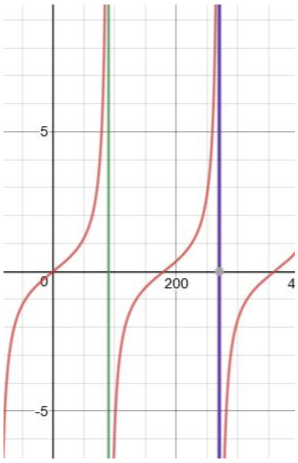
Therefore, there is a point of discontinuity at $x = 0$ as

$$\lim_{x \to 0^-} f(x) = 0 \text{ but } f(0) = 1$$

2.2 Asmptotes

vertical Asymptotes

For example, it is clear from consideration of the graph of the tangent function that it is discontinuous at $90^o, 270^o \ldots$.



At the values $x = 90^o, 270^o$ etc, a vertical line can be drawn which does not touch any point on the graph of the function. This line is called a **vertical asymptote**.The value of the function tends to $\pm\infty$ as x gets closer and closer to the value for which the function is undefined.

A line $x = a$ is a vertical asymptote to a function $f(x)$ if:

$$\lim_{x \to a^{\pm}} f(x) = \pm\infty$$

This means that the function tends to infinity as x gets closer to the point a from either values greater than a (a^+) or below a (a^-).

To find vertical asymptotes, factorise the denominator and simplify the function as far as possible. Then identify any points for which the function is undefined and at which a vertical asymptote will exist. Asymptotes are usually drawn as a dashed line.

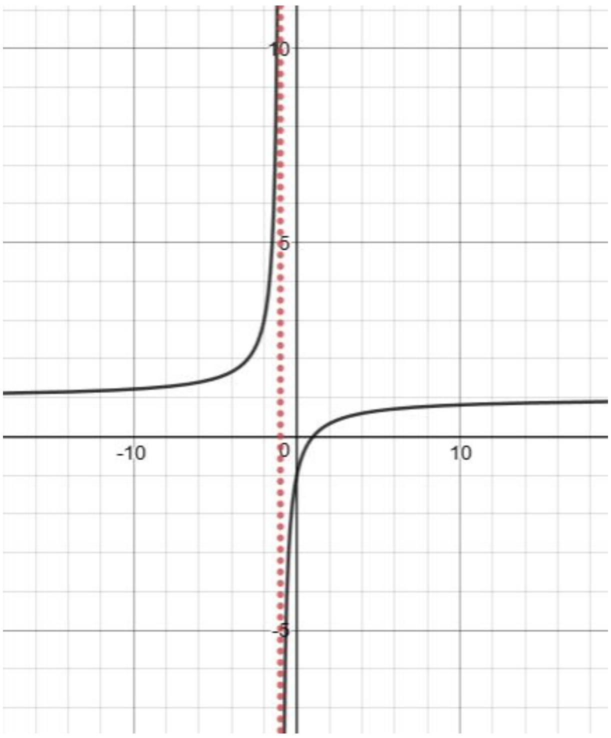
Example 5. Find the equations of any vertical asymptotes of

$$f(x) = \frac{x-1}{x+1}$$

.

Process

The denominator $x + 1$ is zero for $x = -1$. Moreover, $x = -1$ is not a root if the numerator and so $x = -1$ is a vertical asymptote.



Example 6. Find the equations of any vertical asymptotes(if they exist) of the graph

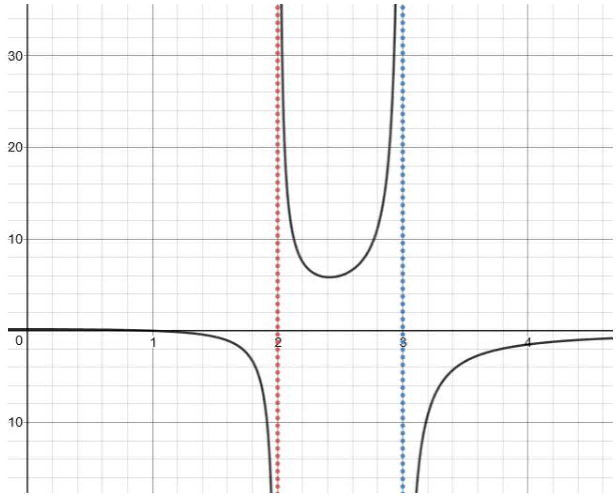
$$y = \frac{1-x}{x^2-5x+6}$$

Process

The denominator can be factorised as

$$x^2 - 5x + 6 = (x-3)(x-2)$$

Therefore, the roots of the denominator are $x = 3$ and $x = 2$ Neither of these roots are roots of the numerator and so there are vertical asymptotes at $x = 3$ and $x = 2$.



Example 7. Find the equations of any vertical asymptotes (if they exist) of the graph of

$$f(x) = \frac{x^2-4}{x^4-3x^2-4}$$

Process

The denominator $x^4 - 3x^2 - 4$ has a root $x = 2$. Using synthetic division,

2	1	0	−3	0	−4
		2	4	8	4
	1	2	1	2	0

Hence

$$x^4 - 3x^2 - 4 = (x-2)(x^3 + 2x^2 + x + 2)$$

As $x = -2$ is a root of $x^3 + 2x^2 + x + 2$, this can be factorised using synthetic division again:

−2	1	2	1	2
		−2	0	−2
	1	0	1	0

Therefore,

$$x^4 - 3x^2 - 4 = (x-2)(x+2)(x^2+1)$$

So, after factorising the numerator, the function $f(x)$ becomes:

$$\begin{aligned} f(x) &= \frac{x^2-4}{x^4-3x^2-4} \\ &= \frac{(x-2)(x+2)}{(x-2)(x+2)(x^2+1)} \\ &= \frac{1}{x^2+1} \end{aligned}$$

As, x^2+1 is positive for all values of x, there are no values of x for which this function is undefined and hence there is no vertical asymptote.

