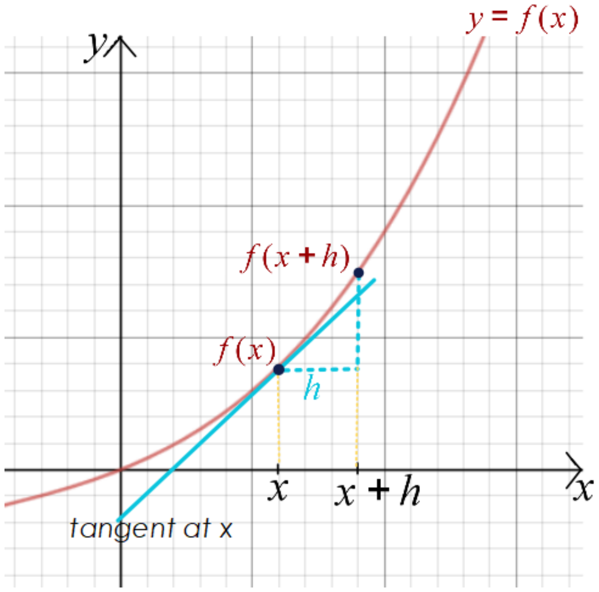
St Columba's High School
Advanced Higher Maths
Differentiation

Contents

1 Differentiation from First Principles

Differentiation is defined as finding the rate of change of a function. The derivative at a point is also the gradient of the tangent to the function at that point.



Looking at the graph of the function shown and recalling that gradient is defined as vertical ÷ horizontal, it is clear that the gradient of a straight line joining $f(x)$ and $f(x+h)$ is given by:

$$\text{gradient} = \frac{\text{vertical}}{\text{horizontal}} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h} \tag{1}$$

As the value of h gets smaller, the gradient of the line joining $f(x)$ and $f(x+h)$ will become increasingly close to the gradient of the tangent at x. Therefore, to find this derivative at the point x, find the value of (1) as the value of h gets smaller i.e $h \to 0$.

This method is called differentiating from first principles and is defined by the formula:

$$f'(x) = \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \right]$$

It is important to note that some functions are not differentiable. You should not be asked to differentiate from first principles in the exam. However, it is expected that you are familiar with the concept and understand how the process works.

Example 1. Find the derivative of x^2 from first principles.

Process

$$\begin{aligned} f(x) &= x^2 \\ \Rightarrow f'(x) &= \lim_{h \to 0} \left[\frac{(x+h)^2 - x^2}{h} \right] \\ &= \lim_{h \to 0} \left[\frac{x^2 + 2hx + h^2 - x^2}{h} \right] \\ &= \lim_{h \to 0} \left[\frac{h(2x+h)}{h} \right] \\ &= \lim_{h \to 0} 2x + h \\ &= 2x. \end{aligned}$$

2 The Chain Rule for Differentiation

The chain rule for differentiation is used to differentiate functions of the form $f(g(x))$. The chain rule is given by:

The Chain Rule for Differentiation.

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x) \tag{2}$$

In reality, this means differentiate the f function first, then differentiate the g function and then multiply the derivatives together.

Example 2. Differentiate $\sin x^2$using the chain rule.

Process

$$\begin{aligned}\frac{d}{dx}\sin x^2 &= \frac{d}{dx}(\sin x)^2 \\ &= 2\sin x \times \frac{d}{dx}(\sin x) \\ &= 2\sin x \cos x \\ &= \sin 2x.\end{aligned}$$

Example 3. Differentiate $(3x-5)^3$ using the chain rule.

Process

$$\begin{aligned}\frac{d}{dx}(3x-5)^3 &= 3(3x-5)^2 \times \frac{d}{dx}(3x-5) \\ &= 3(3x-5)^2 \times 3 \\ &= 9(3x-5)^2\end{aligned}$$

3 Differenting Logarithms and Exponentials

The natural log function, $\ln x$, and its inverse, the natural exponential e^x are important functions that were encountered in the Higher Maths course. They are used in many real life applications including radioactive decay and probability models.

The exponential function has the unique property that its derivative is the same as the function itself.

$$\frac{d}{dx}\mathrm{e}^x = \mathrm{e}^x$$

and

$$\frac{d}{dx}\ln x = \frac{1}{x}$$

Example 4. Differentiate $3\mathrm{e}^{x^2}$ with respect to x.

Process

$$\begin{aligned}\frac{d}{dx}(3\mathrm{e}^{x^2}) &= 3\mathrm{e}^{x^2}.2x \\ &= 6x\mathrm{e}^{x^2}\end{aligned}$$

Example 5. Differentiate e^{x^2+4} .

