St Columba's High School
Advanced Higher Maths
Integration

Contents

1 Introduction and Revision

Integration is the inverse process to differentiation. Integration can be used to find the area under a curve. Some basic differentiation rules are:

$$\begin{aligned}
\int x^a dx &= \frac{x^{a+1}}{a+1} + c \\
\int f(x) + g(x) dx &= \int f(x)dx + \int g(x)dx \\
\int_a^b f(x)dx &= \int_a^c f(x)dx + \int_c^b f(x)dx
\end{aligned}$$

Example 1. Find $\int \cos(\pi - x)dx$ given that the integral passes through $(0, 1)$.

Process

$$\begin{aligned}
&\int \cos(\pi - x)dx \\
&= \sin(\pi - x) \div \frac{d}{dx}(\pi - x) + C \\
&= \frac{\sin(\pi - x)}{-1} + C \\
&= -\sin(\pi - x) + C
\end{aligned}$$

At $(0, 1)$, substitue in $x = 0$ and this can be evaluated to give:

$$\begin{aligned}
-\sin(\pi - 0) + C &= 1 \\
-\sin\pi + C &= 1 \\
C &= 1
\end{aligned}$$

Hence

$$\begin{aligned}
&\int \cos(\pi - x)dx \\
&= -\sin(\pi - x) + 1 \\
&= 1 - \sin(\pi - x)
\end{aligned}$$

Trig Identities

Using trigonometric identities can help in finding integrals. Some useful trig identities are shown in the table below:

$$
\begin{aligned}
\sin^2 x + \cos^2 x &= 1 \\
\frac{\sin x}{\cos x} &= \tan x \\
\sin(2x) &= 2\sin x \cos x \\
\cos(2x) &= 1 - 2\sin^2 x \Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x) \\
\cos(2x) &= 2\cos^2 x - 1 \Rightarrow \cos^2 x = \frac{1}{2}(1 + \cos 2x)
\end{aligned}
$$

A list of some standard integrals is given on the formula sheet. Remember that the integral is the opposite of the derivative and so standard integrals can be found from standard derivatives. . Some standard integrals are listed for you below:

$$
\begin{aligned}
\int x^n dx &= \frac{1}{n+1}x^{n+1} + C \\
\int \frac{1}{\sqrt{a^2 - x^2}} dx &= \sin^{-1}\frac{x}{a} + c \\
\int \frac{1}{a^2 + x^2} dx &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C \\
\int \sec^2(ax) dx &= \frac{1}{a}\tan(ax) + C \\
\int e^{ax} dx &= \frac{1}{a}e^{ax} + C \\
\int \frac{1}{x} dx &= \ln|x| + C
\end{aligned}
$$

Example 2. Integrate $\sin^2 x$ with respect to x.

Process

$$\begin{aligned} \int \sin^2 x dx &= \int \frac{1}{2}(1 - \cos 2x) dx \\ &= \frac{1}{2} \int 1 - \cos 2x dx \\ &= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right] + C \\ &= \frac{1}{2}x - \frac{1}{4}\sin 2x + C \end{aligned}$$

Example 3. Integrate

$$\frac{1}{\cos^2 x}$$

with respect to x.

Process Using the table of standard integrals on the formula sheet:

$$\begin{aligned} \int \frac{1}{\cos^2 x} dx &= \int \sec^2 x dx \\ &= \tan x + C \end{aligned}$$

Example 4. Find

$$\int \frac{1}{(3x+7)^3} dx$$

Process

2 Using Standard Results

Standard integrals are found from standard derivatives. The table provided above gives some standard integrals that can be used. This process is best demonstrated via examples:

Example 5. Find

$$\int \frac{1}{(3x+7)^3}dx$$

Process

$$\begin{aligned}\int \frac{1}{(3x+7)^3}dx &= \int (3x+7)^{-3}dx \\ &= \frac{(3x+7)^{-2}}{-2} \div \frac{d}{dx}(3x+7) + C \\ &= \frac{(3x+7)^{-2}}{-2\times 3} + C \\ &= \frac{-1}{6(3x+7)^2} + C\end{aligned}$$

Example 6. Find $\int \frac{1}{3x}dx$.