Horizontal Asymptotes

A horizontal asymptote is a horizontal line of the form $y = a$ where the function gets closer and closer to the line $y = a$ as $x \to \infty$ and/or $-\infty$.

Horizontal asymptotes occur when the degree of the numerator is less than or equal to the degree of the denominator.

To find the equation of a horizontal asymptote, check what happens to the function as $x \to \pm\infty$.

Example 8. Find the equation of the horizontal asymptote of

$$f(x) = \frac{x-1}{x+1}$$

Process

By polynomial division, the function $f(x)$ can be rewritten in a way that allows for a horizontal asymptote to be identified.

$$\begin{array}{r|r} & 1 \\ \hline x+1 & x-1 \\ & -x-1 \\ \hline & -2 \end{array}$$

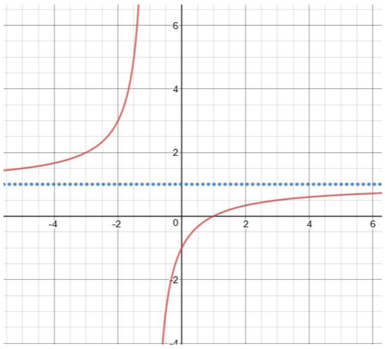
and so

$$f(x) = 1 - \frac{2}{x+1}$$

As $x \to \infty$, $f(x) \to 1$ and so $y = 1$ is a horizontal asymptote.

To identify the shape of the curve, consider how the function tends to 1 as $x \to \infty$ from above and from below.
As $x \to \infty_+$, $f(x) \to 1$ from below.

Similarly, as $x \to \infty_-$, $f(x) \to 1$ from above. This can be seen in the diagram below:



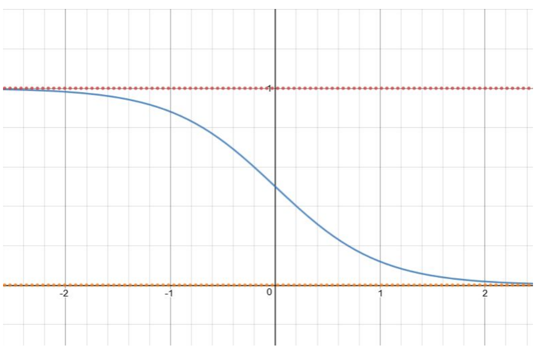
Example 9. Find the equation of any horizontal asymptotes of the function

$$f(x) = \frac{1}{2e^x + 1}$$

Process

As $x \to +\infty, e^x \to \infty$ and so the denominator will become infinitely large. Therefore, the function $f(x) \to 0$ as $x \to \infty$. Hence, $y = 0$ is a horizontal asymptote.

As $x \to -\infty$, $e^x \to 0$ and so the denominator will tend towards 1 and $f(x) \to 1$.Therefore, $y = 1$ is a horizontal asymptote.
These can be seen in the diagram below:



Example 10. Find the equation of any horizontal asymptotes to the curve

$$f(x) = \frac{4x^2 - 3x + 4}{2x^2 + 5}$$

Process

$$\begin{array}{r|l} & 2 \\ \hline 2x^2 + 5) & 4x^2 - 3x + 4 \\ & -4x^2 - 10 \\ \hline & - 3x - 6 \end{array}$$

Hence

$$f(x) = 2 - \frac{(3x + 6))}{(2x^2 + 5)} = 2 - \frac{3(x + 2)}{(2x^2 + 5)}$$

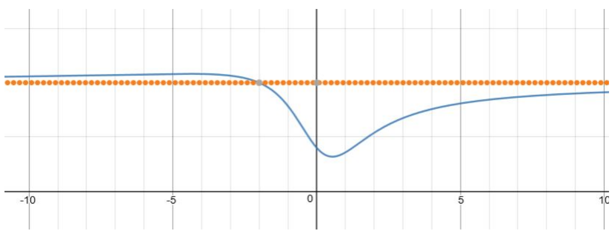
Therefore, as $x \to +\infty$,

$$\frac{3(x + 2)}{(2x^2 + 5)} \to 0$$

and, as $x \to -\infty$,

$$\frac{3(x + 2)}{(2x^2 + 5)} \to 0$$

Therefore $f(x) \to 2$ as $x \to \pm\infty$ and so $y = 2$ is a horizontal asymptote. This can be seen in the diagram below:



Slant or Oblique Asymptotes

A slant asymptote is a non-constant straight line graph to which a function, $f(x)$, tends as $x \to \pm\infty$. It occurs when the degree of the numerator is exactly one greater than the degree of the denominator. To find the slant asymptote, use polynomial division to find the equation of the asymotote.

Example 11. Find the equation of the slant asymptote to the curve

$$f(x) = \frac{x^3 - 3x}{x^2 + 1}$$

Process

$$\begin{array}{r|l} & x \\ \hline x^2 + 1 & x^3 - 3x \\ & -x^3 - x \\ \hline & -4x \end{array}$$

Therefore, $f(x)$ can be rewritten as:

$$f(x) = x - \frac{4x}{x^2 + 1}$$

As $x \to \infty$,

$$\frac{4x}{x^2 + 1} \to 0_+$$

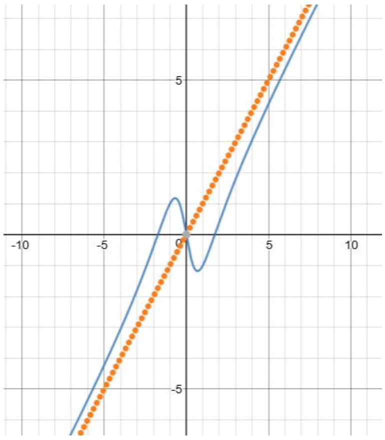
and so $f(x) \to x$ from below.

