St Columba's High School
Advanced Higher Maths
Complex Numbers

Contents

1 Introduction

In this topic, we delve into a new area of maths: complex and imaginary numbers. All maths and numbers are based on a system that helps us model real life applications. However, sometimes the familiar whole numbers and integers are not sufficient to model intricate and complicated real life scenarios. Hence, imaginary and complex numbers were devised as a means to solve problems which could not be solved in a traditional way

Imaginary numbers are of enormous use in applied maths and physics. Complex numbers consist of the sum of real and imaginary numbers and complex analysis is the study of functions made up of complex variables. Complex numbers occur quite naturally in the study of quantum physics. They are also useful for modelling periodic motions (such as water or light waves) as well as alternating currents. Understanding complex analysis has enabled mathematicians to solve fluid dynamic problems, understand how to pump oil in oilrigs, model how earthquakes shake buildings and determine how electronic devices work.

Consider the equation $x^2 + 1 = 0$.

$$\begin{aligned} x^2 + 1 &= 0 \\ x^2 &= -1 \\ x &= \pm\sqrt{-1} \end{aligned}$$

Until now, we would have said that there are no solutions as we can't take the square root of a negative number (try it on your calculator, it will say error). Mathematicians devised a way to obtain solutions to problems like this by defining a new *imaginary* number $i = \sqrt{-1}$. With this new definition, the equation above has solutions $x = \pm i$.

The imaginary number, i is defined as $i = \sqrt{-1}$. Hence $i^2 = -1$.

2 Complex Numbers

Complex numbers do not mean complicated numbers. It simply means that a number is made up of a real and an imaginary part. The letter z is usually used to denote an imaginary number. Generally;

For $a,b \in \mathbb{R}$, the complex number z is given by

$$z = a + ib$$

where i is defined as $\sqrt{-1}$.
a is the real part of z and b is the imaginary part of z.

2.1 Arithmetic for Complex Numbers

Complex numbers may be added or subtracted by adding or subtracting the real and imaginary parts separately. For example: If $z_1 = 2+4i$ and $z_2 = 14 - 3i$

$$\begin{aligned} z_1 + z_2 &= 2 + 4i + 14 - 3i = 2 + 14 + 4i - 3i = 16 + i \\ z_2 - z_1 &= (14 - 3i) - (2 + 4i) = 14 - 2 - 3i - 4i = 12 - 7i \end{aligned}$$

Similarly, to multiply complex numbers, apply the normal rules for multiplying algebraic expressions.

$$\begin{aligned} 3z_1 &= 3(2 + 4i) = 6 + 12i \\ z_2 \times z_1 &= (14 - 3i)(2 + 4i) \\ &= 28 + 56i - 6i - 12i^2 \\ &= 28 + 50i - 12(-1) \quad \text{as} \quad i^2 = -1 \\ &= 28 - 50i + 12 \\ &= 40 - 50i. \end{aligned}$$

2.2 Solving Equations with Complex numbers

Note that two complex numbers are only equal if their real and imaginary parts are equal. This fact will allow complex equations to be solved by equating real and imaginary parts.

Example 1. Solve $5 + 6i = 3 - i + z$ for z where $z = a + ib$

Process

$$\begin{aligned} 5 + 6i &= 3 - i + z \\ 3 - i + z &= 5 + 6i \\ z &= 5 + 6i - 3 + i \\ z &= 2 + 7i \end{aligned}$$

Example 2. Solve $4 + 2i = (2 - i)z$ for z where $z = a + ib$

Process

$$\begin{aligned} 4 + 2i &= (2 - i)z \\ 4 + 2i &= (2 - i)(a + ib) \\ 4 + 2i &= 2a + 2ib - ai - i^2b \\ 4 + 2i &= 2a + (2b - a)i - (-1)b \\ 4 + 2i &= 2a + (2b - a)i + b \\ 4 + 2i &= 2a + b + (2b - a)i \end{aligned}$$

Equating real and imaginary parts gives:

$$\begin{aligned} 2a + b &= 4 \\ -a + 2b &= 2 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2a + b &= 4 \\ -2a + 4b &= 4 \end{aligned}$$

Solving simultaneously gives: $5b = 8 \Rightarrow b = \frac{8}{5}$. Substituting gives:

$$\begin{aligned} -a + 2(\frac{8}{5}) &= 2 \\ a &= \frac{16}{5} - 2 = \frac{6}{5} \end{aligned}$$

Hence $z = \frac{6}{5} + \frac{8}{5}i = \frac{1}{5}(6 + 8i)$.

