St Columba's High School
Advanced Higher Maths
Partial Fractions

Contents

1 Introduction to Partial Fractions

In integration and differentiation, it can be difficult to deal with complex algebraic fractions. Partial fractions are used to rewrite algebraic fractions in separate terms where the denominator of each of the new terms is a factor of the original denominator.

There are different ways to find partial fractions depending on the form and degree of the original denominator. Each of these will now be shown.

2 Denominators with Linear factors

When the denominator of the original fraction is made up of distinct linear factors, we **decompose** the function into the sum of two simpler functions using the general form

$$\frac{A}{ax+b}$$

Note

There may be more than two factors in the denominator, in these cases, use three or more partial fractions.

Example 1. Decompose

$$\frac{4x+1}{(x+1)(x-2)}$$

into partial fractions.

Process

The fraction is written in a general form as:

$$\frac{4x+1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)}$$

Multiplying through by $(x+1)(x-2)$ gives:

$$4x+1 = (x-2)A + (x+1) \tag{1}$$

To find the value of A, look for the value of x that will make $(x+1) = 0$. Hence, set $x = -1$ in (1).

When $x = -1$,

$$\begin{aligned} 4x+1 &= (x-2)A + (x+1)B \\ 4(-1)+1 &= (-1-2)A + (-1+1)B \\ -4+1 &= -3A \\ -3 &= -3A \\ A &= 1 \end{aligned}$$

To find the value of B, look for the value of x that will make $x-2) = 0$. Hence, choose $x = 2$ and sub into (1).
When $x = 2$,

$$\begin{aligned} 4x+1 &= (x-2)A + (x+1)B \\ 4(2)+1 &= (2-2)A + (2+1)B \\ 8+1 &= 3B \\ 9 &= 3B \\ B &= 3 \end{aligned}$$

Hence,

$$\frac{4x+1}{(x+1)(x-2)} = \frac{1}{(x+1)} + \frac{3}{(x-2)}$$

Example 2. Sometimes the denominator must first be factorised before partial fractions can be found.

Decompose

$$\frac{5x-11}{x^2-4x+3}$$

into partial fractions.

Process

Factorising $x^2 - 4x + 3$ gives $(x-3)(x-1)$. Hence,

$$\frac{5x-11}{x^2-4x+3} = \frac{5x-11}{(x-3)(x-1)}$$

This can be rewritten as:

$$\frac{5x-11}{(x-3)(x-1)} = \frac{A}{(x-3)} + \frac{B}{(x-1)}$$

.
Multiplying through by $(x-3)(x-1)$ gives:

$$5x - 11 = (x-1)A + (x-3)B \tag{2}$$

Let x = 3, then (2) becomes:

$$\begin{aligned} 5x - 11 &= (x-1)A + (x-3)B \\ 5(3) - 11 &= (3-1)A + (3-3)B \\ 4 &= 2A \\ A &= 2 \end{aligned}$$

Let x = 1, then (2) becomes:

$$\begin{aligned}
5x - 11 &= (x-1)A + (x-3)B \\
5(1) - 11 &= (1-1)A + (1-3)B \\
-6 &= -2B \\
B &= 3
\end{aligned}$$

Hence,

$$\frac{5x-11}{(x-3)(x-1)} = \frac{2}{(x-3)} + \frac{3}{(x-1)}$$

3 Irreducible Quadratic in denominator

Some quadratic functions cannot be factorised into linear factors. This is called an irreducible quadratic. Remember that the discriminant can be used to check if a quadratic has real roots. If an irreducible quadratic appears as the denominator, an alternative method must be used to find partial fractions using the general form

$$\frac{A}{(ax+b)} + \frac{Bx+C}{cx^2+dx+e}$$

.

Example 3. Express

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)}$$

in partial fractions.

