

St Columba's High School
Advanced Higher Maths
The Binomial Theorem

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1 Introduction to Binomials

A binomial is an algebraic expression for the sum (or difference) of strictly two terms. From Higher and National 5 maths, the term trinomial will be familiar to describe an expression that is the sum or difference of three terms, for example $x^2 + 3x + 2$. The following expressions are examples of binomials:

$$x + 2y, \quad 3x - 7y, \quad 6 - 8y$$

Previously, to expand a binomial of the form

$$(x + y)^3$$

the brackets would be written out fully as

$$(x + y)(x + y)(x + y)$$

and then multiplied out as follows:

$$\begin{aligned}(x + y)(x + y)(x + y) &= (x + y)(x^2 + 2xy + y^2) \\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3.\end{aligned}$$

When considering binomials raised to higher powers, eg $(x + 2y)^9$, writing out the brackets in full and multiplying out as shown above becomes very time consuming. The binomial theorem provides an efficient way of writing out the expansion of these brackets.

However, some definitions are necessary prior to considering the binomial theorem and binomial expansions.

2 Factorial Notation

Factorials

Factorials are a quick way of writing down the product of all of the integers starting with the given number.

Definition 1. The product of the integers

$$n! = n(n - 1)(n - 2)\dots \times 3 \times 2 \times 1$$

where $n!$ is pronounced n factorial.

Definition 2. $n! = n \times (n - 1)!$

Definition 3. $0! = 1$

Example 1. Find $4!$.

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

Example 2. Find $7!$.

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

3 Combinations and ${}_nC^r$

Another notation that is used in the binomial theorem is ${}_nC_r$. This allows us to work out the number of possible ways of choosing r components from a total of n . For example, consider the integers 1 to 5. How many different combinations of 3 numbers can be made from these?

$$\begin{array}{cccc} 1 & 2 & 3 & 1 & 3 & 4 & 2 & 3 & 4 & 3 & 4 & 5 \\ 1 & 2 & 4 & 1 & 3 & 5 & 2 & 3 & 5 & & & \\ 1 & 2 & 5 & 1 & 4 & 5 & 2 & 4 & 5 & & & \end{array}$$

Hence, there are 10 possible ways of combining three numbers out of 5.

However, there is a more efficient way to calculate these combinations without writing out a list in full.

Definition 4.

$${}^n C_r = \frac{n!}{(n-r)!r!}$$

An alternative notation for this is:

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

From the discussion above, the number of possible ways of combining three numbers out of 5 is

$$\binom{5}{3} = \frac{5!}{(5-3)!3!} = 10$$

Example 3. Find the value of ${}^7 C_4$

Process

$$\begin{aligned} {}^7 C_4 &= \frac{7!}{4!(7-4)!} \\ &= \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)} \\ &= \frac{7 \times 6 \times 5}{3 \times 2 \times 1} \\ &= 35 \end{aligned}$$

Example 4. Find $\binom{10}{5}$

Process

$$\begin{aligned}\binom{10}{5} &= \frac{10!}{(10-5)!5!} \\ &= \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1)(5 \times 4 \times 3 \times 2 \times 1)} \\ &= \frac{10 \times 9 \times 8 \times 7}{5 \times 4} \\ &= \frac{1260}{5} \\ &= 252\end{aligned}$$

4 Pascal's Triangle & Binomial Coefficient

Another means of finding coefficients in the binomial expansion is to use Pascal's triangle. This is formed by adding the two terms directly above to make the next row. The triangle starts at row 0.

				1				Row 0
				1	1			Row 1
			1	2	1			Row 2
		1	3	3	1			Row 3
	1	4	6	4	1			Row 4
1	5	10	10	5	1			Row 5

This triangle can be linked to the $\binom{n}{r}$ notation as shown:

			$\binom{0}{0}$				Row 0
			$\binom{1}{0}$		$\binom{1}{1}$		Row 1
		$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$	Row 2
$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$	Row 3

Hence, $\binom{3}{2}$ means the second entry in Row 3 of Pascal's triangle. (Note that the initial entry in each row is classed as the 0th entry). Looking at the triangle, it is clear that this is 3. This can be seen to

be true as

$$\begin{aligned} \binom{3}{2} &= \frac{3!}{2!(3-2)!} \\ &= \frac{3 \times 2 \times 1}{(2 \times 1) \times 1} \\ &= 3 \end{aligned}$$

Definition 5. As each term in Pascal's triangle is obtained by adding together the two entries directly above, in general:

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Example 5. Solve $\binom{n}{2} = 3$ for $n \in \mathbb{N}$ where \mathbb{N} is the set of natural numbers which includes only all positive integers.