2.3 Division of Complex Numbers

To divide by a complex number, we use a process which makes use of what is known as the **complex conjugate**.

The complex conjugate, $\bar{z}$ (pronounced z-bar), of a complex number $z = a + ib$ is defined as

$$\bar{z} = a - ib$$

When a complex number is multiplied by its conjugate, then the resultant is a whole number i.e. the imaginary part will have been removed as shown below:

$$\begin{aligned} z\bar{z} &= (a + ib)(a - ib) \\ &= a^2 - abi + abi - i^2b^2 \\ &= a^2 - (-1)b^2 \\ &= a^2 + b^2 \end{aligned}$$

To divide by a complex number, multiply numerator and denominator by the complex conjugate, replace i^2 by -1 and simplify.

Example 3. Find $(4+2i) \div (2-3i)$ giving an answer in the form $a+ib$ for $a, b \in \mathbb{R}$.

Process

$$\begin{aligned} (4+2i) \div (2-3i) &= \frac{(4+2i)}{(2-3i)} \\ &= \frac{(4+2i)}{(2-3i)} \times \frac{(2+3i)}{(2+3i)} \\ &= \frac{8+16i+6i^2}{4-6i+6i-9i^2} \\ &= \frac{8+16i+6(-1)}{4-6i+6i-9(-1)} \\ &= \frac{2+16i}{13} \end{aligned}$$

2.4 Square Roots of a Complex Number

To calculate a square root of a complex number, make an equation using $z = a + ib$, square both sides and equate real and imaginary parts.

Example 4. Calculate $\sqrt{3 - 4i}$ giving an answer in the form $a + ib$ where a and b are real numbers.

Process Let $\sqrt{3 - 4i} = a + ib$ and square both sides as shown:

$$\begin{aligned} \sqrt{3 - 4i} &= a + ib \\ 3 - 4i &= (a + ib)^2 \\ 3 - 4i &= a^2 + 2abi + i^2b^2 \\ 3 - 4i &= a^2 - b^2 + 2abi \quad (\text{as } \ i^2 = -1) \end{aligned}$$

Now compare real and imaginary parts:

$$\begin{aligned} a^2 - b^2 = 3 \\ 2ab = -4 \end{aligned} \Rightarrow \begin{aligned} a^2 - b^2 = 3 \\ a = -2/b \end{aligned}$$

Then substitute $a = \frac{1}{b}$ into $a^2 - b^2 = 4$.

$$\begin{aligned} a^2 - b^2 &= 3 \\ \frac{4}{b^2} - b^2 &= 3 \\ 4 - b^4 &= 3b^2 \\ b^4 + 3b^2 - 4 &= 0 \\ (b^2 + 4)(b^2 - 1) &= 0 \\ b &= \pm 1 \quad (\text{as } a, b \in \mathbb{R}) \end{aligned}$$

Substituting back gives:

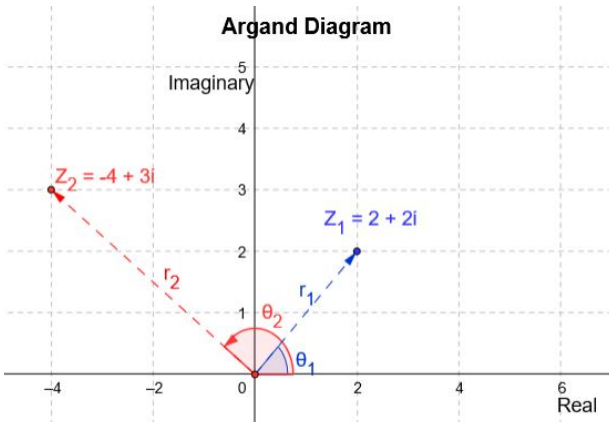
$$\begin{aligned} a &= \frac{-2}{1} \text{ or } a = \frac{-2}{-1} \\ \Rightarrow a &= -2 \text{ or } a = 2 \end{aligned}$$