As $x \to -\infty$,

$$\frac{4x}{x^2 + 1} \to 0_-$$

and so $f(x) \to x$ from above.

Hence, the line $y = x$ is a slant asymptote to the function $f(x)$.This can be seen in the diagram below.



Example 12. Find the equations of all of the asymptotes of the function

$$f(x) = \frac{2x^2 + x - 3}{x + 1}$$

Process

First, factorise the numerator, $2x^2 + x - 3 = (x - 1)(2x + 3)$. Hence, there are no terms in the numerator and denominator that can cancel and so $x = -1$ is a vertical asymptote.

Now, rewrite $f(x)$ using polynomial division.

$$\begin{array}{rr} & 2x - 1 \\ x + 1 \big) & 2x^2 \quad + x - 3 \\ & -\, 2x^2 - 2x \quad\;\; \\ \hline & -\, x - 3 \\ & x + 1 \\ \hline & -\, 2 \end{array}$$

Hence,

$$f(x) = 2x - 1 - \frac{2}{x + 1}$$

As $x \to \infty$,

$$\frac{2}{x + 1} \to 0_+$$

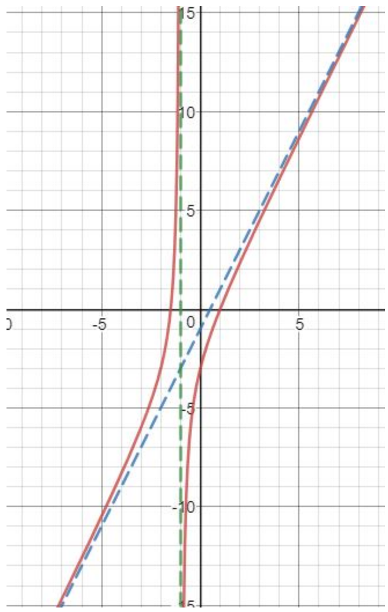
and so $f(x) \to x$ from below.

As $x \to -\infty$,

$$\frac{2}{x+1} \to 0_-$$

and so $f(x) \to x$ from above.

Therefore, $y = 2x - 1$ is a slant asymptote to $f(x)$. The graph of $f(x)$ can be seen in the diagram below with the asymptotes denoted by the dashed lines.



3 Stationary Points and Points of Inflection

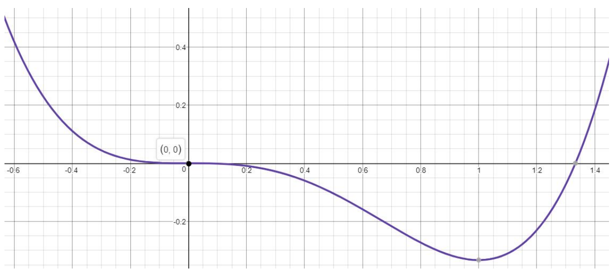
In Higher maths, the derivative was used to find the stationary points of a function and a nature table was used to identify the nature of these points, i.e. whether they were maximum or minimum turning points. However, there are actually three types of stationary point, namely:

- local maximum
- local minimum
- point of inflection

A function may also take its maximum and/or minimum values at the end points of the region for which the function is defined. That is why the terms local maximum and local minimum values are used. The nature of the stationary point may now be identified through the use of the second derivative:

- If $f''(a) > 0$ then a is a local minimum.
- If $f''(a) < 0$ then a is a local maximum.
- If $f''(a) = 0$ then a is a horizontal point of inflection.

A horizontal point of inflection is a point where the function is stationary yet the gradient of the function doesn't change sign when moving through the stationary point. An example of a point of inflection (PI) can be seen in the diagram below. The PI in this case occurs at $x = 0$.



Example 13. Use the second derivative test to identify all of the stationary points, along with their nature, for the graph of the function

$$f(x) = 2x^4 - 8x^3$$

Process

Begin by calculating the first and second derivatives of the function.

$$\begin{aligned} f(x) &= 2x^4 - 8x^3 \\ f'(x) &= 8x^3 - 24x^2 \\ f''(x) &= 24x^2 - 42x \end{aligned}$$

For stationary points (SPs), the derivative equals 0. Hence:

$$\begin{aligned} 8x^3 - 24x^2 &= 0 \\ x^2(8x - 24) &= 0 \\ x = 0 \quad \text{or} &\quad x = 3 \end{aligned}$$

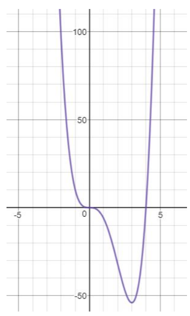
Now use the second derivative test for each of the stationary points identified.

When $x = 0$, $f(x) = 0$ and $f''(x) = 0$. Therefore $(0, 0)$ is a PI.

When $x = 3$, $f(x) = 2 \times 3^4 - 8 \times 3^3 = -54$ and

$$f''(x) = 24 \times 3^2 - 42 \times 3 = 90 > 0$$

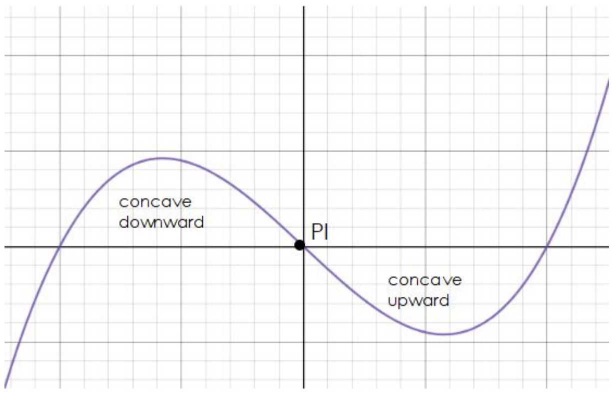
and therefore $(3, -54)$ is a local minimum.