Process

The quadratic in the denominator is irreducible. Write

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)} = \frac{A}{(x+1)} + \frac{Bx+C}{x^2+2x+2}$$

Then multiply through by $(x+1)(x^2+2x+2)$ to get:

$$3x^2+2x+1 = (x^2+2x+2)A + (x+1)(Bx+C) \tag{3}$$

Set $x = -1$ in (3) to find A.

$$\begin{aligned}
3x^2+2x+1 &= (x^2+2x+2)A + (x+1)(Bx+C) \\
3(-1)^2+2(-1)+1 &= ((-1)^2+2(-1)+2)A + (-1+1)(B(-1)+C) \\
3-2+1 &= (1-2+2)A \\
2 &= A \\
A &= 2
\end{aligned}$$

Now choose any values for x to find B and C.
Set $x = 0$ and use A = 2 in (3) to find C.

$$\begin{aligned}
3x^2 + 2x + 1 &= (x^2 + 2x + 2)A + (x + 1)(Bx + C) \\
3(0)^2 + 2(0) + 1 &= ((0)^2 + 2(0) + 2)A + (0 + 1)(B(0) + C) \\
1 &= 2A + C \\
1 &= 4 + c \\
C &= -3
\end{aligned}$$

Set $x = 1$ in (3) and use the values obtained for A and C to find B.

$$\begin{aligned}
3x^2 + 2x + 1 &= (x^2 + 2x + 2)A + (x + 1)(Bx + C) \\
3(1)^2 + 2(1) + 1 &= ((1)^2 + 2(1) + 2)A + (1 + 1)(B(1) + C) \\
6 &= 5A + 2(B + C) \\
6 &= 5(2) + 2(B + -3) \\
6 &= 10 + 2B - 6 \\
2 &= 2B \\
B &= 1
\end{aligned}$$

Hence

$$\frac{3x^2 + 2x + 1}{(x + 1)(x^2 + 2x + 2)} = \frac{2}{(x + 1)} + \frac{x - 3}{x^2 + 2x + 2}$$

Example 4. Sometimes the denominator must be factorised prior to decomposing into partial fractions. Eg.

$$\frac{x^2 - 10x - 8}{x^3 - 8}$$

The denominator is $x^3 - 8$. To factorise, use synthetic division. $(x - 2)$ is a factor and so,

2		1	0	0	-8
			2	4	8
		1	2	4	0

Therefore $x^3 - 8 = (x-2)(x^2+2x+4)$. This quadratic is irreducible and so the general form will be:

$$\frac{x^2-10x-8}{x^3-8} = \frac{x^2-10x-8}{(x-2)(x^2+2x+4)} = \frac{A}{(x-2)} + \frac{Bx+c}{x^2+2x+4}$$

Multiplying through by $(x-2)(x^2+2x+4)$ gives:

$$x^2-10x-8 = (x^2+2x+4)A + (x-2)(Bx+C) \tag{4}$$

Set $x=2$ in 4 gives:

$$\begin{aligned}
x^2-10x-8 &= (x^2+2x+4)A + (x-2)(Bx+C) \\
2^2-10(2)-8 &= (2^2+2(2)+4)A + (2-2)(B(2)+C) \\
4-20-8 &= 12A \\
-24 &= 12A \\
A &= -2
\end{aligned}$$

Setting $x=0$ and substituting $A=-2$ in (4) gives:

$$\begin{aligned}
x^2-10x-8 &= (x^2+2x+4)A + (x-2)(Bx+C) \\
0^2-10(0)-8 &= (0^2+2(0)+4)A + (0-2)(B(0)+C) \\
-8 &= 4A-2C \\
-8 &= -8-2C \\
C &= 0
\end{aligned}$$

Setting $x=1$ and substituting $A=-2, C=0$ in (4) gives:

$$\begin{aligned}
x^2 - 10x - 8 &= (x^2 + 2x + 4)A + (x - 2)(Bx + C) \\
1^2 - 10(1) - 8 &= (1^2 + 2(1) + 4)A + (1 - 2)(B(1) + C) \\
-17 &= 7A - B - C \\
-17 &= -14 - B - 0 \\
B &= 3
\end{aligned}$$

