St Columba's High School
Advanced Higher Maths
Sequences and Series

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1 Introduction to Sequences and Series

- A **Sequence** is an ordered list of terms. The n^{th} term of the sequence if often denoted u_n.
- A sequence can be given as a recurrence relation e.g $u_{n+1} = au_n + b$.
- A **Series** is the sum of the terms in an infinite sequence.

2 Arithmetic Sequence

An arithmetic sequence is a sequence in which the terms differ by a constant amount. For example: $2, 5, 8, 11, \dots$ is an arithmetic sequence. The common difference in this sequence is 3.

]

More generally, for a difference of d and initial value $u_1 = a$, an arithmetic sequence has an n^{th} term given by

$$u_n = a + (n-1)d \tag{1}$$

Example 1. Find a and d for the arithmetic sequence $-3, 2, 7, \dots$ and hence find the n^{th} term.

Process

The difference between each term is 5 and so $d = 5$. The initial value is -3 and so $a = -3$. Therefore,

$$\begin{aligned} u_n &= a + (n-1)d \\ &= -3 + 5(n-1) \\ &= = -3 + 5n - 5 \\ \Rightarrow u_n &= 5n - 8 \end{aligned}$$

Example 2. For a given arithmetic sequence, $u_{12} = 41$ amd $a = 8$. Find the value of d and write down an expression for the n_{th} term.

Process

In this case, $n = 12$, $a = 8$, and $u_{12} = 41$.

$$\begin{aligned} u_n &= a + (n-1)d \\ u_{12} &= 8 + (12-1)d \\ 41 &= 8 + 11d \\ 11d &= 41 - 8 \\ 11d &= 33 \\ d &= 3. \end{aligned}$$

Therefore,

$$\begin{aligned} u_n &= 8 + 3(n-1) \\ &= 8 + 3n - 3 \\ \Rightarrow u_n &= 5 + 3n. \end{aligned}$$

Example 3. An arithmetic sequence has $u_5 = 24$ and $u_{10} = 49$. Find the first four terms of this arithmetic sequence and identify the sequence by finding u_n.

Process

$$\begin{aligned} u_n &= a + (n-1)d \\ u_5 &= a + 4d \\ 24 &= a + 4d \\ a + 4d &= 24 \\ a &= 24 - 4d \qquad (2) \end{aligned}$$

Similarly:

$$\begin{aligned} u_n &= a + (n-1)d \\ u_{10} &= a + 9d \\ 49 &= a + 9d \\ a + 9d &= 49 \\ a &= 49 - 9d \qquad (3) \end{aligned}$$

Therefore, by equating equation(2) and equation(refeq2), we obtain:

$$\begin{aligned} 24 - 4d &= 49 - 9d \\ 5d &= 25 \\ d &= 5. \end{aligned}$$

and hence, substituting back into equation(2) gives

$$\begin{aligned} a &= 24 - 4d \\ &= 24 - 20 \\ &= 4 \end{aligned}$$

and the n_{th} term is therefore $u_n = 4 + 5(n-1) = 5n - 1$. The first four terms are:

$$\begin{aligned} u_1 &= 4 \\ u_2 &= 5 \times 2 - 1 = 10 - 1 = 9 \\ u_3 &= 5 \times 3 - 1 = 15 - 1 = 14 \\ u_4 &= 5 \times 4 - 1 = 20 - 1 = 19 \end{aligned}$$

3 Arithmetic Series

An arithmetic series is the sum of the terms of an arithmetic sequence.

To find the sum of any arithmetic sequence, consider the n_{th} term, $u_n = a + (n-1)d$ for first term a and difference d. The first terms of the sequence are: $a,\ a+d,\ a+2d,\ \ldots$.

Then the sum of the terms, S_n is given by:

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) \\ &\quad + \ldots + (a + (n-2)d) + (a + (n-1)d) \end{aligned} \tag{4}$$

But, this same sum can be written backwards as:

$$\begin{aligned} S_n &= (a + (n-1)d) + (a + (n-2)d) \\ &\quad + \ldots + (a+2d) + (a+d) + a \end{aligned} \tag{5}$$

Adding equations (4) and (5) gives:

$$\begin{aligned} 2S_n &= (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d) \\ 2S &= n\,(2a + (n-1)d) \\ \Rightarrow S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \end{aligned} \tag{6}$$

The sum of the first n terms of an arithmetic series is given by:

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

where a is the initial term and d is the difference.