3.1 Concavity

A function can also have non-horizontal points of inflection. These occur at a point a when $f''(a) = 0$ but $f'(a) \neq 0$. However, it is possible to find a point a where the second derivative equals zero yet a is not a point of inflection. In this case, it is necessary to look at the **concavity** of the function in order to determine if the point is a PI.

Generally, a PI occurs when a function f changes concavity from from concave upwards to concave downward or vice versa. The diagram below illustrates this, the concavity changes but the PI is not a SP:



For any function f at a point a,

- $f''(a) > 0 \Rightarrow f(x)$ is concave upwards at a.
- $f''(a) < 0 \Rightarrow f(x)$ is concave downwards at a.

Hence, once a PI has been identified by the method of finding the points where the second derivative is zero, it is necessary to consider the sign of $f''(x)$ at points just above and just below the possible PI.

Example 14. Identify all the stationary points (and their nature) and points of inflection for the function

$$f(x) = x^5 - 20x^2$$

Process

Calcuate the derivatives:

$$\begin{aligned} f'(x) &= 5x^4 - 40x \\ f''(x) &= 20x^3 - 40 \end{aligned}$$

Stationary points occur when $f'(x) = 0$. Hence,

$$\begin{aligned} 5x^4 - 40x &= 0 \\ \Rightarrow 5x(x^3 - 8) &= 0 \\ \Rightarrow x = 0 \quad \text{or} \quad x &= \sqrt[3]{8} = 2 \end{aligned}$$

Using the second derivative test, when $x = 0$,

$$f''(x) = -60 < 0$$

and so there is a maximum SP at $(0, 0)$.

When $x = 2$,

$$f''(x) = 20 \times 2^3 - 40 = 120 > 0$$

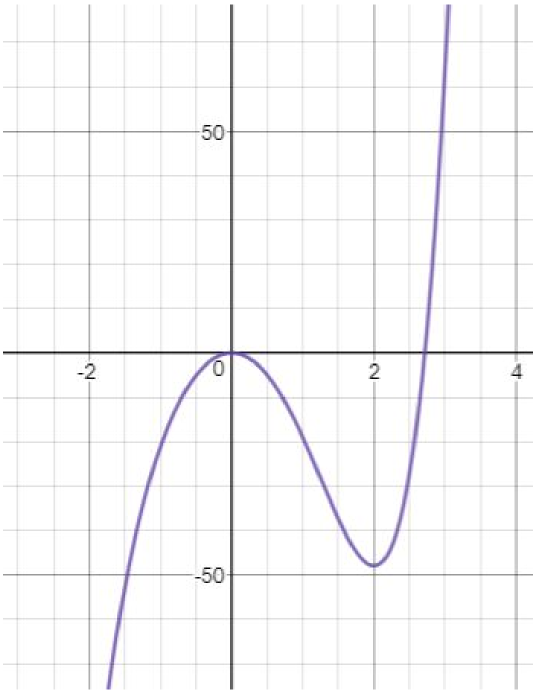
and so there is a minimum SP at $x = 2$. $f(x) = 2^5 - 20 \times 2^2 = -48$ Therefore the minimum SP is at $(2, -48)$.

To check for any non-horizontal PIs, examine when the second derivative is 0:

$$\begin{aligned} f''(x) &= 0 \\ 20x^3 - 40 &= 0, \\ 20(x^3 - 2) &= 0 \\ x^3 &= 2 \Rightarrow x = \sqrt[3]{2} \end{aligned}$$

For x just below $\sqrt[3]{2}$, $f''(x) < 0$.
For x just above $\sqrt[3]{2}$, $f''(x) > 0$.
When $x = \sqrt[3]{2}$, $f(x) = \sqrt[3]{2}^5 - 20 \times \sqrt[3]{2} = -22.0$

Therefore, as $f(x)$ changes concavity at $x = \sqrt[3]{2}$, $(\sqrt[3]{2}, -22.0)$ is a non-horizontal point of inflection.



4 Maxima and Minima

As mentioned earlier, a function defined for a given closed interval will always have a maximum and minimum value. These extreme values will occur at either a maximum or minimum stationary point, at one of the end points of the interval or at a point where the derivative is not defined.

Example 15. Find the maximum and minimum values of the function

$$f(x) = x^3 - 3x^2$$

for x values in the closed interval $[-2, 3]$.

Process

First find the derivatives to identify the stationary points.

$$\begin{aligned} f(x) &= x^3 - 3x^2 \\ f'(x) &= 3x^2 - 6x \\ f''(x) &= 6x - 6 \end{aligned}$$

Stationary points therefore occur when $f'(x) = 0$ i.e.

$$\begin{aligned} 3x^2 - 6x &= 0 \\ 3x(x-2) &= 0 \\ \Rightarrow x = 0 &\quad \text{or} \quad x = 2 \end{aligned}$$

Using the second derivative test, now identify the nature of these SPs:
When $x = 0$, $f''(x) = -6 < 0$ therefore $(0, 0)$is a local maximum.
When $x = 2$, $f''(x) = 6 > 0$ therefore $(2, -4)$is a local minimum.

Now find the values of $f(x)$ at the end points of the closed interval.
$f(-2) = (-2)^3 - 3 \times (-2)^2 = -20$
$f(3) = 3^3 - 3 \times 3^2 = 0$
Therefore $f(x)$ has maximum value 0 and minimum value -20 in the closed interval $[-2, 3]$.

Sometimes functions are defined in multiple parts. An example of this is shown below.

Example 16. Find the maximum and minimum values of

$$f(x) = \begin{cases} 3x, & \text{if } x < 0 \\ x^2 - 1, & \text{if } x \geq 0 \end{cases}$$

in the closed interval $[-2, 2]$.