Process

$$\begin{aligned}\frac{d}{dx}\mathrm{e}^{x^2+4} &= \mathrm{e}^{x^2+4}.2x \\ &= 2x\mathrm{e}^{x^2+4}\end{aligned}$$

Example 6. Differentiate $2\mathrm{e}^{x^3}$.

Process

Using the chain rule,

$$\begin{aligned}\frac{d}{dx}2e^{x^3} &= 2e^{x^3} \times \frac{d}{dx}x^3 \\ &= 2e^{x^3}.3x^2 \\ &= 6x^2e^{x^3}\end{aligned}$$

Example 7. Differentiate $\ln(7x)$.

Process

Using the chain rule,

$$\begin{aligned}\frac{d}{dx}\ln(7x) &= \frac{1}{7x} \times \frac{d}{dx}(7x) \\ &= \frac{1}{7x} \times 7 \\ &= 7\frac{1}{7x} \\ &= \frac{1}{x}\end{aligned}$$

Example 8. Differentiate $\ln(\ln x)$.

Process

Using the chain rule,

$$\begin{aligned}\frac{d}{dx}\ln(\ln x) &= \frac{1}{\ln x} \times \frac{d}{dx}(\ln x) \\ &= \frac{1}{\ln x} \times \frac{1}{x} \\ &= \frac{1}{x \ln x}\end{aligned}$$

Example 9. Differentiate $\frac{1}{2}\ln(\sin x)$.

Process

Using the chain rule,

$$\begin{aligned}\frac{d}{dx}\frac{1}{2}.\ln(\sin x) &= \frac{1}{2}\ \frac{1}{\sin x} \times \frac{d}{dx}(\sin x) \\ &= \frac{1}{2}.\ \frac{1}{\sin x} \times \cos x \\ &= \frac{\cos x}{2\sin x}\end{aligned}$$

4 Differentiation Using The Product Rule

The product rule is used to differentiate a product of functions eg $f(x)g(x)$.

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x) \qquad (3)$$

This can also be written in Leibniz notation for the product of the functions u and v as shown:

$$\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (4)$$

Proof

The product rule can be proven using differentiation from first principles as shown below:
Let $k(x) = f(x)g(x)$. Then,

$$
\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h)-f(x)]\,g(x) + f(x)\,[g(x+h)-g(x)]}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} g(x) + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} f(x) \\
&= f'(x)g(x) + g'(x)f(x)
\end{aligned}
$$

Therefore, $k'(x) = \frac{d}{dx} f(x)g(x) = f'(x)g(x) + g'(x)f(x)$ as required.

Example 10. Differentiate $x^2(x+2)^3$.

Process

$$
\begin{aligned}
& \frac{d}{dx} x^2(x+2)^3 \\
=& \left[\frac{d}{dx}(x^2)\right](x+2)^3 + x^2\left[\frac{d}{dx}(x+2)^3\right] \\
=& \ 2x.(x+2)^3 + x^2.3(x+2)^2 \\
=& \ 2x(x+2)^3 + 3x^2(x+2)^2 \\
=& \ x(x+2)^2\,[2(x+2)+3x] \\
=& \ x(x+2)^2\,[2x+4+3x] \\
=& \ x(x+2)^2(5x+4)
\end{aligned}
$$

Example 11. Differentiate $y = x \sin x$

Process

Using Leibniz notation for this example. Let

$$\begin{array}{ll} u = x & v = \sin x \\ \frac{du}{dx} = 1 & \frac{dv}{dx} = \cos x \end{array}$$

$$\begin{aligned} \frac{d}{dx} x \sin x &= \frac{d}{dx} uv \\ &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x \cos x + \sin x.1 \\ &= \sin x + x \cos x \end{aligned}$$

Example 12. Differentiate $y = x\sqrt{x^2+4}$.

Process - In this example, both methods will be shown.