Hence,

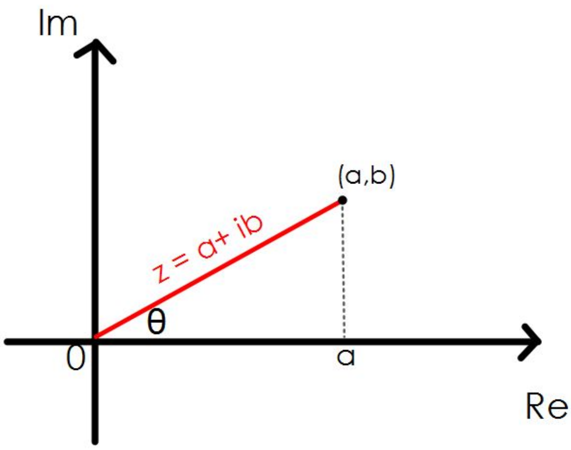
$$\sqrt{4 + 2i} = -2 + i \text{ or } 2 - i$$

3 Geometrical Representation

In many cases, it is useful to represent real numbers using a number line. In the case of complex numbers, this idea can be extended and the complex number $z = x + iy$ can be thought of as the point in a plane with cartesian co-ordinates $P = (x, y)$ and position vector $\vec{OP}$. This is known as an *Argand Diagram*. The x-axis represents the real part of the complex number and the y-axis represents the imaginary part.



Note that adding complex numbers can be considered as geometrically equivalent to adding together the position vectors of two points.



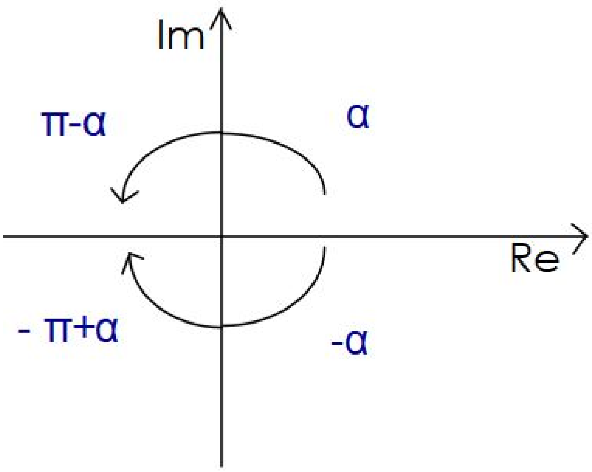
In a similar manner to finding the magnitude of a vector, we can find the **modulus**of a complex number.

> The **modulus**, $|z|$ of a complex number is the length of the line in the argand diagram and can be found by Pythagoras as
>
> $$|z| = \sqrt{a^2 + b^2}$$

For complex numbers, another aspect that can be calculated is the **argument**.

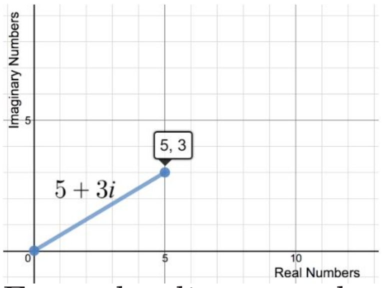
> The **argument**, θ, of a complex number is the size of the anti-clockwise angle (in radians) between the positive x-axis and the vector on the argand diagram.
>
> $$\tan\theta = \frac{b}{a}$$

The argument has values between $-pi$ and pi. This means that the angle must be obtained by considering the direction from the positive direction of the real axis and the following diagram:



Example 5. Let $z = 5 + 3i$. Represent z on an argand diagram and find the modulus and argument of z.

Process



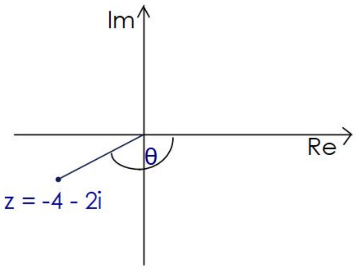
$$
\begin{aligned}
|z| &= \sqrt{5^2 + 3^2} \\
&= \sqrt{34}
\end{aligned}
$$

From the diagram, the complex number z is located in the first quadrant. The argument is calculated as follows:

$$\theta = \tan^{-1}\frac{3}{5} = 0.54$$

Example 6. Let $z = -4 - 2i$. Represent z on an argand diagram and find the modulus and argument of z.

Process



$$
\begin{aligned}
|z| &= \sqrt{(-4)^2 + (-2)^2} \\
&= \sqrt{20} \\
&= 2\sqrt{5}
\end{aligned}
$$

1.