Process

First differentiate both parts of the function:

$$f'(x) = \begin{cases} 3, & \text{if } x < 0 \\ 2x, & \text{if } x \geq 0 \end{cases}$$

For $x < 0$, there are no stationary points of $f(x)$. For $x \geq 0$, stationary points occur when $2x = 0 \Rightarrow x = 0$. Using the second derivative, the nature of the SP can be identified.

$$f''(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2, & \text{if } x \geq 0 \end{cases}$$

Hence, as $f''(x) > 0$ for $x \geq 0$, there is a local minimum.
Evaluating the function to find the value of $f(x)$ at the point 0 gives $f(0) = 0^2 - 1 = 0$ and so $(0, 0)$ is a minimum SP.

Now consider the value of $f(x)$ at the end points of the closed interval.
When $x = -2$, $f(x) = 3 \times (-2) = -6$.
When $x = 2$, $f(x) = 2^2 - 1 = 3$.

Hence, the function $f(x)$ has maximum value 3 and minimum value -6 in the closed interval $[-2, 2]$.

Example 17. Find the maximum and minimum values of

$$f(x) = \begin{cases} 3x, & \text{if } x < 1 \\ -3x, & \text{if } x \geq 1 \end{cases}$$

in the closed interval $[-3, 3]$.

Process Identify any SPs by finding the derivatives of $f(x)$.

$$f'(x) = \begin{cases} 3, & \text{if } x < 1 \\ -3, & \text{if } x \geq 1 \end{cases}$$

Therefore, $f''(x) = 0$ for all values of x.

There are no stationary points as there are no points at which the derivative is 0.

The function is discontinuous at $x = 1$ and the derivative is therefore not defined at $x = 1$. Therefore, this is another point at which a maximum or minimum value could occur.

$$\begin{aligned} f(3) &= -3 \times 3 = -9 \\ f(-3) &= 3 \times -3 = -9 \\ f(1) &= -3 \times 1 = -3. \end{aligned}$$

Therefore, the function has a maximum value of -3 and a minimum value of -9 in the region $[-3, 3]$.

5 Odd and Even Functions

A function is **even** if it has reflectional symmetry across the y-axis. An **odd** function has rotational symmetry about the origin. Note that many functions are neither odd nor even. Knowing if a function is odd or even can help when sketching the function. These definitions can be defined more mathematically as:

A function is Even if $f(-x) = f(x)$ for all x in the domain of $f(x)$.
A function is Odd if $f(-x) = -f(x)$ for all x in the domain of $f(x)$.

The cosine function $\cos x$ is an example of an even function whereas $\sin x$ is an odd function.

Example 18. Show that $f(x) = x^4 - 6x^2 + 2$ is an even function.

Process

$$\begin{aligned} f(x) &= x^4 - 6x^2 + 2 \\ \Rightarrow f(-x) &= (-x)^4 - 6(-x)^2 + 2 \\ \rightarrow f(-x) &= x^4 - 6x^2 + 2 \\ \Rightarrow f(-x) &= f(x) \end{aligned}$$

Therefore $f(x)$ is an even function.

Example 19. Show that $f(x) = 3x^3 - 5x$ is an odd function

Process

$$\begin{aligned} f(x) &= 3x^3 - 5x \\ \Rightarrow f(-x) &= 3(-x)^3 - 5(-x) \\ \rightarrow f(-x) &= -3x^3 + 5x \\ \Rightarrow f(-x) &= -f(x) \end{aligned}$$

Therefore $f(x)$ is an odd function.

Example 20. Is the function

$$f(x) = 1 - \frac{1}{x}$$

odd, even or neither?

Process

$$\begin{aligned} f(x) &= 1 - \frac{1}{x} \\ \Rightarrow f(-x) &= 1 - \frac{1}{-x} \\ \Rightarrow f(-x) &= 1 + \frac{1}{x} \\ \Rightarrow f(-x) &\neq -f(x) \text{ and } f(-x) \neq f(x) \end{aligned}$$

Therefore the function is neither odd nor even.

Note that, for a product or sum of functions, it is possible to decide if the composite function is odd or even by looking at the function components.

- The sum of two even functions is even.
- The sum of two odd functions is odd.
- The sum of odd and even functions is neither even nor odd.
- The product of two even functions is even.
- The product of two odd functions is odd.
- The product of an odd and even function is odd.

Example 21. Is the function

$$f(x) = (3x^3 - 5x)(x^4 - 6x^2 + 2)$$

odd or even?

Process From the previous examples, $3x^3 - 5x$ is an odd function. $x^4 - 6x^2 + 2$ is an even function. Hence, $f(x)$ is odd.

6 Sketching Graphs of Functions

To sketch a graph of a function, use all of the previously learned strategies for identifying important features of a function. The following are vital in a sketch:

- Find points of intersection with the x and y axes.
- Identify any stationary points and their nature.
- Identify any non-horizontal points of inflection.
- Identify any asymptotes and consider the behaviour of the function near the asymptotes and as $x \to \pm\infty$.
- Determine if the function is odd, even or neither.

Example 22. Sketch the graph of the function

$$f(x) = \frac{1}{x+3}$$

Process

- On the x axis, $y = 0$. However, there are no values of x for which $y = 0$ and so there are no points of intersection of the x-axis. On the y-axis, $x = 0$ and so $f(x) = \frac{1}{3}$. Points on the axes are therefore $(0, \frac{1}{3})$.
- Identify any stationary points and their nature.

$$\begin{aligned} f(x) &= \frac{1}{x+3} = (x+3)^{-1} \\ f'(x) &= -(x+3)^{-2} = \frac{-1}{(x+3)^2} \\ f''(x) &= 2(x+3)^{-3} = \frac{2}{(x+3)^3} \end{aligned}$$

For SPs, $f'(x) = 0$. Therefore:

$$\frac{-1}{(x+3)^2} = 0$$

There are no values of x that would make $f'(x) = 0$ and so there are no stationary points.

- Identify any non-horizontal points of inflection.

$$\begin{aligned} f''(x) &= 0 \\ \Rightarrow \frac{2}{(x+3)^3} &= 0 \end{aligned}$$

There are no values of x which will make $f''(x) = 0$ and so there are no non-horizontal points of inflection.