Example 4. Find the sum of the first 12 terms of the arithmetic sequence that starts$4, 7, 10, \ldots$.

Process

In this example, $a = 4, d = 3$ and $n = 12$. Therefore,

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{12}{2}[2 \times 4 + (12-1) \times 3] \\ &= 6[8 + 33] \\ &= 6 \times 41 \\ &= 246 \end{aligned}$$

Example 5. After how many terms does the sum of the arithmetic sequence $2, 8, 14, \ldots$ first exceed 200?

Process

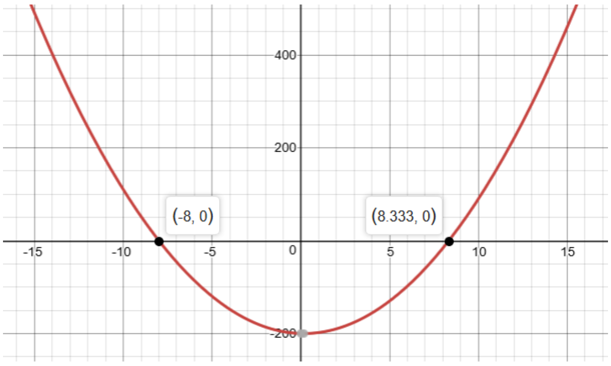
In this case, $a = 2$ and $d = 6$.Then,

$$\begin{aligned} S_n &= \frac{n}{2}[2a + (n-1)d] \\ &= \frac{n}{2}[2 \times 2 + (n-1) \times 6] \\ &= \frac{n}{2}[4 + 6n - 6] \\ &= \frac{n}{2}[6n - 2] \\ &= 3n^2 - n \end{aligned}$$

Hence, it is now necessary to find the value of n such that $s_n > 200$.

$$\begin{aligned} 3n^2 - n &> 200 \\ 3n^2 - n - 200 &> 0 \end{aligned}$$

To solve the inequality, sketch the graph of the function $y = 3n^2 - n - 200$. Solving $y = 3n^2 - n - 200 = 0$ (using the quadratic formula or competing the square) gives $n = -8$ or $n = 8.333$.



From the graph, it is clear that it is > 0 for $n > 8.33$ or $n < -8$. As it is not possible to have decimal or negative term numbers, it is clear that the first value of n for which the arithmetic series is greater than 200 is 9.

Example 6. The sum of the first 10 terms of an arithmetic series is 25 and the common difference is 0.3. What is the first term?

Process

$$
\begin{aligned}
S_n = \frac{n}{2}\left[2a + (n-1)d\right] &= 25 \\
\Rightarrow \frac{10}{2}\left[2a + (10-1)0.3\right] &= 25 \\
\Rightarrow 5[2a + 2.7] &= 25 \\
10a + 13.5 &= 25 \\
10a &= 11.5 \\
a &= 1.15
\end{aligned}
$$

4 Geometric Sequences

If a sequence consists of terms that are obtained by multiplying the previous term by a constant, then it is called a geometric sequence. Some exampled of geometric sequences are:

- 4, 12, 36, 108, ... The terms in this sequence are obtained by multiplying the previous term by 3.
- 200, 40, 8, 1.6, ... The The terms in this sequence are obtained by multiplying the previous term by $\frac{1}{5}$.
- 3, -12, 48, ... The terms in this sequence are obtained by multiplying the previous term by -4.

In a geometric series, the common ratio is denoted by the letter r. The first term is given by a.
To find the n_{th} term, look for a pattern:

$$\begin{aligned} u_1 &= a \\ u_2 &= ar \\ u_3 &= ar^2 \\ u_4 &= ar^3 \\ \\ \therefore u_n &= ar^{n-1} \end{aligned}$$

The n_{th} term of a geometric sequence is given by

$$u_n = ar^{n-1}$$

Example 7. For the geometric sequence $4, 8, 16, 32, \ldots$, identify a and r and find an expression for the n_{th} term.