Process

$$\begin{aligned} \binom{n}{2} &= 3 \\ \Rightarrow \frac{n!}{2!(n-2)!} &= 3 \\ \Rightarrow \frac{(n(n-1)(n-2) \cdots \times 3 \times 2 \times 1)}{(2 \times 1)((n-2)(n-3) \cdots \times 3 \times 2 \times 1)} &= 3 \\ \frac{n(n-1)}{2} &= 3 \\ n^2 - n &= 6 \\ n^2 - n - 6 &= 0 \\ (n-3)(n+2) &= 0 \\ \Rightarrow n = 3 \text{ or } n = -2 \end{aligned}$$

However, as $n \in \mathbb{N}$, the solution must be positive and so $n = 3$.

Example 6. For $n \in \mathbb{N}$, solve $\binom{n}{1} + \binom{n}{2} = 28$

Process

$$\begin{aligned}\binom{n}{1} + \binom{n}{2} &= 28 \\ \Rightarrow \binom{n+1}{2} &= 28 \\ \Rightarrow \frac{(n+1)!}{2!(n+1-2)!} &= 28 \\ \Rightarrow \frac{(n+1)n(n-1)(n-2)\cdots \times 3 \times 2 \times 1}{(2 \times 1)((n-1)(n-2)\cdots \times 3 \times 2 \times 1)} &= 28 \\ \frac{(n+1)n}{2} &= 28 \\ n^2 + n &= 56 \\ n^2 + n - 56 &= 0 \\ (n-7)(n+8) &= 0 \\ \Rightarrow n = 7 \text{ or } n = -8\end{aligned}$$

However, as $n \in \mathbb{N}$, the solution must be positive and so $n = 7$.

5 The Binomial Theorem

A binomial of the form $(x + y)^n$, can be expanded (multiplied out) using the coefficients from Pascal's triangle. For example $(x + y)^4$ can be expanded using the numbers found in the fourth row of Pascal's triangle as shown:

$$(x + y)^4 = \mathbf{1}x^4 + \mathbf{4}x^3y + \mathbf{6}x^2y^2 + \mathbf{4}xy^3 + \mathbf{1}y^4$$

In general, for any binomial, the expansion can be found using the binomial theorem.

Definition 6. The Binomial Theorem

$$\begin{aligned}(x + y)^n &= \sum_{r=0}^n \binom{n}{r} x^{n-r} y^r \\ &= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \cdots + \binom{n}{n} y^n\end{aligned}$$

where \sum is pronounced "sigma" and is the notation to define the sum of terms (this notation was used before in calculating standard deviation).

Note that the coefficients in the expansion come from the row of Pascal's triangle that corresponds to the power of the binomial.

As the powers of the first term increase, the powers of the second term decrease.

The sum of the total powers in each part of the expansion must equal the power of the original binomial.

Example 7. Expand $(x + 2y)^4$ using the Binomial Theorem.

Process

$$\begin{aligned} & (x + y)^4 \\ &= \sum_{r=0}^4 \binom{4}{r} x^{4-r} (2y)^r \\ &= \binom{4}{0} x^4 + \binom{4}{1} x^3(2y) + \binom{4}{2} x^2(2y)^2 + \binom{4}{3} x(2y)^3 + \binom{4}{4} (2y)^4 \\ &= x^4 + 4 \times 2x^3y + 6 \times 4x^2y^2 + 4 \times 8xy^3 + 16y^4 \\ &= x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4 \end{aligned}$$

Example 8. Expand $(3x - 2y)^3$ using the binomial expansion.

Process

$$\begin{aligned} & (3x - 2y)^3 \\ &= \sum_{r=0}^3 \binom{3}{r} (3x)^{3-r} (-2y)^r \\ &= \binom{3}{0} (3x)^3 + \binom{3}{1} (3x)^2(-2y) + \binom{3}{2} 3x(-2y)^2 + \binom{3}{3} (-2y)^3 \\ &= 1 \times (3x)^3 + 3 \times (3x)^2(-2y) + 3(3x)(-2y)^2 + (-2y)^3 \\ &= 27x^3 - 54x^2y + 36xy^2 - 8y^3 \end{aligned}$$

Example 9. Find the coefficient of x^6 in the expansion of $(3 - x)^9$.