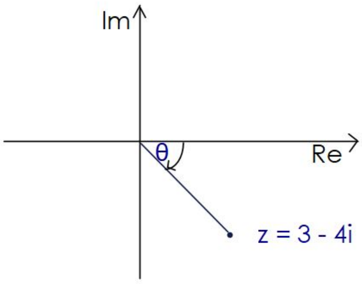
From the diagram, the complex number z is located in the third quadrant. Therefore, to find the argument, we first find the first quadrant acute angle,α, and use this to obtain the third quadrant obtuse angle θ.

$$
\begin{aligned}
\alpha &= \tan^{-1}\frac{2}{4} = \tan^{-1}\frac{1}{2} \\
&= 0.4636 \text{ radians}
\end{aligned}
$$

Hence, $\theta = -(\pi - 0.4636) = -2.6780$ radians. Note that the negative shows that the movement from the x-axis has been in a clockwise direction (the negative of the anti-clockwise direction).

Example 7. Let $z = 3 - 4i$. Represent z on an argand diagram and find the modulus and argument of z.

Process



$$
\begin{aligned}
|z| &= \sqrt{(3)^2 + (-4)^2} \\
&= \sqrt{25} \\
&= 5
\end{aligned}
$$

1.

From the diagram, the complex number z is located in the fourth quadrant. Therefore, to find the argument, we the fourth quadrant angle θ and make it negative to account for the fact that it is a clockwise direction rather than anti-clockwise.

$$
\begin{aligned}
\theta &= \tan^{-1} \frac{4}{3} \\
&= 0.9273 \text{ radians}
\end{aligned}
$$

4 Polar Form

Any complex number $z = a + ib$ can be written in **polar form** as

$$z = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ and $\theta = \arg z$.

Example 8. Write the complex number $z = 1 + i$ in polar form.

Process Calculate the modulus and argument.

$$\begin{aligned} r &= |z| \\ &= \sqrt{1^2 + 1^2} \\ &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \theta &= \tan^{-1}\frac{1}{1} \\ &= \tan^{-1} 1 \\ &= \frac{\pi}{4} \end{aligned}$$

Hence

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

Example 9. Write the complex number $z = 3 - 2i$ in polar form.

Process Calculate the modulus and argument.

$$\begin{aligned} r &= |z| \\ &= \sqrt{3^2 + (-2)^2} \\ &= \sqrt{13} \end{aligned}$$

$$\begin{aligned} \alpha &= \tan^{-1}\frac{2}{3} \\ &= 0.927 \\ \theta &= -\alpha = -0.927. \end{aligned}$$

Hence

$$z = \sqrt{13}\,(\cos -0.927 + i\sin -0.927)$$

5 Multiplication and Division

When multiplying or dividing complex numbers z_1 and z_2 given in polar form, the following rules apply:

Rule 1	Ensure that both z_1 and z_2 are written in polar form $z = a + ib$.
Rule 2	$\lvert z_1 z_2\rvert = \lvert z_1\rvert \times \lvert z_2\rvert$ and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
Rule 3	$\lvert z_1\rvert \div \lvert z_2\rvert = \lvert z_1\rvert \div \lvert z_2\rvert$ and $\arg(z_1 \div z_2) = \arg(z_1) - \arg(z_2)$
Rule 4	Multiply moduli and add arguments when multiplying. Divide moduli and subtract arguments when dividing
Rule 5	Ensure that the principle argument (between $-\pi$ and π) is in the final answer. This may mean that you have to add or subtract 2π from the argument to ensure that the final answer has argument between $-\pi$ and π.

Example 10. Simplify the expression

$$3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \times 4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$$

and give an answer in polar form.

Process

$$\begin{aligned}
&3(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}) \times 4(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})\\
&= 6\left[\cos(\frac{\pi}{2} + \frac{\pi}{6}) + i\sin((\frac{\pi}{2} + \frac{\pi}{6})\right]\\
&= 6\left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]
\end{aligned}$$

Example 11. Simplify the expression

$$5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \div 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

and give an answer in polar form.

$$\begin{aligned}
&5(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) \div 4(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) \\
&= \frac{5}{4}\left[\cos(\frac{\pi}{4} - \frac{\pi}{3}) + i\sin(\frac{\pi}{4} - \frac{\pi}{3})\right] \\
&= \frac{5}{4}\left[\cos(\frac{-\pi}{12}) + i\sin(\frac{-\pi}{12})\right]
\end{aligned}$$

Example 12. Simplify the expression

$$3(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}) \times \sqrt{2}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6})$$

and give an answer in polar form.