Process

For $4, 8, 16, 32, \ldots$, the common ratio, r is $8 \div 4 = 2$. The first term a is 4. Hence, the n_{th} term is:

$$u_n = 4 \times 2^{n-1}$$

In this special case, as $4 = 2^2$, this can be rewritten as:

$$u_n = 2^2 \times 2^{n-1} = 2^{n-1+2} = 2^{n+1}$$

Example 8. Find the first four terms of the geometric sequence whose third term is 18 and whose sixth term is 486.

Process

$$\begin{aligned} u_3 &= ar^{3-1} \\ \Rightarrow 18 &= ar^2 \\ \Rightarrow a &= \frac{18}{r^2} \end{aligned}$$

$$\begin{aligned} u_6 &= ar^{5-1} \\ \Rightarrow 486 &= ar^5 \\ \Rightarrow a &= \frac{486}{r^5} \end{aligned}$$

Equating the two expressions for a gives:

$$\begin{aligned} \frac{18}{r^2} &= \frac{486}{r^5} \\ 18r^3 &= 486 \quad \text{(multiplying through by } r^5\text{)} \\ r^3 &= 27 \\ r &= 3 \end{aligned}$$

Substituting back gives

$$a = \frac{18}{3^2} = 2$$

Hence, the first four terms in the sequence are 2, 6, 18, 54 and

$$u_n = 2 \times 3^{n-1}$$

Example 9. Find the first term in the geometric sequence $5, 15, 45, \ldots$ to exceed 5000.

Process

For this sequence, $a = 5$ and $r = 3$. Hence, the sequence has n_{th} term $u_n = 5 \times 3^{n-1}$.
Therefore, we need to find the value of n so that $5 \times 3^{n-1} > 5000$.

$$\begin{aligned}
5 \times 3^{n-1} &> 5000 \\
3^{n-1} &> 1000 \\
\ln 3^{n-1} &> \ln 1000 \\
(n-1)\ln 3 &> \ln 1000 \\
n - 1 &> \frac{\ln 1000}{\ln 3} \\
n - 1 &> 6.2877 \\
n &> 7.2877
\end{aligned}$$

As n must be an integer, the first value of n for which the geometric sequence is greater than 5000 is $n = 8$. For $n = 8$, the term is:

$$u_8 = ar^{8-1} = 5 \times 3^7 = 10935$$

5 Sum of a geometric series

The sum of a geometric series can be found via the formula:

Sum of a geometric series is

$$S_n = \frac{a(1-r^n)}{1-r} \tag{7}$$

Example 10. Find the sum of the first 5 terms of the geometric series that starts 4, 12, 36, ...

Process

In this example, $n = 5, a = 4$, and $r = 3$. Hence,

$$\begin{aligned} s_5 &= \frac{a(1-r^n)}{1-r} \\ &= \frac{4(1-3^5)}{1-3} \\ &= 484 \end{aligned}$$

Example 11. Find the sum of the geometric series
$78125 - 15625 + 3125 + \cdots + 5$.

Process

In this case, $a = 78125$ and $r = -\frac{1}{5}$. Hence, it is necessary to find the value of n which gives the term with value 5 and then use this to find the sum of the series.

$$\begin{aligned} u_n &= ar^{n-1} \\ u_n &= 78125 \times \left(-\frac{1}{5}\right)^{n-1} \end{aligned}$$

This can now be equated to 5 and solved to find n.

$$
\begin{aligned}
78125 \times \left(-\frac{1}{5}\right)^{n-1} &= 5 \\
\left(-\frac{1}{5}\right)^{n-1} &= \frac{5}{78125} \\
(-1)^{n-1}\left(\frac{1}{5}\right)^{n-1} &= \frac{5}{78125} \\
\left(\frac{1}{5}\right)^{n-1} &= \frac{5}{78125} \quad \text{as} (-1)^{n-1} \text{ must be positive} \\
\ln\left(\frac{1}{5}\right)^{n-1} &= \ln\frac{5}{78125} \\
(n-1)\ln\left(\frac{1}{5}\right) &= \ln\frac{5}{78125} \\
n-1 &= \frac{\ln\frac{5}{78125}}{\ln\left(\frac{1}{5}\right)} \\
n-1 &= 6 \\
n &= 7
\end{aligned}
$$