Method 1

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} x\sqrt{x^2+4} \\ &= \left[\frac{d}{dx}(x)\right](x^2+4)^{\frac{1}{2}} + x\left[\frac{d}{dx}(x^2+4)^{\frac{1}{2}}\right] \\ &= 1.(x^2+4)^{\frac{1}{2}} + x.\frac{1}{2}(x^2+4)^{-\frac{1}{2}}.2x \\ &= (x^2+4)^{\frac{1}{2}} + x^2(x^2+4)^{-\frac{1}{2}} \\ &= (x^2+4)^{-\frac{1}{2}}\left[x^2+4+x^2\right] \\ &= (x^2+4)^{-\frac{1}{2}}(2x^2+4) \\ &= \frac{2(x^2+2)}{(x^2+4)^{\frac{1}{2}}} \end{aligned}$$

Method 2

$$\begin{array}{lll} u = x & v = \sqrt{x^2+4} & = (x^2+4)^{\frac{1}{2}} \\ \frac{du}{dx} = 1 & \frac{dv}{dx} = \frac{1}{2}(x^2+4)^{-\frac{1}{2}}.2x & = x(x^2+4)^{-\frac{1}{2}} \end{array}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}x\sqrt{x^2+4} \\ &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= x.x(x^2+4)^{-\frac{1}{2}} + (x^2+4)^{\frac{1}{2}}.1 \\ &= (x^2+4)^{-\frac{1}{2}}\left[x^2 + (x^2+4)\right] \\ &= \frac{2(x^2+2)}{(x^2+4)^{\frac{1}{2}}} \end{aligned}$$

5 Differentiating using the Quotient rule

The quotient rule is used to differentiate a function that is made up of one function divided by another. To differentiate using the quotient rule, use the following rule:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2} \tag{5}$$

where $v' = \frac{dv}{dx}$ and $u' = \frac{du}{dx}$.

Example 13. Differentiate

$$\frac{x^3}{x+5}$$

Process

Let

$$\begin{aligned} u &= x^3 & v &= x+5 \\ \tfrac{du}{dx} &= 3x^2 & \tfrac{dv}{dx} &= 1 \end{aligned}$$

$$\begin{aligned} \frac{d}{dx}\left(\frac{x^3}{x+5}\right) &= \frac{vu' - uv'}{v^2} \\ &= \frac{(x+5).3x^2 - x^3.1}{(x+5)^2} \\ &= \frac{x^2\left(3(x+5) - x\right)}{(x+5)^2} \\ &= \frac{x^2\left(2x+15\right)}{(x+5)^2} \end{aligned}$$

Example 14. Use the quotient rule to differentiate

$$]\frac{x+1}{\cos x}$$

Process

Let

$$u = x+1 \qquad v = \cos x$$
$$\tfrac{du}{dx} = 1 \qquad \tfrac{dv}{dx} = -\sin x$$

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x+1}{\cos x}\right) &= \frac{vu' - uv'}{v^2} \\
&= \frac{\cos x.1 - (x+1).(-\sin x)}{\cos x^2} \\
&= \frac{\cos x + (x+1)\sin x}{\cos x^2}
\end{aligned}$$

Example 15. Find f'(0) when

$$f(x) = \frac{x^2+5}{x^3+3}$$

Process

Let

$$u = x^2+5 \quad v = x^3+3$$
$$\tfrac{du}{dx} = 2x \qquad \tfrac{dv}{dx} = 3x^2$$

$$
\begin{aligned}
\frac{d}{dx}\left(\frac{x^2+5}{x^3+3}\right) &= \frac{vu'-uv'}{v^2} \\
&= \frac{(x^3+3).2x-(x^2+5).3x^2}{(x^3+3)^2} \\
&= \frac{2x(x^3+3)-3x^2(x^2+5)}{(x^3+3)^2} \\
&= \frac{x\left[2(x^3+3)-3x(x^2+5)\right]}{(x^3+3)^2}
\end{aligned}
$$

6 Higher Derivatives

A function may be differentiated more than once. The derivative of a derivative is called the second derivative. Similarly, the derivative of the second derivative is the third derivative etc.

The second derivative is useful as it can identify the nature of stationary points; find the acceleration at a point and they are used in Maclaurin series that will be seen later on in the course.

There are two notations that can be used for higher derivatives.

1. In function notation, use multiple dashes, eg $f''(x)$ represents the second derivative, $f'''(x)$ is the third derivative etc.

2. In Leibniz notation, use powers, e.g. $\frac{d^2y}{dx^2}$ is the second derivative of y. Similarly, $\frac{d^3y}{dx^3}$ is the third derivative of y.

Example 16. For example, consider the function $f(x) = 4x^5$. Then:

$$\begin{aligned} f'(x) &= 20x^4 \\ f''(x) &= \frac{d}{dx}f'(x) \\ &= 80x^3 \end{aligned}$$

Example 17. Find all of the derivatives of $y = 5x^3 + 2x^2 + 3x - 1$

Process

$$\begin{aligned}
\frac{dy}{dx} &= 15x^2 + 4x + 3 \\
\frac{d^2y}{dx^2} &= 30x + 4 \\
\frac{d^3y}{dx^3} &= 30 \\
\frac{d^4y}{dx^4} &= 0.
\end{aligned}$$

Example 18. Find the first four derivatives of $f(x) = 3\sin 2x$.