Process First rewrite the second complex number in the form $z = a + ib$ using the fact that cos is an even function and sin is an odd function. Hence, $\sin\frac{\pi}{6} = -\sin(\frac{-\pi}{6})$ and $\cos\frac{\pi}{6} = \cos(\frac{-\pi}{6})$.

$$\begin{aligned}
&3(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}) \times \sqrt{2}(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}) \\
&= 3\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right) \times \sqrt{2}\left(\cos(\frac{-\pi}{6}) + i\sin(\frac{-\pi}{6})\right) \\
&= 3\sqrt{2}\left(\cos(\frac{3\pi}{5} - \frac{\pi}{6}) + i\sin(\frac{3\pi}{5} - \frac{\pi}{6})\right) \\
&= 3\sqrt{2}\left(\cos(\frac{13\pi}{30}) + i\sin(\frac{13\pi}{30})\right)
\end{aligned}$$

6 De Moivre's Theorem

De Moivre's Theorem is used to calculate powers of a complex number written in polar form. It states that:

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

Example 13. Use De Moivre's Theorem to find z^5 where $z = 1 + \sqrt{3}i$ giving an answer in both polar form and cartesian form.

Process First write in polar form.

$$\begin{aligned} r &= |z| \\ &= \sqrt{(\sqrt{3})^2 + 1^2} \\ &= \sqrt{4} \\ &= 2 \end{aligned} \qquad \begin{aligned} \theta &= \tan^{-1} \frac{\sqrt{3}}{1} \\ &= \frac{\pi}{3} \end{aligned}$$

Hence $z = 2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})$

$$\begin{aligned} z^5 &= \left(2(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})\right)^5 \\ &= 2^5 \left(\cos 5\frac{\pi}{3} + i \sin 5\frac{\pi}{3}\right) \\ &= 32 \left(\frac{1}{2} + i(\frac{-\sqrt{3}}{2}\right) \\ &= 16 - 16\sqrt{3}i \end{aligned}$$

Note Remember to check your final answer in polar form to ensure that the argument is the principal argument (lying between $-\pi$ and π.

Example 14. Simplify the expression

$$\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^2 \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3$$

leaving answer in polar form.

Process Using De Moivre's Theorem.

$$\begin{aligned} & \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^2 \\ = \;& \cos\frac{2\pi}{2} + i\sin\frac{2\pi}{2} \\ = \;& \cos\pi + i\sin\pi \end{aligned}$$

$$\begin{aligned} & \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^3 \\ = \;& \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} \end{aligned}$$

Now, multiplying gives:

$$\begin{aligned} & (\cos\pi + i\sin\pi) \times \left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) \\ = \;& \cos(\pi + \frac{3\pi}{4}) + i\sin(\pi + \frac{3\pi}{4}) \\ = \;& \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} \\ = \;& \cos(\frac{7\pi}{4} - 2\pi) + i\sin(\frac{7\pi}{4} - 2\pi) \\ = \;& \cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4}) \end{aligned}$$

7 Roots of a Complex Number

Equations of the form $z^n = a\cos\theta + i\sin\theta$ can be solved by using De Moivre's theorem and noticing that adding multiples of 2π to θ will maintain the same position on an argand diagram. In general, if $z = r(\cos\theta + i\sin\theta)$, then:

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right), \quad k = 0, 1, \ldots n-1$$

Example 15. Solve the equation $z^3 = 8(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$ leaving the answers in polar form.

Process

$$\begin{aligned}
z_k &= z^{\frac{1}{3}} \qquad k = 0, 1, 2 \\
&= 8^{\frac{1}{3}}\left(\cos\frac{1}{3}\left(\frac{3\pi}{4} + 2k\pi\right) + i\sin\frac{1}{3}\left(\frac{3\pi}{4} + 2k\pi\right)\right),
\end{aligned}$$

When $k = 0$,

$$\begin{aligned}
z_0 &= 8^{\frac{1}{3}}\left(\cos\frac{1}{3}\left(\frac{3\pi}{4}\right) + i\sin\frac{1}{3}\left(\frac{3\pi}{4}\right)\right) \\
&= 2\left(\cos\frac{3\pi}{12} + i\sin\frac{3\pi}{12}\right) \\
&= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)
\end{aligned}$$