The sum of the first 7 terms can now be evaluated as follows:

$$
\begin{aligned}
S_7 &= \frac{a(1-r)^7}{1-r} \\
&= \frac{78125\left(1-\left(-\frac{1}{5}\right)^7\right)}{1-\left(-\frac{1}{5}\right)} \\
&= 65105
\end{aligned}
$$

6 Sum to Infinity of a Geometric Series

For geometric series with a common ratio greater than 1, the sum of the series grows larger at each successive term. However, if geometric series has a common ratio with absolute value < 1 (i.e. $-1 < r < 1$ then the sum of the geometric series will never exceed a certain number. This number is known as the **limit** or the **sum to infinity** of the series. The sum to infinity of a geometric series can be found as follows:

$$\begin{aligned} S_n &= \frac{a(1-r^n)}{1-r} \\ & \text{However, as } n \to \infty, r^n \to 0 \ \text{ for } |r| < 1 \\ \Rightarrow S_\infty &= \frac{a(1-0)}{1-r} \\ &= \frac{a}{1-r} \end{aligned}$$

For any geometric series with initial value a and common ratio r (where $-1 < r < 1$), the sum to infinity is given by:

$$S_\infty = \frac{a}{1-r}$$

Example 12. Find the sum to infinity of the geometric series $12, 6, 3, \ldots$.

Process

For the geometric series $12, 6, 3, \ldots$, the initial value is $a = 12$ and the common ratio is $r = \frac{1}{2}$.

Therefore,

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ &= \frac{12}{1-\frac{1}{2}} \\ &= \frac{12}{\frac{1}{2}} \\ &= 24 \end{aligned}$$

Example 13. A geometric series has a sum to infinity of 18. If the common ratio is $\frac{1}{3}$, what is the first term of the series?

Process

$$\begin{aligned} S_\infty &= \frac{a}{1-r} \\ \therefore 18 &= \frac{a}{1-\frac{1}{3}} \\ \frac{a}{\frac{2}{3}} &= 18 \\ a &= 18\left(\frac{2}{3}\right) \\ a &= 12 \end{aligned}$$

Example 14. Given that 24 and 16 are two adjacent terms of an infinite geometric series with a sum to ininity of 243, find the first term.

Process

The sum to infinity exists as $16 \div 24 = \frac{2}{3}$ which is less than 1.
Substituting $r = \frac{2}{3}$ and $S_\infty = 243$ into the formula for the sum to inifinty gives:

$$\begin{aligned}
S_\infty &= \frac{a}{1-r} \\
\Rightarrow \frac{a}{1-\frac{2}{3}} &= 243 \\
\frac{a}{\frac{1}{3}} &= 243 \\
a &= 243 \times \frac{1}{3} \\
a &= 81
\end{aligned}$$

7 Expanding $(1-x)^{-1}$ and Related Functions

The sum to infinity of a geometric sequence is given by

$$S_\infty = \frac{a}{1-r}$$

where a is the initial term and r is the common ratio. Hence, when $|r| < 1$, the expression

$$(1-r)^{-1} = \frac{1}{1-r}$$

can be linked to the sum to infinity of a geometric sequence with initial term 1 and common ratio r.

Hence, when $|r| < 1$, the geometric sequence produced from $(1-r)^n$ is

$$1 + r + r^2 + r^3 + r^4 + \dots$$

Discussion

To explain this, consider the binomial theorem. The binomial theorem encountered previously states that

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r, \quad n \in \mathbb{Z}$$

Now, for non integer values of n such as $n = -1$, is it possible to find an expansion for $(x+y)^n$?

The binomial theorem can be rewritten as:

$$
\begin{aligned}
(x+y)^n &= \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r \\
&= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n} y^n \\
&= x^n + \frac{n!}{1!(n-1)!} x^{n-1} y + \frac{n!}{2!(n-2)!} x^{n-2} y^2 + \cdots + y^n \\
&= x^n + \frac{n}{1!} x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2 + \cdots + y^n
\end{aligned}
\tag{8}
$$

This way of writing the expression means that n is now no longer limited to non-negative integers and can also be evaluated for any rational values of n. However, it is only valid for $|x| < y$.