Process

$$\begin{aligned}&\int \frac{1}{3x}dx \\ &= \frac{1}{3}\int \frac{1}{x}dx \\ &= \frac{1}{3}\ln|x| + C\end{aligned}$$

Example 7. Find $\int 5e^{3x+1}dx$.

Process

$$\begin{aligned} \int &\quad 5e^{3x+1}dx \\ = &\quad 5\int e^{3x+1}dx \\ = &\quad 5e^{3x+1} \div \frac{d}{dx}(3x+1) + C \\ = &\quad \frac{5}{3}e^{3x+1} + C \end{aligned}$$

Standard Integrals and Trig. Functions

Example 8. Integrate $\int \sec^2(4x+1)dx$.

Process

$$\begin{aligned} \int &\quad \sec^2(4x+1) \\ = &\quad \frac{\tan(4x+1)}{4} + C \\ = &\quad \frac{1}{4}\tan(4x+1) + C \end{aligned}$$

Example 9. Integrate $\int_0^{\frac{\pi}{4}} 1 - \sin 2x dx$.

Process

$$\begin{aligned} \int_0^{\frac{\pi}{4}} &\quad 1 - \sin 2x dx \\ = &\quad \left[x - \frac{-\cos 2x}{2}\right]_0^{\frac{\pi}{4}} \\ = &\quad \left[\frac{\pi}{4} + \frac{1}{2}\cos\frac{2\pi}{4}\right] - \left[0 + \frac{1}{2}\cos 0\right] \\ = &\quad \frac{\pi}{4} - \frac{1}{2} \end{aligned}$$

Example 10. Integrate $\int_0^{\frac{\pi}{3}} \sin x + \sec^2 x dx$.

Process

$$\begin{aligned}
&\int_0^{\frac{\pi}{3}} \sin x + \sec^2 x dx \\
= &\ [-\cos x + \tan x]_0^{\frac{\pi}{3}} \\
= &\ \left[-\cos \frac{\pi}{3} + \tan \frac{\pi}{3}\right] - [-\cos 0 + \tan 0] \\
= &\ (-\frac{1}{2} + \sqrt{3}) - (-1 + 0) \\
= &\ -\frac{1}{2} + \sqrt{3} + 1 \\
= &\ \frac{1}{2} + \sqrt{3}
\end{aligned}$$

3 Integrals and Inverse Trig. functions

From the tables of standard derivatives, the inverse can be seen to give the standard integrals which are inverse trig functions.

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}dx &= \sin^{-1} x + C \\
\int \frac{1}{\sqrt{a^2-x^2}}dx &= \sin^{-1} \frac{x}{a} + C \\
\int \frac{1}{1+x^2}dx &= \tan^{-1} x + C \\
\int \frac{1}{a^2+x^2}dx &= \frac{1}{a}\tan^{-1} \frac{x}{a} + C
\end{aligned}$$

Example 11. Find

$$\int \frac{3}{\sqrt{16-x^2}}dx$$

Process

$$\begin{aligned}
\int \frac{3}{\sqrt{16-x^2}}dx &= 3\int \frac{1}{\sqrt{16-x^2}}dx \\
&= 3\left[\sin^{-1} \frac{x}{4}\right] + C \\
&= 3\sin^{-1}\left(\frac{x}{4}\right) + C
\end{aligned}$$

Example 12. Find

$$\int \frac{10}{\sqrt{16-49x^2}}dx$$

Process

$$\begin{aligned}
\int \frac{10}{\sqrt{16-49x^2}}dx &= 10\int \frac{1}{\sqrt{16-49x^2}}dx \\
&= 10\int \frac{1}{\sqrt{49(\frac{16}{49}-x^2)}}dx \\
&= 10\int \frac{1}{7\sqrt{\frac{16}{49}-x^2}}dx \\
&= \frac{10}{7}\int \frac{1}{\sqrt{\left(\frac{4}{7}\right)^2-x^2}}dx \\
&= \frac{10}{7}\sin^{-1}\left(\frac{x}{\frac{4}{7}}\right)+C \\
&= \frac{10}{7}\sin^{-1}\left(\frac{7x}{4}\right)+C
\end{aligned}$$

4 Integration and Partial Fractions

It is not always straightforward to integrate an algebraic fraction, particularly with improper fractions. To overcome this, use algebraic division or partial fractions to rewrite in a form that can be integrated.