Hence,

$$\frac{x^2 - 10x - 8}{x^3 - 8} = \frac{-2}{(x - 2)} + \frac{3x}{x^2 + 2x + 4}$$

4 Fractions with a repeated linear factor

When one of the terms in the denominator is repeated, a third method must be used to find partial fractions. The general form that is used is:

$$\frac{A}{ax + b} + \frac{B}{(ax + b)^2}$$

Example 5. Express in partial fractions

$$\frac{x^2 - 7x + 9}{(x + 2)(x - 1)^2}$$

Process

Write

$$\frac{x^2 - 7x + 9}{(x + 2)(x - 1)^2} = \frac{A}{x + 2} + \frac{B}{(x - 1)} + \frac{C}{(x - 1)^2}$$

Multiply through by $(x + 2)(x - 1)^2$ to get:

$$x^2 - 7x + 9 = A(x - 1)^2 + B(x + 2)(x - 1) + C(x + 2) \qquad (5)$$

Let $x = 1$ in (5) to get:

$$\begin{aligned} x^2 - 7x + 9 &= A(x-1)^2 + B(x+2)(x-1) + C(x+2) \\ 1^2 - 7(1) + 9 &= A(1-1)^2 + B(1+2)(1-1) + C(1+2) \\ 3 &= 3C \\ C &= 1. \end{aligned}$$

Let $x = -2$ in (5) to get:

$$\begin{aligned} x^2 - 7x + 9 &= A(x-1)^2 + B(x+2)(x-1) + C(x+2) \\ (-2)^2 - 7(-2) + 9 &= A(-2-1)^2 + B(-2+2)(-2-1) + C(-2+2) \\ 27 &= 9A \\ A &= 3. \end{aligned}$$

Let $x = 0$ in (5) and sustitute in $A = 3, C = 1$ to get:

$$\begin{aligned} x^2 - 7x + 9 &= A(x-1)^2 + B(x+2)(x-1) + C(x+2) \\ (0)^2 - 7(0) + 9 &= 3(0-1)^2 + B(0+2)(0-1) + 1(0+2) \\ 9 &= 3 - 2B + 2 \\ -2B &= 4 \\ B &= -2. \end{aligned}$$

Hence,

$$\frac{x^2 - 7x + 9}{(x+2)(x-1)^2} = \frac{3}{x+2} - \frac{2}{(x-1)} + \frac{1}{(x-1)^2}$$

Example 6. Sometimes the denominator must first be factorised. Consider

$$\frac{2x^2 + 7x + 3}{x^3 + 2x^2 + x}$$

Process

The denominator $x^3 + 2x^2 + x$ can be factorised as shown

$$x(x^2 + 2x + 1) = x(x + 1)(x + 1) = x(x + 1)^2$$

Write

$$\frac{2x^2 + 7x + 3}{x(x + 1)^2} = \frac{A}{x} + \frac{B}{(x + 1)} + \frac{C}{(x + 1)^2}$$

Multiply through by $x(x + 1)^2$ to get:

$$2x^2 + 7x + 3 = (x + 1)^2 A + x(x + 1)B + xC \qquad (6)$$

Let $x = -1$ in (6) to get:

$$\begin{aligned} 2x^2 + 7x + 3 &= (x + 1)^2 A + x(x + 1)B + xC \\ 2(-1)^2 + 7(-1) + 3 &= (-1 + 1)^2 A + x(-1 + 1)B + (-1)C \\ -2 &= -C \\ C &= 2. \end{aligned}$$

Let $x = 0$ in (6) to get:

$$\begin{aligned} 2x^2 + 7x + 3 &= (x + 1)^2 A + x(x + 1)B + xC \\ 2(0)^2 + 7(0) + 3 &= (0 + 1)^2 A + 0(0 + 1)B + (0)C \\ 3 &= A \\ A &= 3. \end{aligned}$$

Let $x = 1$ and sub in $A = 3, C = 2$ in (6) to get:

$$\begin{aligned} 2x^2 + 7x + 3 &= (x + 1)^2 A + x(x + 1)B + xC \\ 2(1)^2 + 7(1) + 3 &= (1 + 1)^2 \times 3 + 1(1 + 1)B + (1)(2) \\ 12 &= 12 + 2B + 2 \\ -2B &= 2 \\ B &= -1. \end{aligned}$$

Hence

$$\frac{2x^2+7x+3}{x(x+1)^2} = \frac{3}{x} - \frac{1}{(x+1)} + \frac{2}{(x+1)^2}$$

5 Dividing Improper Algebraic Fractions

When the degree of the polynomial on the numerator is greater than on the denominator, the fraction must be written as a polynomial and a proper rational fraction. This is similar to writing an improper fraction as the sum of a whole number and a proper fraction. Polynomial division is used to re-write the fraction. Once this is complete, then partial fractions can be found if required.