Process

$$\begin{aligned}
f(x) &= 3\sin 2x \\
f'(x) &= 3\cos 2x.2 = 6\cos 2x \\
f''(x) &= -6\sin 2x.2 = -12\sin 2x \\
f'''(x) &= -12\cos 2x.2 = -24\cos 2x \\
f''''(x) &= 24\sin 2x.2 = 48\sin 2x
\end{aligned}$$

7 Rate of change and Higher Derivatives

The rate of change is the ratio of two quantities and is represented by the gradient/ slope of a line. Differentiating a function finds it's rate of change. Common examples include displacement (s), velocity(v) and acceleration(a) where

$$\begin{aligned} \text{velocity} &= \text{rate of change of displacement} \\ v(t) &= \frac{ds(t)}{dt} \end{aligned}$$

$$\begin{aligned} \text{acceleration} &= \text{rate of change of velocity} \\ a(t) &= \frac{dv(t)}{dt} \end{aligned}$$

Example 19. The distance (in metres) travelled by a vehicle at time, t,is given by $s(t) = 3t^3$. Find the speed and acceleration of the vehicle after 12 seconds.

Process

$$\begin{aligned} v(t) &= \frac{ds(t)}{dt} \\ &= 9t^2 \end{aligned}$$

$$\begin{aligned} a(t) &= \frac{dv(t)}{dt} \\ &= 18t \end{aligned}$$

After 12 seconds,

$$v(t) = 9 \times 12^2 = 1296\text{ms}^{-1}$$

and

$$a(t) = 18 \times 12 = 216\text{ms}^{-1}$$

Example 20. The distance, d, travelled on a theme park attraction is given by

$$d(t) = 12t^2 - 8t$$

Calculate the speed and acceleration 6 seconds after the start of the ride.

Process

$$\begin{aligned} v(t) &= d'(t) \\ &= 24t - 8 \end{aligned}$$

$$\begin{aligned} a(t) &= v'(t) \\ &= 24. \end{aligned}$$

After 6 seconds,

$$v(6) = 24 \times 6 - 8 = 132\text{ms}^{-1}$$

and

$$a(6) = 24\text{ms}^{-1}$$

Example 21. A particular particle motion is given by

$$s(t) = \sin{(3t - \frac{\pi}{6})}, \quad 0 \leq t \leq 2\pi$$

Find the velocity at t=0 and calculate when the acceleration is maximised.

Process

$$\begin{aligned} v(t) &= s'(t) \\ &= 3\cos{(3t - \frac{\pi}{6})} \end{aligned}$$

$$\begin{aligned} a(t) &= v'(t) \\ &= -9\sin{(3t - \frac{\pi}{6})}. \end{aligned}$$

At $t = 0$,

$$v(0) = 3\cos(0 - \frac{\pi}{6}) = \frac{\sqrt{3}}{2}\text{ms}^{-1}.$$

As $-\sin x$ has a maximum at $x = \frac{3\pi}{2}$ (from considering the shape of the sine graph) the maximum acceleration occurs first when

$$\begin{aligned} 3t - \frac{\pi}{6} &= \frac{3\pi}{2} \\ 3t &= \frac{3\pi}{2} + \frac{\pi}{6} \\ 3t &= \frac{10\pi}{6} \\ t &= \frac{10\pi}{18} = \frac{5\pi}{9}\text{seconds.} \end{aligned}$$

The period of $-\sin(3t - \frac{\pi}{6})$ is $\frac{2\pi}{3}$ and so, further maximum acceleration points occur at

$$\frac{5\pi}{9} + \frac{2\pi}{3} = \frac{11\pi}{9}\text{seconds}$$

and

$$\frac{11\pi}{9} + \frac{2\pi}{3} = \frac{17\pi}{9}\text{seconds}$$

8 Higher Derivatives and Stationary Points

The location of the stationary points of a function are found by evaluating where $\frac{dy}{dx} = 0$. To find the nature of the derivative, evaluate the second derivative.

- Minimum stationary point if

$$\frac{d^2y}{dx^2} > 0$$

- Maximum stationary point if

$$\frac{d^2y}{dx^2} < 0$$

- Possible point of inflection if

$$\frac{d^2y}{dx^2} = 0$$

Example 22. Find the nature of the stationary points of $f(x) = x^3 + 4x^2 - 3x - 18$.