Example 13. Integrate with respect to x,

$$\frac{x+5}{x+2}$$

Process First use polynomial division to rewrite the fraction:

$$\begin{array}{r|r} & 1 \\ \hline x+2) & x+5 \\ & -x-2 \\ \hline & 3 \end{array}$$

Hence

$$\frac{x+5}{x+2} = 1 + \frac{3}{x+2}$$

Now integrate:

$$\begin{aligned} \int \frac{x+5}{x+2}dx &= \int 1 + \frac{3}{x+2}dx \\ &= x + 3\ln|x+2| + C \end{aligned}$$

Example 14. Find

$$\int \frac{x^3+3x^2+7}{x^2-2x}dx$$

Process First, as the fraction is improper, divide using polynomial long division.

$$
\begin{array}{r}
x + 5 \\
x^2 - 2x \overline{\big) \; x^3 + 3x^2 \qquad\qquad + 7} \\
- x^3 + 2x^2 \qquad\qquad\quad \\
\overline{5x^2} \qquad\qquad\quad \\
- 5x^2 + 10x \qquad \\
\overline{10x} \qquad
\end{array}
$$

Now integrate as follows:

$$
\begin{aligned}
& \int \frac{x^3 + 3x^2 + 7}{x^2 - 2x} dx \\
= & \int \left(x + 5 + \frac{10x}{x^2 - 2x} \right) dx \\
= & \int \left(x + 5 + \frac{10}{x - 2} \right) dx \\
= & \frac{x^2}{2} + 5x + 10 \ln |x - 2| + C
\end{aligned}
$$

Example 15. Integrate with respect to x,

$$\frac{x^3}{(x+1)(x+2)}$$

Process This is a proper fraction as the degree of the numerator is less than the degree of the denominator. First rewrite using partial fractions. Let

$$\frac{x^3}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

Then, cross multiplying gives

$$x^3 = (x+2)A + (x+1)B$$

When $x = -2$, $-8 = -B \Rightarrow B = 8$. When $x = -1$, $-1 = 1A \Rightarrow A = -1$.

Hence

$$\frac{x^3}{(x+1)(x+2)} = \frac{-1}{x+1} + \frac{8}{x+2}$$

$$\begin{aligned}\int \frac{x^3}{(x+1)(x+2)}dx &= \int \frac{-1}{x+1} + \frac{8}{x+2}dx \\ &= -\ln|x+1| + 8\ln|x+2| + C\end{aligned}$$

5 Integration by Substitution

Sometimes an integration can be made simpler by making a substitution that changes the variable. It is used when integrating composite functions.

Example 16. Integrate $x^3(2x^4-3)^4$ using the substitution $u = 2x^4 - 3$.

Process Let $u = 2x^4 - 3$. Then;

$$\begin{aligned} u &= 2x^4 - 3 \\ \frac{du}{dx} &= 8x^3 \\ \Rightarrow du &= 8x^3 dx \\ \Rightarrow dx &= \frac{du}{8x^3} \end{aligned}$$

Hence, substututing in $2x^4 - 3 = u$ and $dx = \frac{du}{8x^3}$.

$$\begin{aligned} & \int x^3(2x^4-3)^4 dx \\ = & \int x^3.u^4.\frac{du}{8x^3} \\ = & \frac{1}{8}\int u^4 du \\ = & \frac{1}{8}.\frac{1}{5}u^5 + C \\ = & \frac{1}{40}u^5 + C \\ = & \frac{1}{40}(2x^4-3)^5 + C \end{aligned}$$

Example 17. Find the integral $\int \cos^3 x \sin x dx$ using the substitution $u = \cos x$.

Process

$$\begin{aligned} u &= \cos x \\ \frac{du}{dx} &= -\sin x \\ du &= -\sin x dx \\ \Rightarrow dx &= \frac{-du}{\sin x} \end{aligned}$$

Now subsitute $u = \cos x$ and $dx = \frac{-du}{\sin x}$.

$$\begin{aligned} & \int \cos^3 x \sin x dx \\ = & \int u^3 \sin x . \frac{-du}{\sin x} \\ = & -\int u^3 du \\ = & \frac{1}{4} u^4 + C \\ = & \frac{1}{4} \cos^4 x + C \end{aligned}$$

Definite Integrals

When using substitution to find a definite integral, it is important to remember to change the limits of the integral.