- Identify any asymptotes and consider the behaviour of the function near the asymptotes and as $x \to \pm\infty$.

A vertical asymptote exists at $x = -3$.

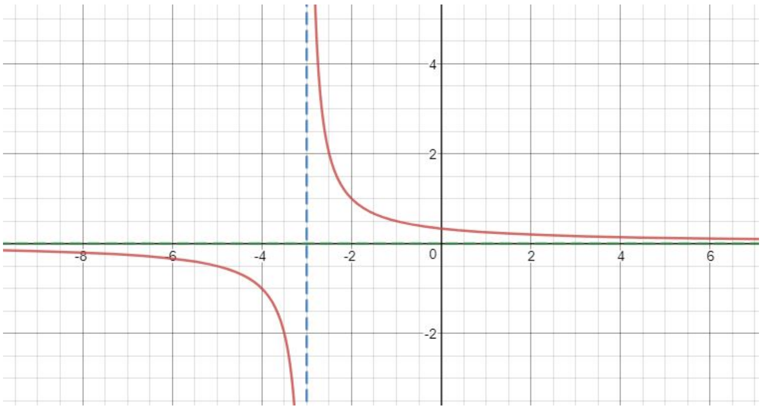
As $x \to \infty, f(x) \to 0^+$.
As $x \to -\infty, f(x) \to 0^-$.
Hence, the graph of $f(x)$ tends to 0 from above as $x \to \infty$ and the graph of $f(x)$ tends to 0 from below as $x \to -\infty$.
Thus, $x = 0$ is a horizontal asymptote of $f(x)$.

- Determine if the function is odd, even or neither.

$$\begin{aligned} f(x) &= \frac{1}{x+3} \\ f(-x) &= \frac{1}{-x+3} \end{aligned}$$

$f(x) \neq f(-x)$ and $f(x)$ is neither odd nor even.

Sketching the graph therefore gives.



Example 23. Sketch the graph of the function

$$f(x) = \frac{x^2}{1-x}$$

Process

- On the x-axis, $y = 0$. Hence,

$$\frac{x^2}{1-x} = 0 \Rightarrow x = 0$$

. Therefore $(0, 0)$ is the point on the x -axis.
On the y-axis, $x = 0$. Hence,

$$f(x) = \frac{0^2}{1-0} = 0.$$

This means that the only point on either axis is $(0, 0)$.

- Identify any stationary points and their nature.

$$\begin{aligned} f(x) &= \frac{x^2}{1-x} \\ f'(x) &= \frac{2x(1-x)-x^2(-1)}{(1-x)^2} \\ &= \frac{2x-2x^2+x^2}{(1-x)^2} \\ &= \frac{2x+x^2}{(1-x)^2} \\ &= \frac{x(2+x)}{(1-x)^2} \end{aligned}$$

For stationary points, $f'(x) = 0$ and so :

$$\begin{aligned} \frac{x(2+x)}{(1-x)^2} &= 0 \\ \Rightarrow x = 0 \ \text{ or } \ x = -2 \end{aligned}$$

There are stationary points at $x = 0$ and $x = 2$. Use the second derivative test to determine the nature of these SPs.

$$\begin{aligned} f'(x) &= \frac{2x+x^2}{(1-x)^2} \\ f''(x) &= \frac{(1-x)^2(2+2x)-(2x+x^2).2(1-x).(-1)}{(1-x)^4} \\ &= \frac{2(1-x)^2(1+x)+2x(2+x)(1-x)}{(1-x)^4} \\ &= \frac{2(1-x)\left[(1-x)(1+x)+x(2+x)\right]}{(1-x)^4} \\ &= \frac{2(1-x)\left[1-x^2+2x+x^2\right]}{(1-x)^4} \\ &= \frac{2(1+2x)}{(1-x)^3} \end{aligned}$$

When $x = 0$, $f''(x) = 2$ and so there is a minimum SP at $x = 0$. $f(0) = 0$ and so there is a local minimum SP at $(0, 0)$.
When $x = 2$, $f''(x) = -10$ and so there is a maximum SP at $x = 2$. $f(2) = -8$ and so there is a local minimum SP at $(2, -8)$.

- Identify any non-horizontal points of inflection. Look at points where the second derivative is 0 i.e.

$$\begin{aligned} \frac{2(1+2x)}{(1-x)^3} &= 0 \\ \Rightarrow 2(1+2x) &= 0 \\ \Rightarrow x &= -\frac{1}{2} \end{aligned}$$

  There is a point of inflection at $x = -\frac{1}{2}$.

- Identify any asymptotes and consider the behaviour of the function near the asymptotes and as $x \to \pm\infty$. To do this, first look at the denominator and identify any points for which the function is undefined. In this case, $x = 1$ is a vertical asymptote of the funtion. Next, use polynomial division to rewrite the function.

$$\begin{array}{r l} & \; - x - 1 \\ \cline{2-2} -x+1 \big) & x^2 \\ & -x^2 + x \\ \cline{2-2} & \phantom{-x^2 +{}} x \\ & - x + 1 \\ \cline{2-2} & \phantom{-x^2 - x + {}} 1 \end{array}$$