When $k = 1$,

$$\begin{aligned}
z_1 &= 2\left(\cos\left(\frac{3\pi}{4} + 2\pi\right) + i\sin\left(\frac{3\pi}{4} + 2\pi\right)\right) \\
&= 2\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)
\end{aligned}$$

When $k = 2$,

$$\begin{aligned} z_2 &= 2\left(\cos\left(\frac{3\pi}{4} + 4\pi\right) + i\sin\left(\frac{3\pi}{4} + 4\pi\right)\right) \\ &= 2\left(\cos\frac{19\pi}{12} + i\sin\frac{19\pi}{12}\right) \\ &= 2\left(\cos\frac{-5\pi}{12} + i\sin\frac{-5\pi}{12}\right) \\ &= 2\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right) \end{aligned}$$

The solutions are:

$$\begin{aligned} z_0 &= 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \\ z_1 &= 2\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right) \\ z_2 &= 2\left(\cos\frac{5\pi}{12} - i\sin\frac{5\pi}{12}\right) \end{aligned}$$

Note that the solutions will divide a circle with radius 2 into three equal sectors.

Example 16. Find the third roots of unity. (Note that this is equivalent to solving the equation $z^3 = 1$.

Process: First find the modulus and argument of $z = 1$ and rewrite in polar form. As $|z| = 1, \arg z = 0$, then $z = 1(\cos 0 + i \sin 0)$. Hence,

$$\begin{aligned}
z_k &= z^{\frac{1}{3}} \\
&= [1(\cos 0 + i \sin 0)]^{\frac{1}{3}} \\
&= 1^{\frac{1}{3}} \left(\cos \frac{1}{3}(0 + 2k\pi) + i \sin \frac{1}{3}(0 + 2k\pi) \right) \\
&= \left(\cos \frac{1}{3}(0 + 2k\pi) + i \sin \frac{1}{3}(0 + 2k\pi) \right) \\
\text{When } k = 0, \quad z_0 &= (\cos 0 + i \sin 0) \\
&= 1 \\
\text{When } k = 1, \quad z_1 &= \left(\cos \frac{1}{3}(0 + 2\pi) + i \sin \frac{1}{3}(0 + 2\pi) \right) \\
&= \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right) \\
&= -\frac{1}{2} + \frac{\sqrt{3}}{2} i \\
\text{When } k = 2, \quad z_2 &= \left(\cos \frac{1}{3}(0 + 4\pi) + i \sin \frac{1}{3}(0 + 4\pi) \right) \\
&= \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right) \\
&= \left(\cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3} \right) \\
&= -\frac{1}{2} - \frac{\sqrt{3}}{2} i
\end{aligned}$$

The solutions are:

$$\begin{aligned} z_0 &= 1 \\ z_1 &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ z_2 &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

These solutions divide a circle of radius 1 into three equal sectors.

8 Roots of a Polynomial

A polynomial has the same number of complex roots as its degree. Eg a cubic has three complex roots, a quartic has 4 etc. This is based on the Fundamental Theorem of Algebra. Note that if $z = a + ib$ is a root(solution), then the complex conjugate $z = a - ib$ is also a solution.

The process for finding all of the roots is the same as for finding roots of a polynomial covered in higher. Identify a root and use synthetic/polynomial division to find the other factor. Then use the quadratic formula, factorising or long division to find the remaining roots.

Example 17. Find all of the roots of the equation $z^3 - z^2 - z - 2 = 0$.

Process: By inspection, $(z - 2)$ is a root. Using polynomial division gives:

2	1	-1	-1	-2
	0	2	2	2
	1	1	1	0

Hence

$$(z-2)(z^2+z+1) = 0$$

and using the quadratic formula gives:

$$\begin{aligned} z &= \frac{-1 \pm \sqrt{1^2 - 4 \times 1 \times 1}}{2 \times 1} \\ &= \frac{-1 \pm \sqrt{-3}}{2} \\ &= \frac{-1 \pm \sqrt{3}i}{2} \\ &= \frac{-1}{2} + \frac{\sqrt{3}}{2}i \quad \text{or} \quad \frac{-1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

Example 18. Find the roots of $z^4 - 2z^3 + 8z^2 - 2z + 7 = 0$ given that one of the roots is $1 + i\sqrt{6}$