Process

$$\begin{aligned} f(x) &= x^3 + 4x^2 - 3x - 18 \\ f'(x) &= 3x^2 + 8x - 3 \\ f'(x) &= (3x-1)(x+3) \end{aligned}$$

$$\begin{aligned} f'(x) = 0 &\Rightarrow (3x-1)(x+3) = 0 \\ &\Rightarrow x = \frac{1}{3} \quad \text{or} \quad x = -3 \end{aligned}$$

$$\begin{aligned} f''(x) &= 6x + 8 \\ f''(\frac{1}{3}) &= 10 > 0, \quad \text{minimum S.P.} \\ f''(-3) &= -10 < 0, \quad \text{maximum S.P.} \end{aligned}$$

Example 23. Find the nature of the stationary points of $y = 3\cos(2x - \frac{\pi}{4}),\ 0 \leq x \leq 2\pi$.

Process

$$\begin{aligned} y &= 3\cos(2x - \frac{\pi}{4}) \\ \frac{dy}{dx} &= -3\sin(2x - \frac{\pi}{4}).2 \\ &= -6\sin(2x - \frac{\pi}{4}) \end{aligned}$$

For S.P.s, $\frac{dy}{dx} = 0$, therefore,

$$
\begin{aligned}
-6\sin(2x - \frac{\pi}{4}) &= 0 \\
\Rightarrow 2x - \frac{\pi}{4} &= 0 \\
2x &= \frac{\pi}{4} \\
x &= \frac{\pi}{8}
\end{aligned}
$$

or

$$
\begin{aligned}
2x - \frac{\pi}{4} &= \pi \\
2x &= \frac{5\pi}{4} \\
x &= \frac{5\pi}{8}
\end{aligned}
$$

or

$$
\begin{aligned}
2x - \frac{\pi}{4} &= 2\pi \\
2x &= \frac{9\pi}{4} \\
x &= \frac{9\pi}{8}
\end{aligned}
$$

To find the nature of these stationary points, use the second derivative.

$$
\frac{d^2y}{dx^2} = -12cos(2x - \frac{\pi}{4})
$$

At

$$
x = \frac{\pi}{8}, \frac{d^2y}{dx^2} = -12\cos 0 = -12 < 0, \text{ Hence max SP}
$$

At

$$
x = \frac{5\pi}{8}, \frac{d^2y}{dx^2} = -12\cos\pi = 12 > 0, \text{ Hence min SP}
$$

At

$$
x = \frac{9\pi}{8}, \frac{d^2y}{dx^2} = -12\cos 2\pi = -12 < 0, \text{ Hence max SP}
$$

9 Derivatives of $\tan x, \cot x, \sec x$ and cosec$x$

The functions $\cot x, \sec x$ and cosec$x$ are defined by:

$$\begin{aligned} \cot x &= \frac{1}{\tan x}, \ x \neq 0, \pi, 2\pi, \ldots \\ \sec x &= \frac{1}{\cos x}, \ x \neq \frac{\pi}{2}, \frac{3\pi}{2}, \ldots \\ \text{cosec} x &= \frac{1}{\sin x}, \ x \neq 0, \pi, 2\pi, \ldots \end{aligned}$$

Unlike $\sin, \cos$ and tan, the graphs of sec and cosec have "breaks" in them. At these points, the functions sec(x) and cosec(x) are undefined. The derivatives can be obtained using previously learned rules for differentiation.

Example 24. Find $\frac{d}{dx} \sec x$.

Process

Using the chain rule for differentiation:

$$\begin{aligned} \frac{d}{dx} \sec x &= \frac{d}{dx} \frac{1}{\cos x} \\ &= \frac{d}{dx} (\cos x)^{-1} \\ &= -(\cos x)^{-2} . - \sin x \\ &= \frac{\sin x}{\cos^2 x} \\ &= \frac{\sin x}{\cos x} . \frac{1}{\cos x} \\ &= \frac{\tan x}{\cos x} \\ &= \tan x \sec x \end{aligned}$$

Example 25. Find $\frac{d}{dx}\text{cosec } x$.