Example 18. Using the substitution $u = x^2 + 4x$, find

$$\int_0^1 (2x+4)(x^2+4x)^5 dx$$

Process

$$\begin{aligned} u &= x^2 + 4x \\ \frac{du}{dx} &= 2x + 4 \\ du &= (2x+4)dx \end{aligned}$$

When $x = 0,\ u = 0$ and when $x = 1,\ u = 5.$ Hence,

$$\begin{aligned} & \int_0^1 (2x+4)(x^2+4x)^5 dx \\ = & \int_0^1 (x^2+4x)^5 (2x+4) dx \\ = & \int_0^5 u^5 du \\ = & \left[\frac{1}{6}u^6\right]_0^1 \\ = & \frac{1}{6}5^5 - 0 \\ = & \frac{5^6}{6} \\ = & \frac{15625}{6} \end{aligned}$$

Example 19. Using the substitution $u = \sin x$, find $\int_0^{\frac{\pi}{6}} 10 \sin^4 x \cos x dx$.

Process

$$\begin{aligned} u &= \sin x \\ \frac{du}{dx} &= \cos x \\ du &= \cos x dx \end{aligned}$$

When $x = 0,\ u = 0$ and when $x = \frac{\pi}{6},\ u = \sin \frac{\pi}{6} = \frac{1}{2}$.

$$\begin{aligned} & \int_0^{\frac{\pi}{6}} 10 \sin^4 x \cos x dx \\ = & \int_0^{\frac{1}{2}} 10u^4 du \\ = & 10 \left[\frac{1}{5} u^5 \right]_0^{\frac{1}{2}} \\ = & 2 \left[(\frac{1}{2})^5 = 0^5 \right] \\ = & 2(\frac{1}{32}) \\ = & \frac{1}{16} \end{aligned}$$

Choosing an appropriate substitution

If a substitution is obvious, it may be expected to be identified without being expressly given. Look for functions that differentiate to give other parts of the composite function.

Example 20. Find $\int(6x+5)(3x^2+5x)^6dx$.

Process Notice that the derivative of $3x^2+5x$ is $6x+5$. therefore, a good choice for a substitution is $u = 3x^2+5x$.

$$\begin{aligned} u &= 3x^2+5x \\ \frac{du}{dx} &= 6x+5 \\ du &= (6x+5)dx \\ . \end{aligned}$$

Then, rearranging and substituting gives:

$$\begin{aligned} & \int (6x+5)(3x^2+5x)^6 dx \\ = & \int (3x^2+5x)^6 \times (6x+5)dx \\ = & \int u^6 du \\ = & \frac{1}{7}u^7 + C \\ = & \frac{1}{7}(3x^2+5x)^7 + C. \end{aligned}$$

Example 21. Find

$$\int \frac{(x+2)}{(3x^2+12x-7)^3}dx$$

Process Noting that the derivative of $3x^2+12x-7 = 6x+12 = 6(x+2)$, an appropriate substitution is $u = 3x^2+12x-7$.

$$\begin{aligned}
u &= 3x^2 + 12x - 7 \\
\frac{du}{dx} &= 6x + 12 \\
\frac{du}{dx} &= 6(x+2) \\
du &= 6(x+2)dx \\
\frac{du}{6} &= (x+2)dx
\end{aligned}$$

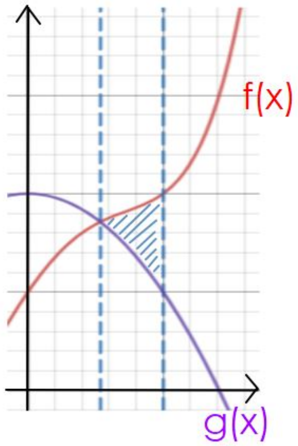
Rearranging and substituting gives:

$$\begin{aligned}
& \int \frac{(x+2)}{(3x^2+12x-7)^3}dx \\
= & \int \frac{1}{(3x^2+12x-7)^3} \times (x+2)dx \\
= & \int \frac{1}{u^3} \times \frac{1}{6}du \\
= & \frac{1}{6}\int u^{-3}du \\
= & \frac{1}{6}.\frac{1}{-2}u^{-2} + c \\
= & -\frac{1}{12u^2} + c \\
= & -\frac{1}{12(3x^2+12x-7)^2} + c
\end{aligned}$$

6 Applications of Integration

6.1 Revision Area under a curve

The integral of a function can be used to find the area enclosed between the function and the x-axis. Similarly, it can be used to find the are between two curves by subtracting the integrals of two functions.