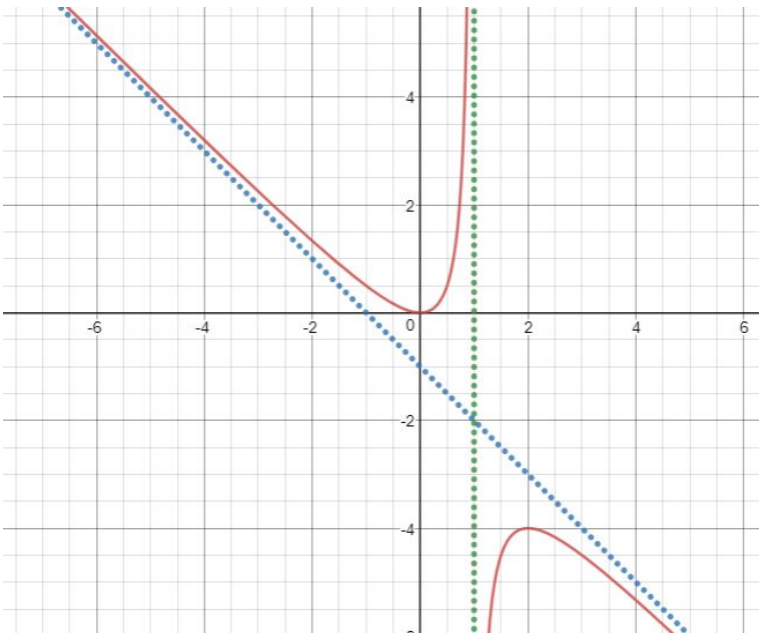
  Hence:

$$f(x) = -x - 1 + \frac{1}{1-x}$$

  Therefore, there is a slant asymptote at $y = -x-1$. As $x \to \infty^+$, $f(x) \to -x - 1$ from below. As $x \to \infty^-$, $f(x) \to -x - 1$ from above.

- Determine if the function is odd, even or neither.
  x^2 is an even function. $1 - x$ is an odd function. Therefore, $f(x)$ is odd.

- Now sketch the graph.



7 Graphs of Related Functions

When the graph of a function, $f(x)$, is known, the graphs of related functions can be relatively easily sketched without going through the time consuming sketching process detailed in the last section. For example:

- Reflect the graph.
  For the graph of $-f(x)$ reflect in the x-axis.
  For the graph of $f(-x)$ reflect in the y-axis.
  For the graph of $f^{-1}(x)$ reflect in the line $y = x$.

- Translate the graph.
  For the graph of $f(x - a)$, move a units in the x direction.
  For the graph of $f(x) + a$, move a units in the y direction.

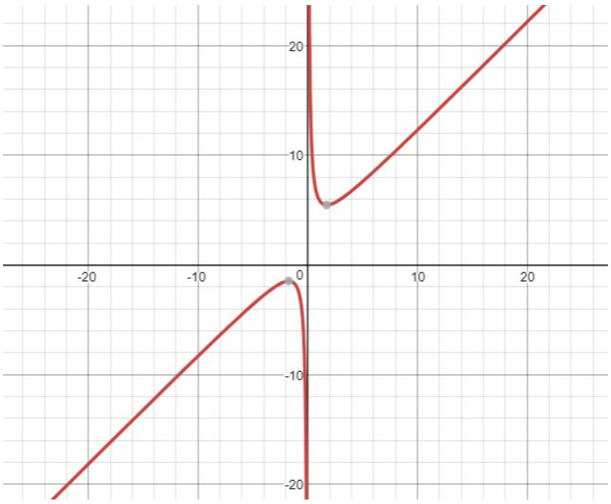
- Scale the graph.
  For the graph of $y = f(kx)$, stretch out in x direction for $0 < k < 1$.
  For the graph of $y = f(kx)$, squash/compress in x direction for $k > 1$.
  For the graph of $y = kf(x)$, stretch in the y direction.

It is useful to use an online graphing calculator tool such as Desmos (www.desmos.com) to see examples of how graphs of related functions are obtained.

Example 24. The graph of

$$f(x) = \frac{\left(x^2 \ + \ 2x \ + \ 3\right)}{x}$$

is shown below.

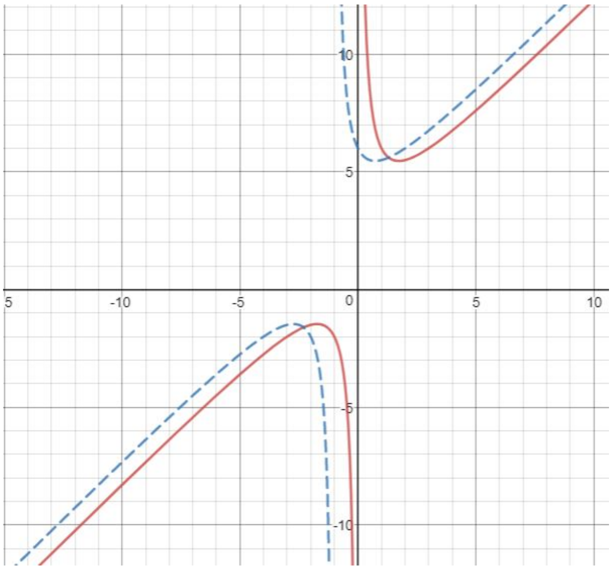


Use this to make sketches of (a) $f(x+1)$ and (b) $-f(x)$.

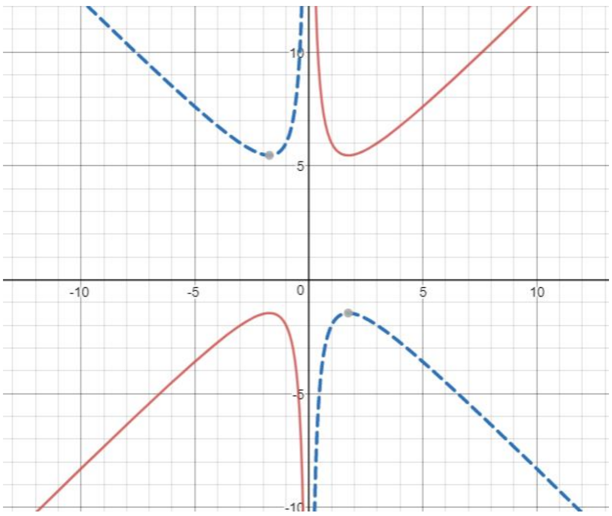
(a)

$$g(x) = \frac{\left((x+1)^2 + 2(x+1) + 3\right)}{x+1}$$

To sketch this graph, translate every point one unit to the left.