Process

As cosec

$$x = \frac{1}{\sin x}$$

$$\begin{aligned}
\frac{d}{dx}\text{cosec}x &= \frac{d}{dx}\frac{1}{\sin x} \\
&= \frac{d}{dx}(\sin x)^{-1} \\
&= -(\sin x)^{-2}.\cos x \\
&= \frac{-\cos x}{\sin^2 x} \\
&= \frac{-\cos x}{\sin x}.\frac{1}{\sin x} \\
&= \frac{\text{cosec}x}{\tan x} \\
&= -\cot x\text{cosec}x
\end{aligned}$$

10 Differentiating Inverse Functions

The inverse of a function is the reflection of the graph of the function in the line $y = x$. If a function, f, has an inverse, it is denoted by f^{-1}. Consider $f(x)$. Then

$$f(f^{-1}(x)) = x$$

Therefore,

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx} x = 1$$

But, by the chain rule, the left hand side is differentiable and so, if $y = f^{-1}(x)$ we can use the chain rule to get:

Differentiating Inverse Fuctions

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

For trigonometric functions, the inverse functions have standard derivatives which are provided on the formula sheet. It may be necessary to use the chain rule when differentiating using the standard derivatives shown below:

Differentiating Inverse Trigonometric Functions

$$\begin{aligned} \frac{d}{dx} \sin^{-1} x &= \frac{1}{\sqrt{1 - x^2}} \\ \frac{d}{dx} \cos^{-1} x &= \frac{-1}{\sqrt{1 - x^2}} \\ \frac{d}{dx} \tan^{-1} x &= \frac{1}{x^2 + 1} \end{aligned}$$

Example 26. If $f(x) = x^2$ find $f'(x)$ and the derivative of $f^{-1}(x)$.

Process

As $f(x) = x^2$, let $y = x^2$, Then:

$$\begin{aligned} y &= x^2 \\ x^2 &= y \\ x &= \sqrt{y} \\ \therefore f^{-1}x &= \sqrt{x}. \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{d}{dx}f^{-1}x &= \frac{d}{dx}\sqrt{x} \\ &= \frac{d}{dx}x^{\frac{1}{2}} \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Example 27. If $y = \sin^{-1}(x)$ find $\frac{dy}{dx}$.

Process

This example will show the steps required to obtain the standard derivative provided in the formula sheet.

$$\begin{aligned} y &= \sin^{-1} x \\ \Rightarrow \sin^{-1} x &= y \\ \Rightarrow x &= \sin y \end{aligned}$$

Hence,

$$\frac{dx}{dy} = \cos y$$

.
To find $\frac{dy}{dx}$, use the fact that

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}.$$

It is necessary to make substitutions later on in the calculations in order to change the y terms back to x.

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} \\
&= \frac{1}{\cos y} \\
&= \frac{1}{\sqrt{1-\sin^2 y}} \qquad \text{as} \sin^2 y + \cos^2 y = 1 \\
&= \frac{1}{\sqrt{1-x^2}}.
\end{aligned}$$

Example 28. Find the derivative of $\sin^{-1} 3x$.

Process

Let $y = \sin^{-1} 3x$. Then, by the standard derivatives and the chain rule:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sqrt{1-(3x)^2}} \times \frac{d}{dx}(3x) \\
&= \frac{3}{\sqrt{1-9x^2}}
\end{aligned}$$

Example 29. Find the derivative of $\ln(\sin^{-1} 2x)$.

Process

$$\begin{aligned}
\frac{d}{dx}\ln(\sin^{-1} 2x) &= \frac{1}{\sin^{-1} 2x} \times \frac{d}{dx}\sin^{-1} 2x \\
&= \frac{1}{\sin^{-1} 2x} \times \frac{1}{\sqrt{1-(2x)^2}} \times 2 \\
&= \frac{2}{\sqrt{1-4x^2}.\ \sin^{-1} 2x}
\end{aligned}$$

11 Implicit Differentiation

A function can be expressed either *explicitly*, where the subject of the formula is clearly identifiable (for example $y = 2x - 3$, $y = x^2 + 5x - 1$) or *implicitly* where the subject of the formula is not clear, variables are linked together or the dependent variable isn't the subject of the formula (eg $2y - 7x + 6 = 0, x^2 + y^2 = 16z$).
Sometimes, an implicit function can be rearranged to be written explicitly. Otherwise, *implcit differentiation* must be used to differentiate the function.

Process of Implicit Differentiation

To differentiate implicitly, follow the steps detailed below:

- first differentiate every term in the function with respect to x.
- For every term involving variables other than x, eg y, differentiate with respect to y and write $\frac{dy}{dx}$ after.
  This is because the derivative with respect to x is $\frac{d}{dx} = \frac{d}{dy} \times \frac{dy}{dx}$
- Rearrange to make $\frac{dy}{dx}$ the subject of the formula and simplify.
- Note that the derivative may be expressed either implicity or explicitly.
- If an explicit derivative is required, it may be easier to rearrange the function into an explicit form before differentiating.
- If a term is a combination of x and y, then the product rule should be applied to differentiate.

Example 30. Find the derivative with respect to x of $4y^2 - 2x^2 + 16x = 0$.