Now consider the expression

$$\frac{1}{(1-r)}.$$

If, $|r| < 1$, then this can be rewritten as:

$$\frac{1}{(1-r)} = \frac{1}{(1+(-r)} = (1+(-r))^{-1}$$

and expanded using equation(8) above as follows:

$$
\begin{aligned}
(1+(-r))^{-1} &= 1^{-1} + \frac{-1}{1!} 1^{-1-1}(-r) + \frac{-1(-1-1)}{2!} 1^{-1-2}(-r)^2 + \ldots \\
&= 1 + r + r^2 + r^3 + \ldots
\end{aligned}
$$

Hence, $(1-r)^{-1}$ can be considered as the sum to infinity of a geometric sequence. with first term 1 and common ratio r.

Expanding $(1-x)^{-1}$ can be interpreted as the sum to infinity of a geometric sequence with first term 1 and common ratio x. I.e.

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

Example 15. Expand $\frac{1}{0.8}$ to four decimal places.

Process

In this case,

$$\frac{1}{0.8} = (1-0.2)^{-1}.$$

Hence, the expansion $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$ can be used as follows:

$$\begin{aligned}(1-0.2)^{-1} &= 1 + 0.2 + 0.2^2 + 0.2^3 + 0.2^4 + 0.2^5 + 0.2^6 \dots \\ &= 1 + 0.2 + 0.04 + 0.008 + 0.0016 + 0.00032 + 0.000064 + \dots \\ &= 1.24998 \\ &= 1.2500 \text{ to four decimal places}\end{aligned}$$

Example 16. Expand $(1-2x)^{-1}$, $|x| < \frac{1}{2}$ in ascending powers of x.

Process

$$\begin{aligned}(1-2x)^{-1} &= 1 + 2x + (2x)^2 + (2x)^3 + (2x)^4 + \dots \\ &= 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots\end{aligned}$$

Example 17. By using an appropriate factorisation, find the first four terms of the expansion of

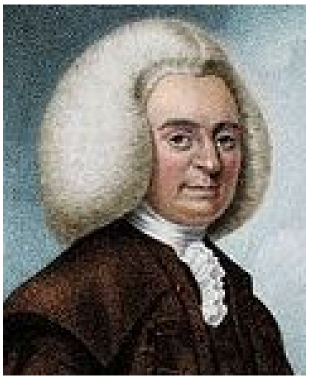
$$\frac{1}{5+3x}.$$

Process

$$\begin{aligned}
\frac{1}{5+3x} &= \frac{1}{5(1+\frac{3}{5}x)} \\
&= \frac{1}{5}\left(\frac{1}{1+\frac{3}{5}x}\right) \\
&= \frac{1}{5}\left[\left(1+\frac{3}{5}x\right)^{-1}\right] \\
&= \frac{1}{5}\left[\left(1-(-\frac{3}{5}x)\right)^{-1}\right] \\
&= \frac{1}{5}\left[1+\frac{-3}{5}x+\left(\frac{-3}{5}x\right)^{2}+\left(\frac{-3}{5}x\right)^{3}+\left(\frac{-3}{5}x\right)^{4}+\dots\right] \\
&= \frac{1}{5}\left[1-\frac{3}{5}x+\frac{9}{25}x^{2}-\frac{27}{125}x^{3}+\dots\right] \\
&= \frac{1}{5}-\frac{3}{25}x+\frac{9}{125}x^{2}-\frac{27}{625}x^{3}+\dots
\end{aligned}$$

8 MacLaurin Series

Colin MacLaurin was a Scottish mathematician who lived in the 18th Century and made important contributions to the development of maths in algebra and geometry. In this section, we will consider his series which is a particular form of a "power"series developed by the mathematician Brook Taylor.



Power Series

A **power series** is a series of the form

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_rx^r + \ldots$$

where $a_0, a_1, \ldots a_r$ are real constants and x is a real variable.

Generally, any function $f(x)$ can be expressed approximately in a infinite series expansion of the form shown above as follows:

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots +$$

Note that the power series may only converge for certain values of x. This power series can be used by computers, calculators etc to calculate values of functions such as logs, trig functions etc.