Area between curves

$$= \int_a^b f(x) - g(x)dx$$

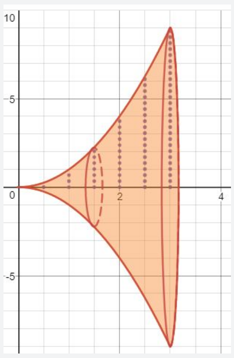
Example 22. Find the area enclosed by the curves $y = x^2$ and $y = 2x$ between the points $x = 0$ and $x = 1$.

Process Between $x = 0$ and $x = 1$, the line $y = 2x$ lies above the curve $y = x^2$. Therefore;

$$\begin{aligned} &\int_0^1 2x - x^2 dx \\ &= \left[\frac{2x^2}{2} - \frac{1}{3}x^3\right]_0^1 \\ &= (1 - \frac{1}{3}) - (0 - 0) \\ &= \frac{2}{3} \end{aligned}$$

6.2 Volumes - Rotation about the x-axis

If a curve is rotated around the x-axis, it makes a 3D solid. The volume of this solid can be calculated by calculating the area of a circle with radius equal to the distance from the x-axis to the curve and then integrating this area between two given points.



Volume of revolution

$$= \int_a^b \pi y^2 dx$$

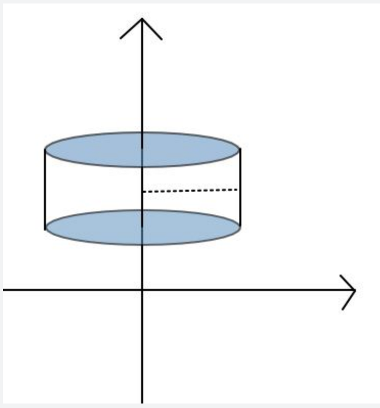
Example 23. Find the volume of the solid obtained by rotating the curve $y = x^2 + 2$ through 360^o about the x-axis between $x = 1$ and $x = 2$.

Process

$$\begin{aligned}
\text{Volume} &= \int_1^2 \pi(x^2+2)^2 dx \\
&= \int_1^2 \pi(x^4+4x^2+4)dx \\
&= \pi \int_1^2 x^4+4x^2+4dx \\
&= \pi \left[\frac{1}{5}x^5+\frac{4}{3}x^3+4x\right]_1^2 \\
&= \pi \left[\left(\frac{1}{5}2^5+\frac{4}{3}2^3+4(2)\right)-\left(\frac{1}{5}1^5+\frac{4}{3}1^3+4(1)\right)\right] \\
&= \frac{293}{15}\pi
\end{aligned}$$

6.3 Volume - Rotation about the y - axis

Similarly, if a curve or line is rotated around the y-axis, the volume can be calculated by finding the area of the circle with radius equal to the distance between the y-axis and a point on the line or curve and then integrating . This is calculated using the following formula:



$$\text{Volume} = \int_a^b \pi x^2 dy$$

where x is a function given in terms of y.

Example 24. Find the volume of the solid obtained by rotating the curve $y = x^2 + 2$ through 360^o about the y-axis between $x = 1$ and $x = 2$.

$$\begin{aligned} V &= \int_3^6 \pi \left[\sqrt{y-2}\right]^2 dy \\ &= \pi \int_3^6 y - 2dy \\ &= \pi \left[\frac{1}{2}y^2 - 2y\right]_3^6 \\ &= \frac{15\pi}{2} \end{aligned}$$

$$\begin{aligned} y &= x^2 + 2 \\ \Rightarrow x^2 &= y - 2 \\ \Rightarrow x &= \sqrt{y-2} \end{aligned}$$

$$\begin{aligned} \text{When } x &= 1,\ y = 3 \\ \text{When } x &= 2,\ y = 6 \end{aligned}$$

7 Integration by Parts

In differentiation, the product rule is used to differentiate a product of functions when no obvious substitution can be used. When integrating a product of functions, we use the process of integration by parts. With this, a mixture of integration and can be used to evaluate the integral. It is possible to derive this formula from the product rule (see Leckie and Leckie Textbook, page 84). The formula used for integration by parts is:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

where the function to be integrated is written as $u\frac{dv}{dx}$.