Process

$$\begin{aligned}
4y^2 - 2x^2 + 16x &= 0 \\
\Rightarrow 8y\frac{dy}{dx} - 4x + 16 &= 0 \\
8y\frac{dy}{dx} &= 4x - 16 \\
\frac{dy}{dx} &= \frac{4x - 16}{8y} \\
\frac{dy}{dx} &= \frac{x - 4}{2y}
\end{aligned}$$

Example 31. Differentiate $x^2y + 2x = 0$ with respect to x

Process

$$\begin{aligned}
x^2y + 2x &= 0 \\
\Rightarrow 2xy + x^2 \times 1\frac{dy}{dx} + 2 &= 0 \\
2xy + x^2\frac{dy}{dx} + 2 &= 0 \\
x^2\frac{dy}{dx} &= -2 - 2xy \\
\frac{dx}{dy} &= \frac{-2(1 + xy)}{x^2}
\end{aligned}$$

Example 32. Find an expression for the gradient of the tangent to the curve $x^2 - y^2 = \frac{1}{x}$ at any given point.

Process

$$\begin{aligned}
x^2 - y^2 &= \frac{1}{x} \\
x^3 - xy^2 &= 1 \\
\Rightarrow 3x^2 - \frac{d}{dx}(xy^2) &= 0 \\
3x^2 - (y^2 + 2xy\frac{dy}{dx}) &= 0 \\
3x^2 - y^2 - 2xy\frac{dy}{dx} &= 0 \\
-2xy\frac{dy}{dx} &= y^2 - 3x^2 \\
\frac{dy}{dx} &= \frac{y^2 - 3x^2}{-2xy} \\
\frac{dy}{dx} &= \frac{3x^2 - y^2}{2xy}
\end{aligned}$$

The derivative at the point gives the gradient of the tangent at that point. Therefore, the gradient of the tangent is $\frac{y^2-3x^2}{2y}$.

Second derivatives of implicit functions

Second derivatives for functions defined implicitly can be found by differentiating the first derivative.

Example 33. Find the second derivative of $x^3 + y^2 = y$.

Process First find the first derivative as follows:

$$\begin{aligned} x^3 + y^2 &= y \\ \Rightarrow 3x^2 + 2y\frac{dy}{dx} &= 1 \times \frac{dy}{dx} \\ (2y-1)\frac{dy}{dx} &= -3x^2 \\ \frac{dy}{dx} &= \frac{-3x^2}{2y-1} \end{aligned}$$

Using the quotient rule:

$$\begin{array}{ll} u = -3x^2 & v = 2y-1 \\ u' = -6x & v' = 2\frac{dy}{dX} \end{array}$$

Hence;

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{vu' - uv'}{v^2} \\ &= \frac{-6x(2y-1) - 2(-3x^2)\frac{dy}{dx}}{(2y-1)^2} \\ &= \frac{-6x(2y-1) + 6x^2\left(\frac{-3x^2}{2y-1}\right)}{(2y-1)^2} \\ &= \frac{-6x(2y-1)^2 - 18x^4}{(2y-1)^3} \end{aligned}$$

12 Logarithmic Differentiation

When a function has an awkward index that includes powers or quotients, logarithmic differentiation can be used along with implicit differentiation to find the derivative.

Example 34. Differentiate $y = 7^x$.

ProcessTake logs of both sides and then differentiate:

$$\begin{aligned} y &= 7^x \\ \Rightarrow \ln y &= \ln 7^x \\ \ln y &= x \ln 7 \\ \Rightarrow \frac{d}{dx} \ln y &= \frac{d}{dx}(x \ln 7) \\ \frac{dy}{dx}\frac{1}{y} &= \ln 7 \\ \frac{dy}{dx} &= y \ln 7 \\ &= 7^x \ln 7 \end{aligned}$$

Example 35. Differentiate $y = \frac{3^x}{(2x+3)}$.

ProcessUse the quotient rule to differentiate where u and v are:

$$u = 3^x, \quad v = 2x + 3$$

The derivative $v' = 2$. Use logarithmic differentiation to find u'.

$$\begin{aligned} u &= 3^x \\ \Rightarrow \ln u &= \ln 3^x \\ \ln u &= x \ln 3 \\ \Rightarrow u' = \frac{1}{u}\frac{du}{dx} &= \ln 3 \\ \frac{du}{dx} &= u \ln 3 \\ \frac{du}{dx} &= 3^x \ln 3 \end{aligned}$$

Now apply the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{vu' - uv'}{v^2} \\ &= \frac{(2x+3).3^x \ln 3 - 3^x \times 2}{(2x+3)^2} \\ &= \frac{(2x+3).3^x \ln 3 - 3^x \times 2}{(2x+3)^2}\end{aligned}$$

13 Parametric differentiation

Functions can be defined *parametrically* using a third variable, often t, where $x = x(t)$ and $y = y(t)$. In this case, $x = x(t)$ and $y = y(t)$ are parametric equations and t is the parameter. For example, consider the function defined parametrically as:

$$\begin{aligned} x &= t + 5 \\ y &= t^2 \end{aligned}$$

Each value of t will give a point on the curve.