Example 7. Write as the sum of a polynomial and a proper rational fraction.

$$\frac{x^3+4x^2-x+2}{x^2+x}.$$

Process

A process similar to normal long division is used. First consider the highest power of x in the numerator. What must the denominator be multiplied by to give this power? In this case, multiplying by x gives $x^3 - x^2$ and so x goes on the quotient and $x^3 - x^2$ is written below the original denominator. Subtracting leaves $3x^2 - x + 2$. Now consider what the original denominator can be multiplied by to give the $3x^2$. In this example, multiplying by 3 would give the required result and so 3 is written on the quotient and $3x^2 + 3x$ is written on the next line of the division. Subtracting leaves $-4x + 2$. As this is of a lower degree than x^2+x, the division is finished and $-4x+2$ is the remainder. This can then be written in polynomial and proper rational fraction form.

			x	$+3$
x^2+x	x^3	$+4x^2$	$-x$	$+2$
	x^3	x^2		
		$3x^2$	$-x$	$+2$
		$3x^2$	$+3x$	
			$-4x$	$+2$

Therefore,

$$\frac{x^3+4x^2-x+2}{x^2+x} = (x+3) + \frac{-4x+2}{x^2+x}$$

Example 8. Write as a polynomial function and proper rational fraction.

$$\frac{x^3-3x}{x^2-x-2}$$

Process

			x	-2
x^2-x-2	x^3	$-3x^2$		
	x^3	$-x^2$	$-2x$	
		$-2x^2$	$+2x$	
		$-2x^2$	$+2x$	$+4$
				-4

Therefore,

$$\frac{x^3-3x}{x^2-x-2} = (x-2) - \frac{4}{x^2-x-2}$$

Division of Algebraic Fractions and Partial Fractions

Sometimes, a combination of algebraic division and partial fractions will be necessary to rewrite an expression in the required form. If an algebraic fraction is to be rewritten in partial fractions, first check that the fraction is proper, i.e. that the expression on the numerator is smaller than that on the denominator. If this is not the case, first use algebraic long division and then find partial fractions for the proper fraction part of the resulting expression.

Example 9. To find partial fractions for

$$\frac{x^3 - 3x}{x^2 - x - 2}$$

first use the result of Example 8 to rewrite as:

$$\frac{x^3 - 3x}{x^2 - x - 2} = (x - 2) - \frac{4}{x^2 - x - 2} \tag{7}$$

Now find partial fractions for

$$\frac{4}{x^2 - x - 2}.$$

$$\frac{4}{x^2 - x - 2} = \frac{4}{(x - 2)(x + 1)}$$

Hence, to find partial fractions write in the general form;

$$\frac{4}{(x - 2)(x + 1)} = \frac{A}{(x - 2)} + \frac{B}{(x + 1)}$$

Multiplying through by $(x - 2)(x + 1)$ gives

$$4 = (x + 1)A + (x - 2)B \tag{8}$$

Setting $x = -1$ in Equation(8) gives

$$\begin{aligned} 4 &= (-1+1)A + (-1-2)B \\ 4 &= -3B \\ B &= \frac{-3}{4} \end{aligned}$$

Setting $x = 2$ in Equation(8) gives

$$\begin{aligned} 4 &= (2+1)A + (2-2)B \\ 4 &= 3A \\ A &= \frac{4}{3} \end{aligned}$$

Hence, substituting the partial fractions in to Equation(7) gives:

$$\frac{x^3 - 3x}{x^2 - x - 2} = (x-2) - \frac{4}{3(x-2)} + \frac{3}{4(x+1)}$$