Example 25. Find the integral of $3x\sin x$.

Process

To use integration by parts, let

$$\begin{aligned} u &= 3x \\ \Rightarrow \frac{du}{dx} &= 3 \end{aligned}$$

Similarly, let

$$\begin{aligned} \frac{dv}{dx} &= \sin x \\ \Rightarrow v &= \int \sin x dx = -\cos x \end{aligned}$$

Now, substitute into the formula for integration by parts to obtain:

$$\begin{aligned} & \int 3x\sin x dx \\ = \ & uv - \int v\frac{du}{dx}dx \\ = \ & 3x(-\cos x) - \int(-\cos x)\times 3dx \\ = \ & -3x\cos x + 3\int \cos x dx \\ = \ & -3x\cos x + 3\sin x + C \end{aligned}$$

Example 26. Evaluate $\int_1^3 x^4 \ln x dx$.

Process

To use integration by parts, let

$$\begin{aligned} u &= \ln x \\ \Rightarrow \frac{du}{dx} &= \frac{1}{x} \end{aligned}$$

Similarly, let

$$\begin{aligned} \frac{dv}{dx} &= x^4 \\ \Rightarrow v &= \int x dx = \frac{1}{5}x^5 \end{aligned}$$

Now, substitute into the formula for integration by parts to obtain:

$$\begin{aligned} & \int x^4 \ln x dx \\ &= uv - \int v \frac{du}{dx} dx \\ &= \ln x . \frac{1}{5}x^5 - \int \frac{1}{5}x^5 . \frac{1}{x} dx \\ &= \frac{x^5 \ln x}{5} - \frac{1}{5}\int \frac{x^5}{x} dx \\ &= \frac{x^5 \ln x}{5} - \frac{1}{5}\int x^4 dx \\ &= \frac{x^5 \ln x}{5} - \frac{1}{25}x^5 + C \end{aligned}$$

Now, evaluating the integral for the limits 1 and 3 gives:

$$\begin{aligned} & \int_1^3 x^4 \ln x dx \\ &= \left[\frac{x^5 \ln x}{5} - \frac{1}{25}x^5\right]_1^3 \\ &= \left[\frac{(3)^5 \ln 3}{5} - \frac{1}{25}(3)^5\right] - \left[\frac{(1)^5 \ln 1}{5} - \frac{1}{25}(1)^5\right] \\ &\approx 43.71 \end{aligned}$$

Integration by Parts - Using the process twice

Sometimes the process of integrating by parts must be applied multiple times in order to obtain the derivative.

Example 27. Find $\int x^2 \sin x dx$.

Process Start the process of integrating by parts.

$$\begin{aligned} u &= x^2 \\ \frac{du}{dx} &= 2x \end{aligned} \qquad \begin{aligned} \frac{dv}{dx} &= \sin x \\ v &= \int \sin x dx = -\cos x \end{aligned}$$

Now, using integration by parts:

$$\begin{aligned} &\int x^2 \sin x dx \\ &= uv - \int v \frac{du}{dx} dx \\ &= x^2(-\cos x) - \int (-\cos x) \times 2x dx \\ &= -x^2 \cos x + 2 \int x \cos x dx \end{aligned}$$

Notice that the integral that remains is still a product of functions and so integration by parts must be applied again to the integral $\int x \cos x dx$. In this case, let the substitutions be:

$$\begin{aligned} u &= x \\ \frac{du}{dx} &= 1 \end{aligned} \qquad \begin{aligned} \frac{dv}{dx} &= \cos x \\ v &= \int \cos x dx = \sin x \end{aligned}$$

Using integration by parts for the second time gives:

$$
\begin{aligned}
& \int x^2 \sin x dx \\
= \; & -x^2 \cos x + 2\left[uv - \int v \frac{du}{dx} dx\right] \\
= \; & -x^2 \cos x + 2\left[x \sin x - \int \sin x dx\right] \\
= \; & -x^2 \cos x + 2x \sin x + 2 \cos x dx + C
\end{aligned}
$$

Example 28.