Parametric equations can sometimes be changed back to cartesian equations by eliminating the parameter. In the above example, it is possible to rewrite the the equation for x as $t = x - 5$. The substituting into the parametric equation for y gives

$$y = (x-5)^2 = x^2 - 10x + 25$$

The parametric equation is representing a parabola in this case.

Parametric equations can still be differentiated to find $\frac{dy}{dx}$ using parametric differentiation. Remember that, for any variable t, $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$. Therefore:

For functions $x(t)$ and $y(t)$ defined parametrically in terms of the parameter t,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

where

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$$

Similarly,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}$$

Example 36. Find $\frac{dy}{dx}$ in terms of t when

$$\begin{aligned} x &= \tfrac{2}{t} \\ y &= \sqrt{t^2 - 3} \end{aligned}$$

Process First differentiate both parametric equations $x(t)$ and $y(t)$ with respect to t.

$$\begin{aligned} x &= \frac{2}{t} = 2t^{-1} \\ \frac{dx}{dt} &= -2t^{-2} = -\frac{2}{t^2} \\ \frac{dt}{dx} &= -\frac{t^2}{2} \end{aligned}$$

$$\begin{aligned} y &= \sqrt{t^2 - 3} = (t^2 - 3)^{\frac{1}{2}} \\ \frac{dy}{dt} &= \frac{1}{2}(t^2 - 3)^{-\frac{1}{2}} \times 2t \\ \frac{dy}{dt} &= t(t^2 - 3)^{-\frac{1}{2}} \end{aligned}$$

Hence,

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \times \frac{dt}{dx} \\ &= t(t^2 - 3)^{-\frac{1}{2}} \times -\frac{t^2}{2} \\ &= \frac{-t^3}{2(t^2 - 3)^{\frac{1}{2}}} \\ &= \frac{-t^3}{2\sqrt{t^2 - 3}} \end{aligned}$$

Example 37. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of θ when

$$\begin{aligned} x &= 4\cos\theta \\ y &= -\sin 2\theta \end{aligned}$$

Process First differentiate the parametric equations with respect to θ.

$$\begin{aligned} x &= 4\cos\theta \\ \frac{dx}{d\theta} &= -4\sin\theta \\ \Rightarrow \frac{d\theta}{dx} &= \frac{-1}{4\sin\theta} \end{aligned}$$

$$\begin{aligned} y &= -\sin 2\theta \\ \frac{dy}{d\theta} &= -2\cos 2\theta \end{aligned}$$

Therefore, $\frac{dy}{dx}$ can be found via:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{d\theta} \times \frac{d\theta}{dx} \\ &= -2\cos 2\theta \times \frac{-1}{4\sin\theta} \\ &= \frac{2\cos 2\theta}{4\sin\theta} \\ &= \frac{\cos 2\theta}{2\sin\theta}. \end{aligned}$$

To find the second derivative, differentiate $\frac{dy}{dx}$ with respect to θ and then multiply by $\frac{d\theta}{dx}$.

$$
\begin{aligned}
\frac{d^2y}{dx^2} &= \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \times \frac{d\theta}{dx} \\
&= \frac{d}{d\theta}\left(\frac{\cos 2\theta}{2\sin\theta}\right) \times \frac{-1}{4\sin\theta} \\
&= \frac{d}{d\theta}\left(\frac{1}{2}\cos 2\theta(\sin\theta)^{-1}\right) \times \frac{-1}{4\sin\theta} \\
&= \left(\frac{1}{2} \times (-2)\sin 2\theta(\sin\theta)^{-1} + \frac{1}{2}\cos 2\theta \times (-1) \times \sin\theta)^{-2} \times \cos\theta\right) \times \frac{-1}{4\sin\theta} \\
&= \left(-\frac{\sin 2\theta}{\sin\theta} - \frac{\cos 2\theta\cos\theta}{\sin^2\theta}\right) \times \frac{-1}{4\sin\theta} \\
&= \frac{\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{4\sin^3\theta}
\end{aligned}
$$