A **MacLaurin Series** is used to approximate any function close to the origin. Hence, to find the values of $a_0, a_1, \dots$ etc, evaluate the function and its derivates (provided that they exist) at the point $x = 0$.

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 + \dots \\
f''(x) &= 2a_2 + 3 \times 2 \times a_3 x + \dots \\
f'''(x) &= 3 \times 2 \times a_3 + \dots
\end{aligned}
$$

Hence, evaluating each of the above expressions at $x = 0$ gives:

$$
\begin{aligned}
f(0) &= a_0 \\
f'(0) &= a_1 \\
f''(0) &= 2 \times a_2 = 2!a_2 \\
f'''(0) &= 3 \times 2 \times a_3 = 3!a_3
\end{aligned}
$$

Therefore, the function, f, can be expressed as the MacLaurin series:

$$
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^r(0)}{r!} + \dots
$$

The MacLaurin series for a given function can be found via the expansion

$$
f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^r(0)}{r!} + \dots \quad (9)
$$

Example 18. Find the MacLaurin series expansion for e^{2x} to the term in x^3.

Process

In this case, $f(x) = e^{2x}$. Hence:

$$\begin{aligned} f(x) = e^{2x} &\Rightarrow f(0) = e^0 = 1 \\ f'(x) = 2e^{2x} &\Rightarrow f'(0) = 2e^0 = 2 \\ f''(x) = 4e^{2x} &\Rightarrow f''(0) = 4e^0 = 4 \\ f'''(x) = 8e^{2x} &\Rightarrow f'''(0) = 8e^0 = 8. \end{aligned}$$

Therefore, the MacLaurin expansion is:

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \\ &= 1 + 2x + \frac{4}{2!}x^2 + \frac{8}{3!}x^3 + \ldots \\ &= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \ldots \end{aligned}$$

Example 19. Find the first four terms of the MacLaurin series expansion of $\cos x$.

Process

In this case, $f(x) = \cos x$. Hence:

$$\begin{aligned}
f(x) = \cos x &\Rightarrow f(0) = \cos 0 = 1 \\
f'(x) = -\sin x &\Rightarrow f'(0) = -\sin 0 = 0 \\
f''(x) = -\cos x &\Rightarrow f''(0) = \cos 0 = -1 \\
f'''(x) = \sin x &\Rightarrow f'''(0) = \sin 0 = 0 \\
f''''(x) = \cos x &\Rightarrow f''''(0) = \cos 0 = 1 \\
f^{v}(x) = -\sin x &\Rightarrow f^{v}(0) = -\sin 0 = 0 \\
f^{vi}(x) = -\cos x &\Rightarrow f^{vi}(0) = -\cos 0 = -1
\end{aligned}$$

Therefore, the MacLaurin expansion is:

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots \\
&= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots \\
&= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \ldots
\end{aligned}$$

Example 20. Find the MacLaurin series expansion of $e^{2x}\cos x$ up to the term in x^4.

Process

As the Maclaurin series expansion for $\cos x$ is known, this can be used to find the expansion of $\cos 3x$ as follows:

$$\begin{aligned} \cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \\ \Rightarrow \cos 3x &= 1 - \frac{1}{2}(3x)^2 + \frac{1}{24}(3x)^4 - \frac{1}{720}(3x)^6 + \dots \\ &= 1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 + \dots \end{aligned}$$

Similarly, from the earlier example:

$$e^{2x} = 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$$

Therefore, to find the expansion of the product of $\cos 3x$ and e^{2x}, multiply the two separate expansions together, multiply out and collect like terms.

$$\begin{aligned} e^{2x}\cos 3x &= \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots\right)\left(1 - \frac{9}{2}x^2 + \frac{27}{8}x^4 - + \dots\right) \\ &= (1 - \frac{9}{2}x^2 + \frac{27}{8}x^4) + (2x - 9x^3 + \frac{27}{4}x^5) + (2x^2 - 9x^4) + (\frac{4}{3}x^3) + \dots \\ &= 1 + 2x - \frac{5}{2}x^2 - \frac{23}{3}x^3 - \frac{45}{8}x^4 + \dots\dots \end{aligned}$$