Process

$$(3 - x)^9 = \sum_{r=0}^9 \binom{9}{r} 3^{9-r} (-x)^r$$

The coefficient of x^6 is calculated as follows:

$$\begin{aligned} & \binom{9}{6} 3^{9-6} (-x)^6 \\ &= \frac{9!}{3!6!} \times 3^3 (-x)^6 \\ &= \frac{9 \times 8 \times 7 \times 6!}{3!6!} \times 27x^6 \\ &= \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times 27x^6 \\ &= 84 \times 27x^6 \\ &= 2248x^6 \end{aligned}$$

Hence the coefficient of x^6 in the expansion of $(3 - x)^9$ is 2248.

Example 10. Expand $(y - \frac{2}{y})^3$.

Process

Using the Binomial theorem alongside the entries in the third row of Pascal's triangle gives:

$$\begin{aligned} &= \sum_{r=0}^3 \binom{3}{r} (y)^{3-r} \left(-\frac{2}{y}\right)^r \\ &= 1 \times (y)^3 + 3 \times (y)^2 \left(-\frac{2}{y}\right) + 3 \times y \left(-\frac{2}{y}\right)^2 + 1 \times \left(-\frac{2}{y}\right)^3 \\ &= y^3 - 6\frac{y^2}{y} + 12\frac{y}{y^2} - \frac{8}{y^3} \\ &= y^3 - 6y + \frac{12}{y} - \frac{8}{y^3} \end{aligned}$$

6 General Term of the Binomial Expansion

Definition 7. The general term of $(x + y)^n$ is given by :

$$\binom{n}{r} x^{n-r} y^r$$

Example 11. Find the coefficient of x^3y^4 in the expansion of $(x + y)^7$

Process

The general term is:

$$(x + y)^7 = \binom{7}{r} x^{7-r} y^r$$

As the coefficient of x^3y^4 is to be found and the general term has powers $x^{7-r}y^r$, by equating powers, it is clear that $r = 4$. Therefore, the coefficient is:

$$\binom{7}{4} = \frac{7!}{4!(7-4)!} = 35$$

Example 12. Find the coefficient of x^4 in the expansion of $(1 + 2x^2)^3$.

Process

The general term is

$$\begin{aligned} & (1 + 2x^2)^3 \\ &= \binom{3}{r} 1^{3-r} (2x^2)^r \\ &= \binom{3}{r} 2^r x^{2r} \end{aligned}$$

Hence, for the coefficient of x^4 , $2r$ must be equal to 4 and so $r = 2$.

$$\begin{aligned} & \binom{3}{2} 2^2 x^4 \\ &= \frac{3!}{2!1!} \times 2^2 x^4 \\ &= 12x^4 \end{aligned}$$

Hence, the coefficient of x^4 is 12.

Example 13. Find the coefficient of y^5 in the expansion of $(y - \frac{1}{y})^5$.

Process

The general term is

$$\begin{aligned} (y - \frac{1}{y})^5 &= \binom{5}{r} y^{5-r} \left(-\frac{1}{y}\right)^r \\ &= \binom{5}{r} y^{5-r} (-1)^r y^{-r} \\ &= \binom{5}{r} (-1)^r y^{5-2r} \end{aligned}$$

Hence, for the coefficient of y^5 , $5 - 2r = 5$ and so $r = 0$.

$$\begin{aligned} & \binom{5}{0} (-1)^0 y^5 \\ &= 1 \end{aligned}$$

Hence, the coefficient of y^5 is 1.

Note that in cases where the term independent of x is to be found, it is necessary to look at the general term and identify the value of r that would make the power of x equal zero.

7 Approximations

The binomial theorem can also be used with decimal numbers to find approximations.

Example 14. Find 1.02^3 correct to three decimal places.

Process

$$\begin{aligned}1.02^3 &= (1 + 0.02)^3 \\&= \binom{3}{0} 1^3 + \binom{3}{1} 1^2(0.02) + \binom{3}{2} 1(0.02)^2 + \binom{3}{3} (0.02)^3 \\&= 1 + 3 \times 0.02 + 3 \times 0.02^2 + 0.02^3 \\&= 1 + 0.06 + 0.0012 + 0.000008 \\&= 1.061 \text{ to 3 decimal places}\end{aligned}$$

Example 15. Find 0.9^5 correct to two decimal places.

Process

$$\begin{aligned}0.9^5 &= (1 - 0.1)^5 \\&= \binom{5}{0} 1^5 + \binom{5}{1} 1^4(-0.1) + \binom{5}{2} 1^3(-0.1)^2 + \binom{5}{3} 1^2(-0.1)^3 + \dots \\&= 1 + 5(-0.1) + 10(0.01) - 10(0.001) + \dots \\&= 1 - 0.5 + 0.1 - 0.01 + \dots \\&\approx 0.59 \text{ to 2 decimal places}\end{aligned}$$