(b) $h(x) = f(-x)$. To sketch this graph, reflect the points in the y-axis.



7.1 The Modulus Function

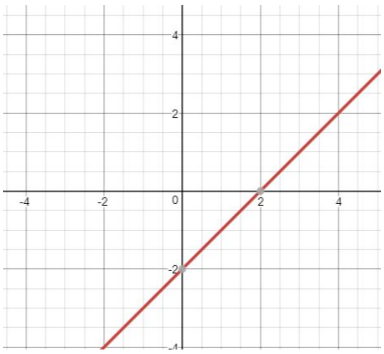
The modulus function defined as $f(x) = |x|$ is always positive. It denotes the absolute value (or size) of the function and is defined as:

$$f(x) == |x| = \begin{cases} x, & \text{if } f(x) \geq 0 \\ -x, & \text{if } f(x) < 0 \end{cases}$$

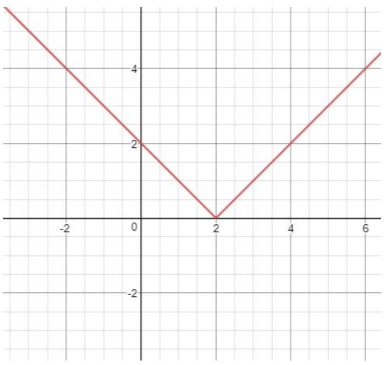
To draw the graph of the modulus function, reflect any part below the x-axis in the x-axis so that the whole graph lies above or on the x-axis.

Example 25. Sketch the graph of $|x - 2|$.

Process The graph of $y = x - 2$ is shown below.

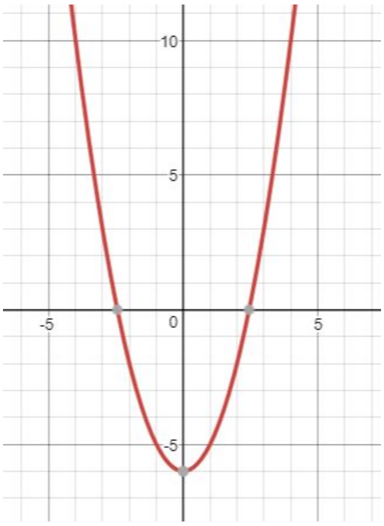


Reflect all of the points below the x-axis to make them positive to sketch the graph of $|x - 2|$.

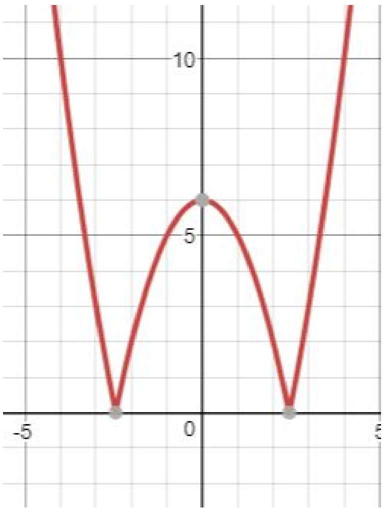


Example 26. Sketch the graph of $|x^2 - 6|$.

Process The graph of $y = x^2 - 6$ is shown below.



Reflecting the points below the x-axis gives:

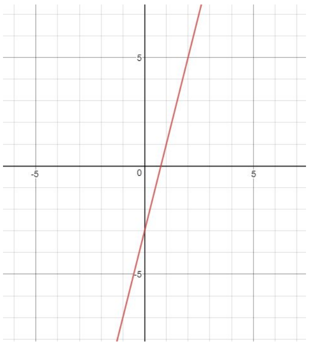


7.2 The Inverse Function

To sketch the graph of an inverse function , reflect in the line $y = x$.

Example 27. The graph of the function $y = 4x - 3$ is shown below. Sketch the graph of $f^{-1}(x)$ showing clearly the points of intersection with $f(x)$.

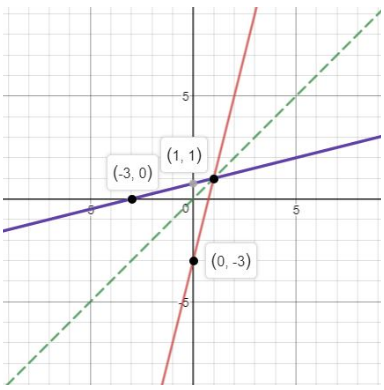
Process The graph of $y = 4x - 3$ is shown below.



Sketch in the line $y = x$. This is shown as the dashed line below. Then reflect to obtain the graph of $f^{-1}(x)$.
Points of intersection occur when:

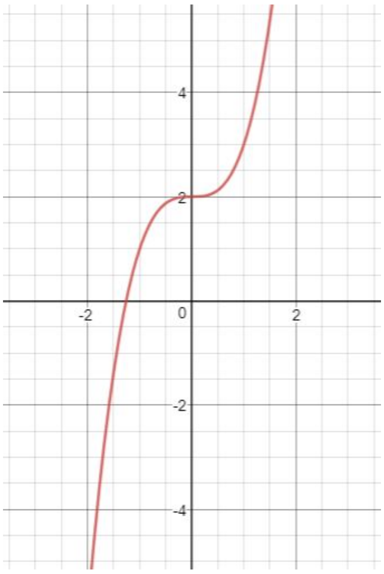
$$4x - 3 \ = \ x \Rightarrow \quad 3x = 3 \quad \Rightarrow \quad x = 1.$$

Therefore the point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ is $(1, 1)$.



Example 28. The graph of the function $y = x^3 + 2$ is shown below. Sketch the graph of $f^{-1}(x)$.

Process



Sketch in the line $y = x$. This is shown as the dashed line below. Then reflect to obtain the graph of $f^{-1}(x)$.