Process: As one root is $1 + i\sqrt{6}$, then, by the fundamental theorem of algebra, the conjugate $1 - i\sqrt{6}$ is also a root. Hence, $(z - (1 + \sqrt{6}i))$ and $(z - (1 - \sqrt{6}i))$ are factors. Multiplying these two factors together gives:

$$\begin{aligned}
(z \quad & -(1+\sqrt{6}i))(z-(1-\sqrt{6}i)) \\
= \quad & z^2 - (1-\sqrt{6}i)z + (1+\sqrt{6}i)z - (1-\sqrt{6}i)(1+\sqrt{6}i) \\
= \quad & z^2 - z - \sqrt{6}iz - z + \sqrt{6}iz - (1 - \sqrt{6}i + \sqrt{6}i - 6i^2) \\
= \quad & z^2 - 2z + 7
\end{aligned}$$

Dividing $z^4 - 2z^3 + 8z^2 - 2z + 7 = 0$ by $z^2 - 2z + 7$ using polynomial division gives:

			z^2		$+1$
$z^2 - 2z + 7$	z^4	$-2z^3$	$+8z^2$	$-2z$	$+7$
	z^4	$-2z^3$	$+7z^2$		
			z^2	$-2z$	$+7$
			z^2	$-2z$	$+7$
					0

So,

$$z^4 - 2z^3 + 8z^2 - 2z + 7 = (z^2 - 2z + 7)(z^2 + 1)$$

and

$$z^4 - 2z^3 + 8z^2 - 2z + 7 = 0 \Rightarrow (z^2 - 2z + 7)(z^2 + 1) = 0$$

The solutions to $z^2 + 1 = 0$ are $z = \pm i$. Therefore the solutions to the equation are:

$$1 + \sqrt{6}i, \ 1 - \sqrt{6}i, \ -i, \ i$$

Example 19. Show that i is a root of $z^4 + z^3 + 2z^2 + z + 1 = 0$

Process: Substitite $z = i$ into $z^4 + z^3 + 2z^2 + z + 1$. If the answer is 0, the i is a root.

$$\begin{aligned} & z^4 + z^3 + 2z^2 + z + 1 \\ = \; & i^4 + i^3 + 2i^2 + i + 1 \\ = \; & (-1)(-1) + i(-1) + 2(-1) + i + 1 \\ = \; & 1 - i - 2 + i + 1 \\ = \; & 0 \end{aligned}$$

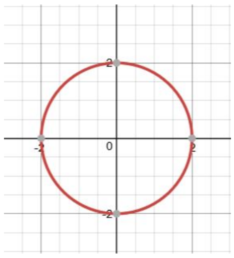
9 Representing Loci Geometrically

Restrictions can be set on complex numbers, eg $|z| < 2$. A set of points that follow such a rule is called a locus. The locus of $|z| = 2$ is the set of points that have a magnitude 2. These can be shown on a diagram.

Example 20. If $z = x + iy$, draw the locus of the point on the complex plane representing $|z| = 4$.

Process:

$$\begin{aligned} |z| &= 4 \\ |x+iy| &= 4 \\ \sqrt{x^2+y^2} &= 4 \\ x^2+y^2 &= 4^2 \end{aligned}$$

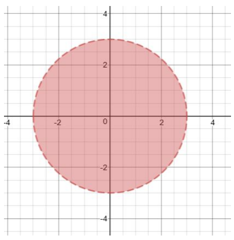


The points lie on the circumference of a circle with centre $(0, 0)$ and radius 4 as shown.

Example 21. If $z = x + iy$, draw the locus of the point on the complex plane representing $|z| < 3$.

Process:

$$\begin{aligned} |z| &< 3 \\ |x+iy| &< 3 \\ \sqrt{x^2+y^2} &< 3 \\ x^2+y^2 &< 9 \end{aligned}$$



The points lie inside circle with centre $(0, 0)$ and radius 3.

Example 22. If $z = x+iy$, find the equation of the locus $|z-2| > 3$and draw this locus on an Argand diagram.

Process

$$\begin{aligned} |z-2| &> 3 \\ |x+iy-2| &> 3 \\ |(x-2)+iy| &> 3 \\ \sqrt{(x-2)^2+y^2} &> 3 \\ (x-2)^2+y^2 &> 9 \end{aligned}$$

The equation $(x-2)^2 + y^2 = 9$ represents a circle with centre $(2, 0)$ and radius 3. Hence, as $(x-2)^2 + y^2 > 9$, the points lie outside of this circle. On the argand diagram, this can be shown as: