Chapter 1

Exercise 1A

1 **a** $\frac{1}{x+2}+\frac{3}{x-4}$ **b** $\frac{5}{x+1}+\frac{2}{x+3}$

c $\frac{2}{x-1}+\frac{3}{x+2}$ **d** $\frac{5}{x}+\frac{4}{x-1}$

e $\frac{1}{x+1}+\frac{1}{x-4}$ **f** $\frac{4}{x+5}-\frac{3}{x+1}$

g $\frac{2}{x}-\frac{1}{x-3}$ **h** $\frac{4}{x-2}+\frac{6}{x+3}$

2 **a** $\frac{1}{x+2}+\frac{1}{x+3}$

b $\frac{1}{x-1}+\frac{1}{x+3}$

c $\frac{3}{x+4}-\frac{2}{x+3}$

d $\frac{5}{2x+1}-\frac{3}{x+4}$

e $\frac{3}{2x-3}-\frac{4}{3x+1}$

f $\frac{1}{2x+1}-\frac{1}{2x-1}$

3 **a** $\frac{1}{2(x+2)}+\frac{3}{2x}$

b $\frac{5}{2(x+3)}-\frac{1}{2(x+1)}$

c $\frac{2}{3(x+6)}-\frac{1}{3(2x+3)}$

4 **a** $\frac{1}{x-1}+\frac{2}{2x-1}-\frac{1}{x+3}$

b $\frac{3}{2(x+1)}-\frac{1}{x+2}-\frac{1}{2(x+3)}$

c $\frac{3}{2x}+\frac{3}{10(x+2)}-\frac{8}{5(2x-1)}$

Exercise 1B

1 **a** $\frac{1}{x+1}+\frac{x+2}{x^2+x+1}$

b $\frac{1}{x+2}+\frac{x+3}{x^2-5x+1}$

c $\frac{3}{x+2}+\frac{x+4}{x^2+1}$

d $-\frac{2}{x+3}+\frac{x+1}{x^2+2x+5}$

e $\frac{1}{2x+1}+\frac{2x+3}{x^2-x+3}$

f $\frac{2}{4x+3}+\frac{1}{2x^2+5x+1}$

g $\frac{5}{2x+1}-\frac{x+1}{x^2+x+3}$

h $\frac{3}{2x+3}-\frac{x-3}{x^2+x+2}$

i $-\frac{2}{x-1}-\frac{5}{x^2+2}$

j $\frac{2}{x-3}-\frac{2x-3}{2x^2-x+1}$

2 **a** $-\frac{1}{2(x+3)}+\frac{x+1}{2x^2+4x+6}$

b $\frac{2}{3(x+1)}+\frac{x+2}{3x^2+6x+15}$

c $\frac{3}{2(x+2)}+\frac{5}{2x^2+4x+8}$

3 **a** $\frac{3}{x+1}+\frac{2x+1}{x^2-2x-1}$

b $\frac{4}{x-2}-\frac{3x+1}{x^2-3x+1}$

c $-\frac{2}{x-3}-\frac{3x-1}{x^2+x+5}$

Exercise 1C

1 **a** $\frac{3}{x+1}+\frac{2}{x+3}-\frac{4}{(x+3)^2}$

b $\frac{1}{2x+1}+\frac{2}{x+1}-\frac{3}{(x+1)^2}$

c $\frac{2}{x+2}+\frac{5}{(x+2)^2}-\frac{1}{x-1}$

d $-\frac{2}{x}-\frac{3}{x^2}+\frac{1}{x-2}$

e $\frac{3}{x}+\frac{13}{(x+1)}-\frac{19}{(x+1)^2}$

f $\frac{5}{9(2-x)}+\frac{5}{9(x+1)}+\frac{5}{3(x+1)^2}$

2 **a** $-\frac{2}{9x}-\frac{2}{3x^2}+\frac{2}{9(x-3)}$

b $\frac{2}{x-1}-\frac{2}{x+2}-\frac{3}{(x+2)^2}$

c $\frac{4}{x-3}-\frac{3}{x-2}-\frac{1}{(x-2)^2}$

d $\frac{12}{3x+2}-\frac{4}{x+1}-\frac{1}{(x+1)^2}$

e $\frac{1}{x+3}-\frac{2}{(x+3)^2}+\frac{4}{(x+3)^3}$

f $-\frac{3}{x+1}+\frac{2}{(x+1)^2}+\frac{4}{x-1}$

Exercise 1D

1 **a** $3+\frac{1}{x+1}$ **b** $4+\frac{3}{x-2}$

c $-1+\frac{2}{x-5}$ **d** $2-\frac{1}{2x-1}$

e $5-\frac{1}{3x-8}$

2 **a** $x+1+\frac{5}{2x-3}$

b $2x-1+\frac{3}{x+4}$

c $3x+1-\frac{5}{x+1}$

d $x^2+2x+8-\frac{7}{x+1}$

e $2x^2-x-5-\frac{7}{x+3}$

3 **a** $3+\frac{x+1}{x^2+2x+3}$

b $4-\frac{2x-2}{2x^2+x+3}$

c $x+4-\frac{x+1}{3x^2+x+2}$

Exercise 1E

1 **a** $2+\frac{1}{x+1}+\frac{2}{x+3}$

b $5+\frac{3}{x+2}+\frac{1}{3x+1}$

c $3-\frac{1}{x-2}+\frac{2}{x+4}$

d $-1-\frac{5}{x-3}-\frac{2}{x-1}$

2 **a** $3+\frac{1}{x^2+2x+5}-\frac{2}{x+3}$

b $2+\frac{3}{x^2+3x+4}-\frac{1}{x-2}$

c $1-\frac{1}{x}-\frac{x}{x^2+x+1}$

3 **a** $x+1+\frac{1}{x-3}-\frac{1}{x+1}$

b $x+3+\frac{1}{x}-\frac{5}{x+1}$

c $x-2+\frac{5}{x-2}-\frac{3}{x-3}$

4 **a** $x+\frac{1}{x}-\frac{2}{x^2+1}$

b $x+\frac{3}{x}+\frac{2}{x-2}-\frac{1}{(x-2)^2}$

Exercise 1F

1 **a** **i** 35 **ii** 3 **iii** 252 **iv** 5

b **i** 35 **ii** 3 **iii** 252 **iv** 5

c $\binom{6}{4}=\binom{6}{2}=15$.

They are the same because $\binom{6}{4}=\frac{6!}{4!2!}$ and $\binom{6}{2}=\frac{6!}{2!4!}$ so, by inspection, the denominators are the same.

2 **a** 133 784 560

b 924

c 455

d 31 073 658 more selections

3 **a** ${}^4C_2 + {}^4C_3 + {}^4C_4 = 11$ bets

b 26 bets

c 120 bets

d 247 bets

Exercise 1G

1 **a** 10, 10

b 45, 45

c 495, 495

d $\binom{n}{r} = \binom{n}{n-r}$

e $\binom{2n}{r} = \binom{2n}{2n-r}$

2 **a** $n = 3$ **b** $n = 5$ **c** $n = 7$

d $n = 11$ **e** $n = 2$ **f** $n = 3$

g $n = 5$ **h** $n = 7$

3 **a** $n = 3$ **b** $n = 8$ **c** $n = 12$

4 **a** $n = 3$ **b** $n = 5$ **c** $n = 5$

d $n = 3$

5 **a**

$$\binom{n}{2} + \binom{n}{3}$$

$$= \frac{n!}{2!(n-2)!} + \frac{n!}{3!(n-3)!}$$

$$= \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!}$$

$$= \frac{3n(n-1)}{3!} + \frac{n(n-1)(n-2)}{3!}$$

$$= \frac{3n(n-1) + n(n-1)(n-2)}{3!}$$

$$= \frac{n(n-1)[3 + (n-2)]}{3!}$$

$$= \frac{(n+1)n(n-1)}{3!}$$

$$= \frac{(n+1)n(n-1)(n-2)}{3!(n-2)!}$$

$$= \frac{(n+1)!}{3!(n-2)!} = \binom{n+1}{3}$$

5 **b-e** Student's own answers, but should follow similar steps to Q5 part **a** given above.

Exercise 1H

1 **a** $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

b $a^3 - 3a^2b + 3ab^2 - b^3$

c $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$

d $16 - 96m + 216m^2 - 216m^3 + 81m^4$

e $16a^4 + 96a^3b + 216a^2b^2 + 216ab^3 + 81b^4$

f $x^8 - 4x^6y^3 + 6x^4y^6 - 4x^2y^9 + y^{12}$

g $3125 - 9375x + 11250x^2 - 6750x^3 + 2025x^4 - 243x^5$

h $125f^3 + 150f^2g + 60fg^2 + 8g^3$

2 **a** $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$

b $16m^4 - 32m^2 + 24 - \frac{8}{m^2} + \frac{1}{m^4}$

c $z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}$

d $729x^6 - 729x^4 + \frac{1215}{4}x^2 - \frac{135}{2} + \frac{135}{16x^2} - \frac{9}{16x^4} + \frac{1}{64x^6}$

e $x^{10} - 15x^7 + 90x^4 - 270x + \frac{405}{x^2} - \frac{243}{x^5}$

3 **a** **i** $\binom{4}{r}5^r x^{4-r}$

ii 20

b **i** $\binom{12}{r}4^{12-r}x^r$

ii 126 720

c **i** $\binom{8}{r}5^{8-r}3^r x^r$

ii 4 725 000

d **i** $\binom{7}{r}(-1)^r\, 2^r\, x^r$

ii 560

e **i** $\binom{18}{r}3^{18-r}2^r x^r$

ii 55 431 806 976

4 **a** −8064 **b** 747 242 496

c 344 064

5 **a** $x^3 + 5x^2 + 8x + 4$

b $x^6 + 11x^5 + 50x^4 + 120x^3 + 160x^2 + 112x + 32$

c 144

6 **a** $1 + 3x - 5x^3 + 3x^5 - x^6$

b $\frac{x^{10}}{32} + \frac{15x^9}{16} + \frac{95x^8}{8} + \frac{165x^7}{2} + \frac{685x^6}{2} + 873x^5 + 1370x^4 + 1320x^3 + 760x^2 + 240x + 32$

Exercise 1I

1 **a** $1 + 20x + 190x^2 + 1140x^3$; 1·220 14

b $1 + 16x + 112x^2 + 448x^3$; 1·171 648

c $1 + 24x + 264x^2 + 1760x^3$; 1·268 16

d $1 - 40x + 720x^2 - 7680x^3$; 0·664 32

e $1 + 10x + 45x^2 + 120x^3$; 0·980 179 04

f $1 - \frac{7}{10}x + \frac{21}{100}x^2 - \frac{7}{200}x^3$; 0·932 065

g $512 + \frac{2304}{5}x + \frac{4608}{25}x^2 + \frac{5376}{125}x^3$; 793 856

2 **a** $1 + 10x + 45x^2 + 120x^3$; 1·104 62

b $1 - 20x + 180x^2 - 960x^3$; 0·817 04

Chapter review

1 **a** $\frac{5}{x+2} - \frac{5}{x+3}$

b $-\frac{5}{x+1} - \frac{2}{(x+1)^2} + \frac{10}{2x+1}$

c $-\frac{1}{4(x+1)} + \frac{x+9}{4\left(x^2+2x+5\right)}$

d $\frac{2}{x-1} + \frac{1}{x-2} - \frac{3}{x-3}$

e $\frac{6}{x} + \frac{1}{x+2} - \frac{4}{x-2}$

f $2 - \frac{3}{2x+1} + \frac{1}{x-1}$

g $x + \frac{3}{x} - \frac{4}{x-1}$

2 **a** $n = 7$

b $n = 5$

c $n = 5$

d Student's own answer, but should follow similar steps to Exercise 1G Q5.

3 **a** $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

b $27x^3 - 54x^2y + 36xy^2 - 8y^3$

c $x^{10} - 15x^7 + 90x^4 - 270x + \frac{405}{x^2} - \frac{243}{x^5}$

4 **a** $\binom{7}{r}(-1)^r\, 3^r 2^{7-r} x^r$; $-20\,412x^5$

b $\binom{6}{r}(-1)^r\, 3^r x^{12-3r}$; $-540x^3$

5 **a** 1·46

b 0·913

Chapter 2

Exercise 2A

1 **a** $4x^3 - \sin x$

b $3\cos 3x + 5x^4$

c $10x - \dfrac{8}{x^5}$

d $-2\sin\left(2x + \dfrac{\pi}{3}\right)$

e $4\cos 4x - 4\sin 4x$

f $\dfrac{-6(x+2)}{x^5}$

g $10x - \dfrac{2}{x^5}$

h $\dfrac{1}{2}\left(\dfrac{9}{x^4} - \sin\left(\dfrac{x}{2}\right)\right)$

i $15(5x+1)^2$

2 **a** $3(4x^2 - 7x)^2\,(8x - 7)$

b $2x\cos(x^2)$

c $\dfrac{-\sin\sqrt[3]{x}}{3\sqrt[3]{x^2}}$

d $-(2x+3)\sin(x^2+2x)$

e $3\cos x\sin^2 x$

f $\dfrac{24}{(1-4x)^3}$

g $-2\sin 3x\cos 3x$ or $-\sin 6x$

h $\dfrac{-\cos x}{\sin^2 x}$ or $\dfrac{-1}{\tan x\sin x}$

i $\dfrac{\sin x}{\cos^2 x}$ or $\dfrac{\tan x}{\cos x}$

3 **a** $-\cos(\cos x)\sin x$

b $4(2x+3)\,(x^2+3x+1)^3$

c $\sin(\cos x)\sin x$

d $\cos(\sin x)\cos x$

e $6\sin 3x\cos 3x$

f $-2\cos(\sin x)\sin(\sin x)\,\cos x$

g $2(x + \sin 3x)\,(1 + 3\cos 3x)$

h $\dfrac{2(x+1)}{(x^2+2x+1)^2}\sin\left(\dfrac{1}{x^2+2x+1}\right)$

i $\dfrac{-6\cos(3x+1)}{\sin^3(3x+1)}$

4 $\dfrac{2(x+1)\sin(x^2+x)}{\cos^2(x^2+x)}$

5 $\dfrac{\cos(\cos x)\sin x}{\sin^2(\cos x)}$

6 $\dfrac{-3\cos(3x+2)}{2\sqrt{\sin^3(3x+2)}}$

7 $3\cos 2x\sqrt{\sin 2x}$

Exercise 2B

1 **a** $3e^{3x+1}$

b $\dfrac{1}{x} - 12x^2$

c $-e^{-x} + 4$

d $\dfrac{1}{x+1} + 2e^{2x+1}$

e $\dfrac{-2}{e^x} - x^2$

f $\dfrac{2}{x+2}$

g $6(x+1)\,e^{x^2+2x}$

h $e^{\frac{x}{2}}$

i $\dfrac{1}{x\ln x}$

2 **a** $\dfrac{x\,e^{x^2}}{\sqrt{e^{x^2} - 3}}$

b $-2\tan 2x$

c $-2\cos 2x\,e^{\sin 2x}$

Exercise 2C

1 **a** $20x\cos 5x + 4\sin 5x$ or $4(5x\cos 5x + \sin 5x)$

b $10x^3(x-2)^5(5x-4)$

c $\frac{1}{2}\sqrt{x}\ \cos x(3\cos x - 4x\sin x)$

d $\frac{1}{2\sqrt{x-4}}(\cos 3x - 6\sin 3x(x-4))$

e $\frac{7(x+3)}{6\sqrt{x}\ \sqrt[3]{(2x+7)}}$

f $(x-1)^3(3x-2)^4(27x-23)$

g $x^4(1 + 5\ln x)$

h $3x^2\cos 2x - 2\sin x(x^3+1)$

i $e^{2x}(\cos x + 2\sin x)$

2 **a** $\frac{1}{x\,e^x}(1 - x\ln x)$

b $-\frac{e^{\frac{1}{x}}\cos x}{x^2}(2x^2\sin x + \cos x)$

c $\frac{1}{2x\ \sqrt{x-3}}\big(x\ln x + 2(x-3)\big)$

d $5x\,e^{\sin x}(2 - x\cos x)$

e $6x(2\cos 3x - 3x\sin 3x)$

f $x^2(17x^2+35)(x^3+5x)^4$

g $\frac{-2\sin 2x}{3+\cos 2x}$

h $e^x(\sin(x^2) + 2x\cos(x^2))$

3 When $x = 0$, $\frac{dy}{dx} = \frac{3 + \ln 4}{2}$

4 When $x = 1$, gradient $= 3$

5 Proof

Exercise 2D

1 **a** $\frac{1}{\cos^2 x}$

b $\frac{2x\cos x - \sin x}{2x\sqrt{x}}$

c $\frac{x(2\cos x + x\sin x)}{\cos^2 x}$

d $\frac{1 - 3\ln x}{x^3}$

e $\frac{1-6x}{2\sqrt{x}\ e^{3x}}$

f $\frac{e^{2x}(2x-3)}{x^4}$

2 **a** $\frac{(x-2)^2(5-x)}{e^x}$

b $-\frac{2\cos x}{\sin^3 x}$

c $\frac{-(3x^2 - 2x + 3)}{2\sqrt{x}\ (x-3)^2\ (x+1)^2}$

d $\frac{(e^x + e^{-x})(e^x - x) - (e^x - e^{-x})(e^x - 1)}{(e^x - x)^2}$

e $\frac{x}{(x^2+1)}$

f $\frac{2}{(1+x)(1-x)}$

3 **a** $\frac{1 - x\,e^x}{(e^x - x)^2}$

b $\frac{\ln 7x - 1}{(\ln 7x)^2}$

c $\frac{-\ln x}{3x^2}$

4 $f'(5) = \frac{55}{16}$

5 $f'(\pi) = -\frac{2}{\pi^3}$

6 When $x = 2$, $\frac{dy}{dx} = 2\frac{2}{9}$

7 $-\left(\frac{1 - x\ln x}{x\,e^x}\right)\sin\left(\frac{\ln x}{e^x}\right)$

Exercise 2E

1 **a** 48

b 72

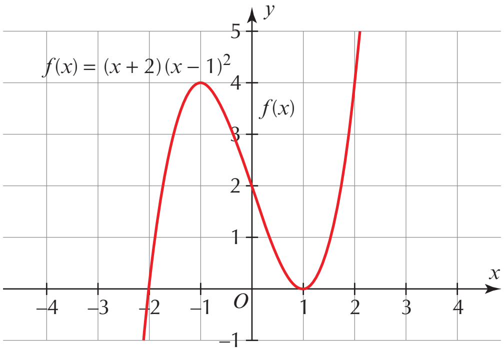
c $48(e^{2x} + \sin 2x)$

d $\frac{6(80x^5 - 1)}{x^4}$

e $\frac{2}{(x - 4)^3}$

f $48e^{2x} + \frac{120}{x^6}$ or $\frac{24}{x^6}(2x^6 e^{2x} + 5)$

2



3 a Minimum SP = $(3,\ 9(1 - \ln 27))$

b Minimum SP = $(\ln 10,\ 10(1 - \ln 10))$

4 a $\frac{d^2y}{dx^2} = 12x^2 - 32x + 16$,

minimum SP when $x = 0$,
rising PI when $x = 2$

b $\frac{d^2y}{dx^2} = -\sin x - \cos x$,

maximum SP when $x = \frac{\pi}{4}$,

minimum SP when $x = \frac{5\pi}{4}$

c $\frac{d^2y}{dx^2} = -\frac{1}{2}\sin\theta - 4\sin 2\theta$, maximum SP when $x = 0{\cdot}6474$, minimum SP when $x = -0{\cdot}7724$

5 acceleration after 3 seconds = $-18\,\text{ms}^{-2}$ (deceleration)

6 acceleration after 5 hours = $60\,\text{mph}^2$

7 a $s(0) = 0$ metres, $v(0) = 9\,\text{ms}^{-1}$
$a(0) = 6\,\text{ms}^{-2}$

b body is instantaneously at rest after 3 seconds

Exercise 2F

1 a $5\sec 5x \tan 5x$

b $3\sec^2 3x$

c $-2\text{cosec}^2\, 2x$

d $-4\text{cosec}4x \cot 4x$

e $-2x\, \text{cosec}^2\,(x^2)$

f $\frac{-\text{cosec}\sqrt{2x + 1}\ \cot\sqrt{2x + 1}}{\sqrt{2x + 1}}$

2 a $\frac{\sec x(x\ln x\tan x - 1)}{x\ln^2 x}$

b $-\sec^2 x\, \text{cosec}^2\,(\tan x)$

c $2x\tan(x^2)$

d $-6\,\text{cosec}^2(3x - 2)\cot(3x - 2)$

e $\frac{-e^{\cot x}(x\ln x\ \text{cosec}^2 x + 1)}{x\ln^2 x}$

f $8\tan 4x \sec^2 4x$

3 a $-2\cot 2x$

b $\text{cosec}^2\,(\cot(x + 7))\,\text{cosec}^2\,(x + 7)$

c $\frac{\tan x - 2x\ln x\sec^2 x}{x\tan^3 x}$

4 $f'(x) =$

$\frac{(x + 1)(\sec x\tan x - \text{cosec}^2 x) - 2(\sec x + \cot x)}{(x + 1)^3}$

5 $\frac{dy}{dx} = 2\sec x\left(1 + \text{cosec}^2 x\right)$

Exercise 2G

1 a $\frac{3}{\sqrt{1 - 9x^2}}$

b $\frac{-1}{2\sqrt{x}\ \sqrt{1 - x}}$

c $\frac{2x}{x^4 + 1}$

d $\frac{-1}{\sqrt{x(2 - x)}}$

e $\frac{3e^{3x}}{\sqrt{1 - e^{6x}}}$

f $\dfrac{-1}{3\sqrt[3]{x^2}\cos^{-1}\left(\sqrt[3]{x}\right)\sqrt{1-\sqrt[3]{x^2}}}$

g $e^{3x}\left(3\tan^{-1}(x)+\dfrac{1}{x^2+1}\right)$ or $\dfrac{e^{3x}\left(3(x^2+1)\tan^{-1}(x)+1\right)}{x^2+1}$

h $\dfrac{x^2\left(3(4x^2+1)\tan^{-1}(2x)-x\right)}{(4x^2+1)\left(\tan^{-1}(2x)\right)^2}$

2 when $x=\dfrac{\pi}{3}$, gradient $=\dfrac{2}{7}$

3 when $t=\dfrac{1}{10}$, rate of change $=\dfrac{40\sqrt{3}}{9}$

4 when $x = 1$, equation of tangent: $\pi y = 368 - 80x$

Exercise 2H

1 **a** $\dfrac{dy}{dx}=-\dfrac{x}{y}$

b $\dfrac{y}{1-y}$

c $\dfrac{-(y+2x)}{x}$

d $\dfrac{1+y}{2y-x}$

e $\dfrac{x\left(2x^2-y^2\right)}{y\left(x^2-2y^2\right)}$

f $\dfrac{e^x-\tan y}{x\sec^2 y}$

g $\dfrac{dy}{dx}=\dfrac{-\left(y+2x^3\right)}{x(2xy-1)}$

h $\dfrac{2y-5x}{3y-2x}$

2 at $P = (1, 2)$, equation of tangent: $2y + x = 5$

3 at $(3, 4)$, equation of tangent: $y + 2x = 10$

4 at $(2, 1)$, equation of tangent: $8y = 5x - 2$

5 at $T = (5, 7)$, equation of tangent: $4y + 3x = 43$

Exercise 2I

1 **a** $\dfrac{dy}{dx}=-\dfrac{x}{y}\quad \dfrac{d^2y}{dx^2}=\dfrac{-\left(x^2+y^2\right)}{y^3}$

b $\dfrac{dy}{dx}=\dfrac{-(2x+y)}{x}\quad \dfrac{d^2y}{dx^2}=\dfrac{2(x+y)}{x^2}$

c $\dfrac{dy}{dx}=\dfrac{e^x-y}{x}\quad \dfrac{d^2y}{dx^2}=\dfrac{e^x(x-2)+2y}{x^2}$

d $\dfrac{dy}{dx}=\dfrac{-y}{x+2y}\quad \dfrac{d^2y}{dx^2}=\dfrac{2y(x+y)}{x+2y}$

2 **a**, **b** Proof

3 Maximum SP = (1, 2)

Exercise 2J

1 **a** Implicit $\dfrac{dy}{dx}=y\ln 16$

Explicit $\dfrac{dy}{dx}=4^{2x}\ln 16$

b Implicit $\dfrac{dy}{dx}=3y\ln\pi$

Explicit $\dfrac{dy}{dx}=3\pi^{3x}\ln\pi$

c Implicit $\dfrac{dy}{dx}=y\ln 2$

Explicit $\dfrac{dy}{dx}=2^x\ln 2$

d Implicit $\dfrac{dy}{dx}=\dfrac{y(x\ln 5-1)}{x}$

Explicit $\dfrac{dy}{dx}=\dfrac{\ln\dfrac{5^x}{x}(x\ln 5-1)}{x}$

e Implicit $\dfrac{dy}{dx}=\dfrac{y}{x}(\sin x+x\ln x\cos x)$

Explicit $\dfrac{dy}{dx}=x^{\sin x-1}(\sin x+x\ln x\cos x)$

2 Proof

3 $\dfrac{dy}{dx}=\dfrac{2x^2}{\sqrt{x^4+1}}\left(\dfrac{1}{x}-\dfrac{x^3}{x^4+1}\right)$

4 when $x = 4$, equation of tangent:
$y = 9x - 9$

5 SP when $x = \frac{9}{10}$

Exercise 2K

1 a $\frac{dy}{dx} = \frac{-t^3}{\sqrt{t^2 + 1}}$

b $\frac{dy}{dx} = \frac{\cos t + \sin t}{\cos t - \sin t}$

c $\frac{dy}{dx} = -\frac{3}{2}\left(\frac{t+1}{t-2}\right)^2$

d $\frac{dy}{dx} = -\left(\frac{1+t^2}{1-t^2}\right)^2$

2 SPs = (6, –16); (2, 16)

3 when $t = \pi$, gradient = –1

4 Proof

5 when $t = 0$, equation of tangent:
$y + 4x + 3 = 0$

6 a SPs = (0, 2); (0, –2)

b $\frac{d^2y}{dx^2} = \frac{4t^3}{(t^2 + 1)^3}$, min SP = (0, 2);
max SP = (0, –2)

7 a SP = (4, –1)

b $\frac{d^2y}{dx^2} = \frac{t(t^2 + 1) - (t - 1)(3t^2 + 1)}{8t^3(t^2 + 1)^3}$,
min SP

8 SPs = (0·2588, 1·1278) and
(0·9610, 6·7308)

Exercise 2L

1 when $x = \pi$, equation of the tangent:
$y = 1{\cdot}23x - 2{\cdot}69$

2 when $t = \pi$, rate of change $= \frac{1}{12}$

3 when $\theta = \frac{\pi}{2}$, stationary point
= minimum TP

4 when $x = 1$, equation of the tangent:
$y = 5{\cdot}5x - 9{\cdot}5$

5 when $t = 3$, gradient of the tangent = 3

6 a $\frac{dy}{dx} = t - 2$ $\frac{d^2y}{dx^2} = \frac{1}{t + 2}$

b min SP when $t = 2$

7 when $t = \frac{\pi}{9}$,

acceleration $A\left(\frac{\pi}{9}\right) = 8\sqrt{3}\,\text{ms}^{-2}$

8 when $x = \frac{\pi}{12}$, gradient of the tangent
$= 2e^{\frac{\pi}{6}}\left(1 - 4\sqrt{3}\right)$

9 when $t = 10$ seconds, acceleration,

$A(10) = 12\sqrt{3}\text{ms}^{-2}$

10 a Greatest F when $x = \frac{3\pi}{4}$,
fuel efficiency = 15 km/litre
Least F when $x = \frac{\pi}{4}$,
fuel efficiency = 11.9 km/litre

b $v = 100$ km/h; $v = 60$ km/h

11 when $t = 1$ s, rate of change = 1808·2Vs^{-1}
when $t = 10$ s, rate of change = 112·9Vs^{-1}

Chapter review

1 a $2\cot 2x$

b $e^{x+2}(\cos 3x - 3\sin 3x)$

c $4(\sin 4x - \operatorname{cosec} 4x)$

d $\frac{2e^{x^2}(x^2 + 2x - 2)}{(x + 2)^5}$

e $\frac{1}{x^2 + 6x + 10}$

f $e^{5x}\operatorname{cosec} x\,(5 - \cot x)$

g $\frac{\cos^{-1}(4x)\sqrt{1 - 16x^2} - 4x\ln x}{x\sqrt{1 - 16x^2}}$

h $(\ln 2 + 3)\,(2^{x+1}\,e^{3x})$

i $\frac{x\ln x\,\sin x\,\sin(\cos x) - \cos(\cos x)}{x\ln^2 x}$

j $\frac{dy}{dx} = 2\sec(4x)e^{2x+3}\,(2\tan(4x) + 1)$

k $\frac{dy}{dx} = \frac{\sec^2(x)(x+3)^5}{\ln x}\left(2\tan x + \frac{5}{x+3} - \frac{1}{x\ln x}\right)$

l $\frac{\cot^2 x}{\tan^2 x}$ $(2\cot x\tan x - 3(2x-1)$
$\text{cosec}^2 x\tan x - (2x-1)\sec^2 x\cot x)$

2 $\frac{d^4y}{dx^4} = \frac{105}{\sqrt{(2x-1)^9}}$

3 $\frac{d^2y}{dx^2} = 6x - 8$, min SP when

$x = \left(\frac{11}{3}, -\frac{400}{27}\right)$, max SP when

$x = (-1, -36)$

4 **a** after $t = 6$ seconds, $v = 23\ \text{ms}^{-1}$

b acceleration $= 0\ \text{ms}^{-2}$ after $1\frac{1}{3}$ seconds

5 $\frac{dy}{dx} = \frac{y-2x}{6y-x}$ and

$\frac{d^2y}{dx^2} = \frac{22\left(xy - x^2 - 3y^2\right)}{(6y-x)^3}$

6 at $(2, -3)$ $\frac{dy}{dx} = 4$

at $(-1, -1)$ $\frac{d^2y}{dx^2} = 3$

7 Proof

8 at $(-2, 3)$ $\frac{dy}{dx} = 4$ and $\frac{d^2y}{dx^2} = 33$

9 Proof

10 $\frac{dy}{dx} = \frac{-5}{2}\cot\theta$

$\frac{d^2y}{dx^2} = \frac{-5}{4}\text{cosec}^3\theta$

Chapter 3

Exercise 3A

1 **a** $\frac{4x^7}{7} + c$ **b** $x^3 - \frac{7x^2}{2} + c$

c $\frac{(2x-3)^3}{6} + c$ **d** $\frac{x^2}{2} + \frac{7}{x} + c$

e $\frac{2\sqrt{x}^3}{3} - 14\sqrt{x} + c$ **f** $\frac{\sin 2x}{2} + c$

g $c - 2\cos 3x$ **h** $\frac{(8t+3)^4}{32} + c$

i $c - \frac{(5-4x)^6}{24}$

j $c - \frac{1}{16(4x-5)^4}$

k $c - \frac{\cos(3\theta - 1)}{3}$

l $c - \frac{\sin(2-4x)}{4}$

2 **a** $1 - \sin(\pi - x)$ **b** $\frac{7}{3} - \frac{2}{3(3x+1)}$

3 **a** $4\sqrt{x} - \frac{2\sqrt{x}^5}{5} + c$

b $\frac{(2x+3)^6}{12} + c$

4 $\frac{9x^5}{5} - 2x^3 + x + 1$

5 **a** 42 **b** 19 $\frac{8}{3}$

c $\frac{\sqrt{3}}{4} - \frac{1}{2\sqrt{2}}$ **d** $\frac{1}{5}\left(8 - 2\sqrt{2}\right)$

e $\frac{33}{4}$

6 **a** 1 **b** $2\sqrt{5}$

Exercise 3B

1 **a** $x + c$

b $\frac{x}{2} + \frac{\sin 2x}{4} + c$

c $\frac{3x}{2} - \frac{3\sin 2x}{4} + c$

d $\sin 2\theta - \theta + c$

e $c - \frac{\cos 2x}{2}$

f $c - \cos\theta$

2 **a** 1 **b** –2

c 0 **d** $\frac{1}{2}$

3 $\frac{1}{2}(\sin x + x) + c$

Exercise 3C

1 **a** $y = 4e^x + c$

b $y = 2e^{3x} + 2e^{-5x} + c$

c $y = \frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2} + c$

d $y = c - \frac{7}{3}\ln|x|$

e $y = \frac{2}{3}\ln|3x+4| + c$

f $y = c - 4\ln|7-2x|$

g $y = \frac{e^{2x}}{2} + 2x - \frac{e^{-2x}}{2} + c$

h $y = x - e^{-x} + c$

i $y = \ln|5x+6| + c$

j $y = -\frac{e^{-2x}}{2} + c$

k $y = -\ln|1-x| + C$

2 **a** 11·8242 (4 d.p.)

b 4·1452 (4 d.p.)

c –0·9486 (4 d.p.)

d $\frac{3}{2}\ln 5 \approx 2\cdot4142$ (4 d.p.)

Exercise 3D

1 **a** $y = 2\tan 3x + \mathrm{c}$

b $y = \frac{1}{6}\tan 2x + c$

c $y = 8\tan\left(\frac{x}{2}\right) + c$

d $y = \tan x + c$

e $y = 3\tan x + c$

f $y = 9\tan x + c$

g $y = \tan x + c$

h $y = \operatorname{cosec} x + c$

i $y = \frac{\tan 4x}{4} - x + c$

2 a 8·9486 (4 d.p.)

b $\frac{1}{\sqrt{3}}$

c $\sqrt{3} - \frac{1}{2} \approx 1{\cdot}2321$ (4 d.p.)

d $2 - \frac{\pi}{2}$

Exercise 3E

1 a $y = \sin^{-1}\left(\frac{x}{5}\right) + c$

b $y = \sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + c$

c $y = \frac{1}{3}\sin^{-1}\left(\frac{x}{2}\right) + c$

d $y = c - \tan^{-1} 3x$

e $y = \tan^{-1} 4x + c$

2 a $y = \frac{\pi}{2\sqrt{5}}$

b 0·6755 (4 d.p.)

c 0·2766 (4 d.p.)

d 0·0413 (4 d.p.)

Exercise 3F

1 a $y = \frac{3}{10}\ln|x-3| - \frac{1}{6}\ln|x-1|$
$-\frac{2}{15}\ln|x+2| + c$

b $y = 2\ln\left|\frac{x+1}{x}\right| - \frac{4}{x+1} + c$

c $y = x + \ln\left|\frac{x-2}{x+2}\right| + c$

d $y = \frac{x^2}{2} - x + 2\ln|x| - \ln|x+1| + c$

e $y = 2x + \ln\left|(x-3)^5\sqrt{x^2+1}\right| + c$

f $y = \ln|3| + 2$

2 a 12·8954 (4 d.p.)

b $y = 7\ln|4| + \frac{3}{5}$

Exercise 3G

1 a $y = e^{\sin x} + c$

b $y = \frac{1}{2}e^{x^2+4x} + c$

c $y = \frac{2}{5}\left(e^x + 1\right)^5 + c$

d $y = -\frac{1}{4}\cos^4 x + c$

e $y = \frac{1}{2}\ln\left|1 - e^{-2x}\right| + c$

f $y = -\frac{1}{2\sin^2 x} + c$

2 $y = \ln|\cos x + \sin x| + c$

3 $y = 3\sin^{-1}\left(\frac{x}{3}\right) + c$

4 $y = \frac{3x}{\sqrt{3}} + \frac{3}{2}\sin^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

5 $y = \frac{2}{3}\sqrt{\left(3 + x^2\right)^3} + c$

6 $y = \frac{2}{3}\sqrt{\left(e^x - 1\right)^3} + c$

7 $y = \sqrt{\left(4 - x^2\right)} + c$

8 $y = \frac{\tan^2(x)}{2} + c$

9 $y = \cot x + c$

Exercise 3H

1 a $y = \ln\left|3x^2 + 5x\right| + c$

b $y = c - \ln|\cos x|$

c $y = \frac{1}{2}\ln\left|e^{2x} + 1\right| + c$

d $y = \frac{1}{4}\left(x^2 - 3\right)^2 + c$

2 **a** $y = \frac{1}{6}\left(x^2+3x-6\right)^6 + c$

b $y = \frac{1}{15}\left(x^3-5\right)^5 + c$

c $y = \frac{1}{32}\left(x^4+2\right)^8 + c$

3 **a** $y = \frac{2}{3}\sqrt{\left(x^2-3\right)^3} + c$

b $y = \frac{2}{3}\sqrt{\left(x^3-8\right)^3} + c$

c $y = \frac{1}{6}\sqrt{\left(5+x^4\right)^3} + c$

4 **a** $y = 2\left(\ln|x|\right)^2 + c$

b $y = 3\ln(\ln|x|) + c$

c $y = \sin\left(\ln|x|\right) + c$

5 **a** $y = \frac{1}{6}(\ln|x|)^2 + c$

b $y = c - \frac{1}{4}\left(1-x^3\right)^4$

c $y = c - \frac{1}{3}\left(x^2+3\right)^{-3}$

d $y = \frac{4}{3}\sqrt{\left(x^4+1\right)^3} + c$

e $y = \frac{3}{4}\sqrt[3]{\left(x^3-3\right)^4} + c$

6 $y = c - \frac{1}{12(3x^2+12x-7)^2}$

7 $y = 2\sqrt{\left(x^4-3x\right)} + c$

8 $y = \frac{1}{2\cos^2 x} + c$

Exercise 3I

1 $\frac{65}{1728}$

2 $\frac{500123}{4}$

3 5166.3302 (4 d.p.)

4 8

5 $2\sqrt{5} - 2$

6 $2\sqrt{3} + \frac{4\pi}{3}$

7 $\frac{1}{3}\ln\left(2+\sqrt{3}\right)$

8 π

9 $\frac{2\pi}{3}$

10 $\ln 2$

Exercise 3J

1 **a** $y = \sin x - \frac{\sin^3 x}{3} + c$

b $y = c - \cos x - \frac{2\cos^3 x}{3} + \frac{\cos^5 x}{5}$

2 **a** $y = \frac{3x}{8} - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + c$

b $y = \frac{x}{2} + \frac{\sin 4x}{8} + c$

3 **a** $y = \frac{1}{2\cos^2(x)} + \ln|\cos(x)| + c$

b $y = \frac{1}{4}\sec^4(x) - \sec^2(x) + \ln|\cos(x)| + c$

Exercise 3K

1 $y = \sin x - x\cos x + \text{c}$

2 $y = 3e + e^{-1}$

3 $y = 2e^2 - 8$

4 $y = x\tan 5x + \frac{1}{5}\ln|\cos 5x| + c$

5 $y = (3x-1)\tan x + 3\ln|\cos x| + c$

6 $y = 3 + 5e^2$

7 $y = x\ \sin^{-1}(x) + \sqrt{1-x^2} + c$

Exercise 3L

1 $y = 3(\sin x(x^2\ \ 2) + 2x\cos x) + c$

2 $y = 11 - \frac{216}{e^5}$

3 $y = \frac{1}{16}\left(\sin(4x)(4x^2-1) + 4x\cos(4x)\right) + c$

4 $y = (x^2-2)\cos x - 2x\sin x + \text{c}$

5 $y = \frac{1}{2}\left((16x+1)\sin 2x - (8x^2+x-5)\cos 2x + 8\cos 2x\right) + c$

6 $y = \frac{e^{4x}}{4}\left(8x^2 - 4x + 1\right) + c$

Exercise 3M

1 $y = \frac{1}{7}(4\sin 3x\sin 4x + 3\cos 3x\cos 4x) + c$

2 $y = \frac{1}{2}e^x(\sin x - \cos x) + c$

3 $y = \frac{1}{13}e^{2x}(2\cos 3x + 3\sin 3x) + c$

4 $y = 2\sin^{-1}\left(\frac{x}{2}\right) + x\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right) + c$

5 $y = \frac{1}{15}(\cos x \sin 4x - 4\sin x \cos 4x) + c$

6 $y = \frac{3}{2}\sin 3x \cos 2x - \sin 2x \cos 3x + c$

Exercise 3N

1 A = 2ln 21 units^2

2 A = 6(3 – 2ln 2) units^2

3 A ≈ 536·8 units^2

4 $A = \frac{14}{3}$ units^2

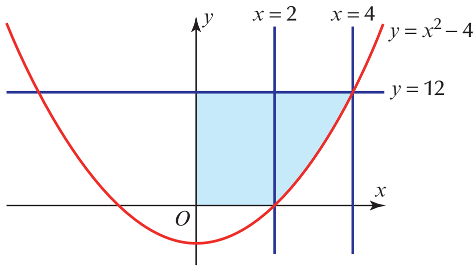
5 A = $e^5 - e^2$ ≈ 141·02 units^2

6 $V = 72\pi \approx 226{\cdot}19$ units^3

7 $V = 15{\cdot}3$ cm^3 > 12·5 cm^3, so specification meets the requirements.

8 $V = 10\pi$ units^3

9 a



b $\int_0^{12} \pi(y + 4)dy$

c $V = \frac{3\pi}{5}$ litres

d V ratio $= \frac{450\pi}{600\pi} = \frac{3}{4}$ full

10 $V = \frac{\pi^2}{4}$ units^3

11 2π units^3

12 a $s(t) = t - \frac{1}{2}\sin 2t + c$

b $s\left(\frac{\pi}{2}\right) = 3.57$ units

13 $s(4) = \frac{1}{4}(e^8 - 105)$ in the positive direction

Chapter 3 review

1 $y = \ln\left|\sqrt[3]{x}\right| + 2\sqrt{e^x} + c$

2 $y = \ln|4x - 1| + c$

3 $\frac{1}{e} - \frac{1}{e^2} + 3\ln 2 \approx 2{\cdot}31$

4 $4\ln\left|\frac{e + 4}{2e + 4}\right| \approx -1{\cdot}36$

5 $y = \frac{1}{6}(3\tan(2x) - 2\ln|\cos(3x)| - 30x) + c$

6 $y = \frac{1}{3}\tan(3x + 1) + c$

7 0·438

8 $\frac{\pi}{6}$

9 $y = 18\sin^{-1}\left(\frac{3}{4}\right) \approx 15{\cdot}3$

10 $y = 6\tan^{-1}(18) \approx 9{\cdot}09$

11 $y = \ln\left|(x^2 + 1)(x + 1)^3\right| + c$

12 $y = 2x^2 - 8x + 7\ln|x + 1| + c$

13 $y = e^{x^2 - 5} + c$

14 $\frac{1}{2}\ln\left|\frac{119}{11}\right| \approx 1{\cdot}19$

15 2·72

16 $\frac{\pi}{12}$

17 $\frac{\pi}{5}\ln|3| \approx 0{\cdot}69$

18 $y = \frac{1}{2}\tan^2(2x) + \pi$

19 a $V \approx 126$ cm^3

b Proof

c $V \approx 674^3$ units^3

Chapter 4 Complex numbers

Exercise 4A

1 **a** $2\sqrt{2}i$ **b** $7i$
c $10i$ **d** $4 + 8i$
e $5i$ **f** $10i$

2 **a** $8 - 2i$ **b** $2 + 6i$
c $-2 - 6i$ **d** $25 - 10i$
e $23 - 14i$ **f** $19 - 8i$
g $-3 + 4i$ **h** $-17 - 24i$
i $-117 - 44i$

3 **a** $9 + i$ **b** $-1 + 9i$
c $10 + 5i$ **d** $-6 + 2i$
e $7 + i$ **f** $27 + 14i$
g $-5 - 31i$ **h** $3 + 4i$
i $-10 + 198i$

4 **a** $\pm 4i$ **b** $0, \pm 5i$
c $2 \pm i$ **d** $-1 \pm 2i$
e $\frac{-1 \pm \sqrt{2}i}{3}$ **f** $\frac{-1 \pm \sqrt{79}i}{4}$
g $\frac{1 \pm \sqrt{35}i}{2}$ **h** $\frac{-3 \pm \sqrt{15}i}{3}$

5 **a** $i^2 = -1, i^3 = -i, i^4 = 1, i^5 = i, i^6 = -1, i^7 = -i, i^8 = 1, i^9 = i$
b **i** $n = 4$ **ii** $n = 3$
c **i** $n = 4, 8, 12, \ldots, 4a$
ii $n = 3, 7, 11, \ldots, 4a - 1$

6 **a** $2 + 4i$ **b** $-4 + 9i$
c $5 + 2i$ **d** $2 + i$

7 **a** $9 + 7i$ **b** $-3 - 5i$
c $4 + i$ **d** $1 - i$
e $\frac{3}{2} - \frac{1}{2}i$ **f** $\frac{19}{29} - \frac{4}{29}i$

Exercise 4B

1 **a** $1 - i$ **b** $2 - 3i$
c $4 + 2i$ **d** $-5 + \frac{1}{2}i$
e $-i$ **f** 8
g $7i$ **h** $-5 - 2i$
i $4i + 7$

2 **a** $2 - i$ **b** $\frac{4}{13} + \frac{6}{13}i$
c $-\frac{3}{5} + \frac{11}{5}i$ **d** $\frac{1}{5} + \frac{7}{5}i$
e $2 + 5i$ **f** $-\frac{7}{5} - \frac{4}{5}i$

3 **a** $23 - 14i$ **b** $\frac{7}{25} + \frac{26}{25}i$
c $\frac{7}{29} - \frac{26}{29}i$ **d** 29
e $7 + 26i$

4 **a** $-\frac{8}{25} + \frac{6}{25}i$ **b** $-\frac{1}{290} + \frac{17}{290}i$
c $\frac{1}{2} + \frac{1}{10}i$

5 $-\frac{16}{5} + \frac{2}{5}i$

6 **a** $\frac{1}{5}$ **b** $\frac{29}{85} + \frac{88}{85}i$
c $\frac{127}{41} + \frac{169}{41}i$

7 Student's own direct proof

8 **a** $3 + i$ **b** $2 - \frac{1}{2}i$
c $-6 + 11i$ **d** $-2 - 3i$

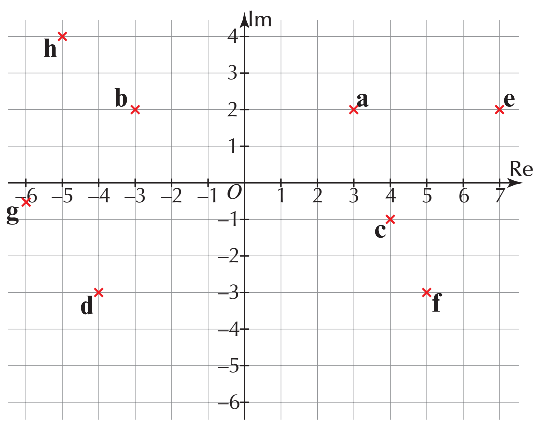
9 All complex numbers with $\text{Re}(z) = 5$

10 $z = 5 + 2i$; $w = 4 - 3i$

11 **a** $3 + 2i, -3 - 2i$
b $4 + 3i, -4 - 3i$
c $4 + i, -4 - i$
d $5 - 4i, -5 + 4i$

Exercise 4C

1



2 **a** z plotted at (4, 3) and $\bar{z}$ plotted at (4, −3)

b z plotted at (−2, 1) and $\bar{z}$ plotted at (−2, −1)

c $\bar{z}$ is a reflection in the x-axis of z

3 **a** $|z| = 5,\ \theta = 0{\cdot}927$

b $|z| = \sqrt{5},\ \theta = -0{\cdot}464$

c $|z| = \sqrt{34},\ \theta = 2{\cdot}6$

d $|z| = 2\sqrt{5},\ \theta = -2{\cdot}03$

e $|z| = \sqrt{53},\ \theta = 0{\cdot}278$

f $|z| = 2,\ \theta = \frac{\pi}{3}$

g $|z| = \frac{4\sqrt{3}}{3},\ \theta = \frac{5\pi}{6}$

h $|z| = \frac{\sqrt{2}}{2},\ \theta = -\frac{3\pi}{4}$

4 **a** $|z| = \sqrt{65},\ \theta = 0{\cdot}124$

b $|z| = 1,\ \theta = \frac{\pi}{2}$

c $|z| = \sqrt{2},\ \theta = -\frac{3\pi}{4}$

5 **a** **i** $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ii $1 + \sqrt{3}i$

b **i** $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

ii i

c **i** $3\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

ii $\frac{3\sqrt{3}}{2} - \frac{3}{2}i$

d **i** $2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

ii $-1 + \sqrt{3}i$

e **i** $5\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

ii $-\frac{5\sqrt{3}}{2} - \frac{5}{2}i$

f **i** $4\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right)$

ii $-2\sqrt{2} - 2\sqrt{2}i$

6 **a** $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

b $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

c $2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

d $2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

e $5(\cos\pi + i\sin\pi)$

f $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

g $2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

h $10\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

7 $z = \pm 2i$

Exercise 4D

1 **a** $6\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

b $8\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

c $5\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$

d $12\left(\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right)$

e $12\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

f $2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

g $5\left(\cos\frac{2\pi}{15} + i\sin\frac{2\pi}{15}\right)$

h $6\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

i $3\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

j $\frac{5}{9}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

2 **a** $5 \cdot 80 - 1 \cdot 55i$ **b** 100

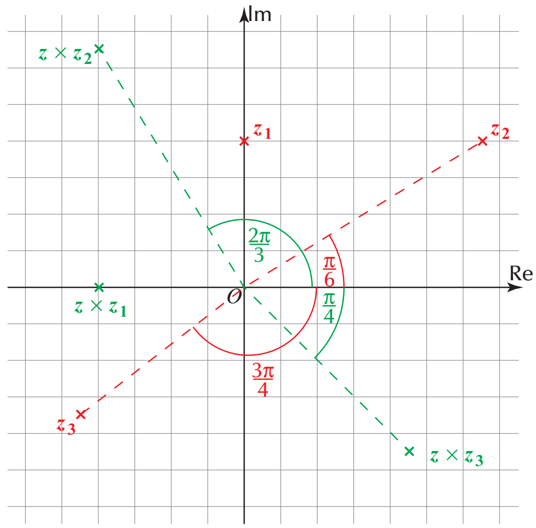
c -10 **d** $4 \cdot 10 + 1 \cdot 10i$

e $-3 \cdot 11 + 2 \cdot 90i$ **f** $-1 \cdot 5i$

g $-1 \cdot 29 - 0 \cdot 966i$

h $-0 \cdot 0518 - 0 \cdot 193i$

3 **a, c**



b $z \times z1 = 2(\cos \pi + i \sin \pi) = -2$

$$z \times z_2 = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$z \times z_3 = 3\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

c The position vector of z_1, z_2 and z_3 have been rotated $\frac{\pi}{2}$ in an anticlockwise direction

d Student's own investigation

e Student's own investigation

4 **a** $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$ **b** $\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

c $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

d $\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}$

e $\cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}$

Exercise 4E

1 **a** $\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

b $81\left(\cos\frac{4\pi}{11} + i\sin\frac{4\pi}{11}\right)$

c $1024\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$

d $1000\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

2 **a** 64

b $243\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

3 **a** $2\sqrt{3}i - 2$ **b** -4

c $-32768\sqrt{3}i - 32768$

d $32i$

4 **a** $512 + 512\sqrt{3}i$ **b** $64\sqrt{3} - 64i$

5 **a** $\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}$

b $\cos\frac{19\pi}{35} + i\sin\frac{19\pi}{35}$

c $\cos\frac{31\pi}{36} + i\sin\frac{31\pi}{36}$

d $\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$

e $\cos\frac{25\pi}{33} + i\sin\frac{25\pi}{33}$

f $\cos\frac{5\pi}{7} + i\sin\frac{5\pi}{7}$

6 **a** Student's own direct proof

b Student's own direct proof

7 **a** $\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta$

b $\cos 2\theta + i\sin 2\theta$

c $\cos 2\theta = \cos^2\theta - \sin^2\theta$

d $\sin 2\theta = 2\sin\theta\cos\theta$

8 **a** $\cos^3\theta - 3\cos\theta\sin^2\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta$

b $\cos 3\theta + i\sin 3\theta$

c $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

d $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

e $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

9 $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

10 $\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

11 **a** Similar to Q8,9,10

b **i, ii** Student's own direct proof

c $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

Exercise 4F

1 a $\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}$; $\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}$;

$\cos\left(-\frac{5\pi}{9}\right) + i\sin\left(-\frac{5\pi}{9}\right)$

Solutions divide a circle of radius 1 into 3 equal sectors $\frac{2\pi}{3}$ radians apart.

b $2\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$;

$2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$;

$2\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

Solutions divide a circle of radius 2 into 3 equal sectors $\frac{2\pi}{3}$ radians apart.

c $2\left(\cos\frac{2\pi}{25} + i\sin\frac{2\pi}{25}\right)$;

$2\left(\cos\frac{12\pi}{25} + i\sin\frac{12\pi}{25}\right)$;

$2\left(\cos\frac{22\pi}{25} + i\sin\frac{22\pi}{25}\right)$;

$2\left(\cos\left(-\frac{18\pi}{25}\right) + i\sin\left(-\frac{18\pi}{25}\right)\right)$;

$2\left(\cos\left(-\frac{8\pi}{25}\right) + i\sin\left(-\frac{8\pi}{25}\right)\right)$

Solutions divide a circle of radius 2 into 5 equal sectors $\frac{2\pi}{5}$ radians apart.

d $4\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$;

$4\left(\cos\frac{11\pi}{12} + i\sin\frac{11\pi}{12}\right)$;

$4\left(\cos\left(-\frac{5\pi}{12}\right) + i\sin\left(-\frac{5\pi}{12}\right)\right)$

Solutions divide a circle of radius 4 into 3 equal sectors $\frac{2\pi}{3}$ radians apart.

e $\sqrt[4]{2}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$;

$\sqrt[4]{2}\left(\cos\left(-\frac{7\pi}{8}\right) + i\sin\left(-\frac{7\pi}{8}\right)\right)$

Solutions divide a circle of radius $\sqrt[4]{2}$ into 2 equal sectors π radians apart.

f $\sqrt[5]{2}\left(\cos\frac{\pi}{30} + i\sin\frac{\pi}{30}\right)$;

$\sqrt[5]{2}\left(\cos\frac{13\pi}{30} + i\sin\frac{13\pi}{30}\right)$;

$\sqrt[5]{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$;

$\sqrt[5]{2}\left(\cos\left(-\frac{23\pi}{30}\right) + i\sin\left(-\frac{23\pi}{30}\right)\right)$;

$\sqrt[5]{2}\left(\cos\left(-\frac{11\pi}{30}\right) + i\sin\left(-\frac{11\pi}{30}\right)\right)$

Solutions divide a circle of radius $\sqrt[5]{2}$ into 5 equal sectors $\frac{2\pi}{5}$ radians apart.

g $\sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$;

$\sqrt{2}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$;

$\sqrt{2}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$;

$\sqrt{2}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

Solutions divide a circle of radius $\sqrt{2}$ into 4 equal sectors $\frac{\pi}{2}$ radians apart.

h $\sqrt[3]{6}\left(\cos\frac{5\pi}{9} + i\sin\frac{5\pi}{9}\right)$;

$\sqrt[3]{6}\left(\cos\left(-\frac{7\pi}{9}\right) + i\sin\left(-\frac{7\pi}{9}\right)\right)$;

$\sqrt[3]{6}\left(\cos\left(-\frac{\pi}{9}\right) + i\sin\left(-\frac{\pi}{9}\right)\right)$

Solutions divide a circle of radius $\sqrt[3]{6}$ into 3 equal sectors $\frac{2\pi}{3}$ radians apart.

2 a $1; -\frac{1}{2} + \frac{\sqrt{3}}{2}i; -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

b $1, i, -1, -i$

c $1;\ \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5};$
$\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5};$
$\cos\left(-\frac{4\pi}{5}\right)+i\sin\left(-\frac{4\pi}{5}\right);$
$\cos\left(-\frac{2\pi}{5}\right)+i\sin\left(-\frac{2\pi}{5}\right)$

3 a $1;\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i;\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i$

b $1;\ \frac{1}{2}+\frac{\sqrt{3}}{2}i;\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i;\ -1;$
$-\frac{1}{2}-\frac{\sqrt{3}}{2}i;\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$

4 a $2^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right);\ 2^{\frac{3}{4}}\left(-\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right);$
$2^{\frac{3}{4}}\left(-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right);\ 2^{\frac{3}{4}}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)$

b $\cos\frac{\pi}{10}+i\sin\frac{\pi}{10};\ \cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i;$
$\cos\frac{9\pi}{10}+i\sin\frac{9\pi}{10};$
$\cos\left(-\frac{7\pi}{10}\right)+i\sin\left(-\frac{7\pi}{10}\right);$
$\cos\left(-\frac{3\pi}{10}\right)+i\sin\left(-\frac{3\pi}{10}\right)$

c $4i;\ -2\sqrt{3}-2i;\ 2\sqrt{3}-2i$

d $5\left(\cos\frac{\pi}{8}+i\sin\frac{\pi}{8}\right);$
$5\left(\cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}\right);$
$5\left(\cos\left(-\frac{7\pi}{8}\right)+i\sin\left(-\frac{7\pi}{8}\right)\right);$
$5\left(\cos\left(-\frac{3\pi}{8}\right)+i\sin\left(-\frac{3\pi}{8}\right)\right)$

e $\cos\frac{\pi}{5}+i\sin\frac{\pi}{5};\ \cos\frac{3\pi}{5}+i\sin\frac{3\pi}{5};$
$\cos\pi+i\sin\pi=-1;$
$\cos\left(-\frac{3\pi}{5}\right)+i\sin\left(-\frac{3\pi}{5}\right);$
$\cos\left(-\frac{\pi}{5}\right)+i\sin\left(-\frac{\pi}{5}\right)$

f $2+2\sqrt{3}i;\ -4;\ 2-2\sqrt{3}i$

Exercise 4G

1 a $i;\ -i$ **b** $2+i;\ 2-i$
c $-1+2i;\ -1-2i$ **d** $1+i;\ 1-i$
e $2+3i;\ 2-3i$ **f** $2+\frac{1}{2}i;\ 2-\frac{1}{2}i$

2 a $-1;\ 2+i;\ 2-i$
b $2;\ -1+2i;\ -1-2i$
c $-3;\ 1+i;\ 1-i$
d $1;\ 2+\frac{1}{2}i;\ 2-\frac{1}{2}i$
e $-2;\ -\frac{1}{2}+\frac{1}{2}i;\ -\frac{1}{2}-\frac{1}{2}i$
f $5;\ 2+3i;\ 2-3i$

3 a $3;\ i;\ -i$ **b** $2;\ 2+i;\ 2-i$
c $-6;\ 1+i;\ 1-i$ **d** $\frac{1}{2};\ 2+3i;\ 2-3i$
e $-\frac{1}{2};\ 3+i;\ 3-i$
f $-1;\ 2+\frac{1}{2}i;\ 2-\frac{1}{2}i$

4 a i 3 solutions **ii** $-5;\ 1+2i;\ 1-2i$
b i 3 solutions **ii** $\frac{1}{2};\ 1+2i;\ 1-2i$
c i 4 solutions **ii** $-1;\ 3;\ 1+i;\ 1-i$
d i 4
ii $3+4i;\ 3-4i;\ -\frac{1}{2}-\frac{3}{2}i;\ -\frac{1}{2}+\frac{3}{2}i$

Exercise 4H

1 a Circle C(0, 0), radius 5 units
b Circle C(0, 0), radius 2 units
c Circle C(3, 0), radius 2 units
d Circle C(0, –1), radius 4 units
e Circle C(1, –3), radius 4 units

f Circle C(–1, 2), radius 5 units

g Circle $C\left(2, -\frac{1}{2}\right)$, radius 2 units

h Circle $C\left(-\frac{2}{3}, 1\right)$, radius $\frac{2}{\sqrt{3}}$ units

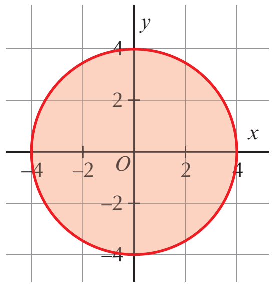
2 **a** Straight line with equation $y = \sqrt{3}x$

b Straight line with equation $y = -x$

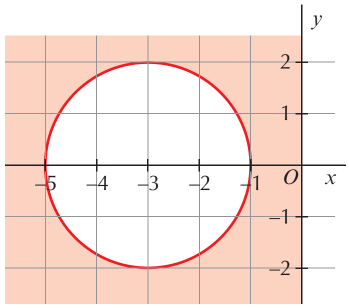
c Straight line with equation $y = \frac{1}{\sqrt{3}}x$

d Straight line with equation $y = -\sqrt{3}x$

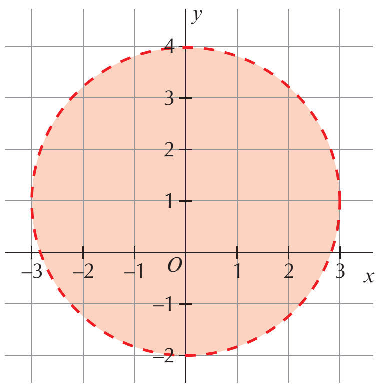
3 **a** Circle C(0, 0), radius 4 units



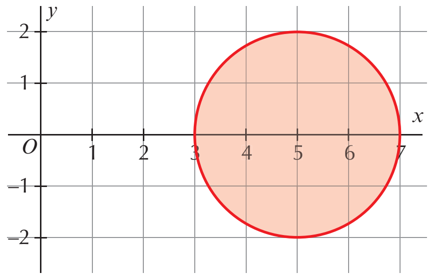
b Circle (–3, 0), radius 2 units



c Circle (0, 1), radius 3 units



d Circle (5, 0), radius 2 units



4 **a** Straight line with equation $y = -x$

b Straight line with equation
$y = -\frac{1}{2}x - \frac{3}{4}$

c Straight line with equation $y = \frac{3}{2}$

d Straight line with equation
$y = -2x - \frac{3}{2}$

Chapter review

1 **a** $9 - 3i$ **b** $15 + 7i$

c $22 - 21i$ **d** $\frac{14}{25} + \frac{27}{25}i$

e $35 + 12i$ **f** $-2 + i, 2 - i$

2 When $a = 1, b = -2$

When $a = \frac{3}{5}, b = -\frac{10}{3}$

3 z_1: Argand diagram showing (5, 3)

z_2: Argand diagram showing (–3, 4)

4 $2\sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

5 **a** $-2{\cdot}59 - 9{\cdot}66i$ **b** $-0{\cdot}448 - 3{\cdot}97i$

c $32\sqrt{2} + 32\sqrt{2}i$

6 $16\sin^5 x - 20\sin^3 x + 5\sin x$

7 $2\left(\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right)$

$2\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$2\left(\cos\left(-\frac{7\pi}{12}\right) + i\sin\left(-\frac{7\pi}{12}\right)\right)$

8 $1; -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

9 $-1 \pm 2i, 2 \pm 3i$

Chapter 5

Exercise 5A

1 a $y = -\frac{1}{2}e^{-2x} + c$ **b** $y = \tan x + c$

c $y = Ae^{\frac{5x}{2}}$ where $A = e^c$

d $y = \ln|x + 2| + c$

d $y = \frac{x^3}{3} - 3x^2 + c$

e $y = Ae^{-\frac{9}{x}}$ where $A = e^c$

f $y = \pm\sqrt{\frac{cx - 18}{x}}$

g $y = \frac{1}{3}\ln|3x^2 + c|$

h $y = A(1 - x)^3 - 1$, where $A = e^c$

2 a $y = \ln\left(\frac{x^2 + 2}{2}\right)$ or

$y = \ln(x^2 + 2) - \ln 2$

b $y = -\frac{1}{2}\ln\left(\frac{2}{3}\cos(3t) + \frac{1}{3}\right)$

c $V = 3e^{t(2 - t)}$

d $P = \frac{t + 2}{t + 1}$

3 a $x = Ae^{kt}$

b when t = 5 hours, $x = 2263$ bacteria

4 a $2\sqrt{h} = -kt + c$

b $h = 0$ when $t = 40$ minutes

5 a $n = Ae^{kt}$

b i $k = 1{\cdot}196$ (to 3 d.p.)

ii $t = 5{\cdot}8$ weeks (approx. 5 weeks 6 days)

6 a $n = Ae^{-kt} + 85$

b i $T = 94{\cdot}2$°C (to 3 s.f.)

ii $t = 1{\cdot}88$ hours (approx. 1 hour 53 minutes)

7 $t = 6{\cdot}7$ hours (approx. 6 hours 41 minutes)

8 a $A = 2,\ k = \frac{\ln 492{\cdot}5}{5}$

b $t = 3{\cdot}9$ days (to nearest 0·1 day)

Exercise 5B

1 $y = \frac{x^2}{3} + \frac{c}{x}$

2 $y = e^{2x}(e^x + c)$

3 $y = \frac{1}{e^{\frac{x}{2}}}\left(\frac{1}{2}e^x + c\right)$

4 $y = \frac{1}{x}(2\sin x + c)$

5 $y = x^2(c - \cos x)$

6 $y = \frac{1}{e^{x^2}}(3\ln x + c)$

7 $y = \frac{1}{x^3}(x^5 + 2)$ or $y = x^2 + \frac{2}{x^3}$

8 $y = x^2(1 - \cos x)$

9 $y = x^3\sin x$

10 a Proof (Substitute $u = \cos x$)

b $y = (x + c)\cos x$

11 a $G = 25k(1 - e^{-\frac{t}{25}})$

b $k = 0{\cdot}132$ (3 d.p.)

c $G(10) = 1{\cdot}09$ m ≈ 1 m, so the claim is justified.

d at $t \to \infty$, $G \to 3{\cdot}6$ m

Exercise 5C

1 a $y = Ae^{2x} + Be^{9x}$

b $y = (Ax + B)e^{-x}$

c $y = Ae^{-5x} + Be^{7x}$

d $y = e^{-3x}(A\sin x + B\cos x)$

e $y = (Ax + B)e^{4x}$

f $y = e^{-2x}(A\cos 2x + B\sin 2x)$

2 a $y = 5e^{-2x} - 4e^{-3x}$

b $y = \sin 2x + \cos 2x)$

c $y = (2 - 5x)e^{5x}$

d $y = 4e^x - e^{-3x}$

Exercise 5D

1 a $y = Ae^{2x} + Be^{-x} - 2x$

b $y = (Ax + B)e^{-3x} - \frac{1}{25}e^{2x}$

c $y = 5x - 2 +$
$e^{-\frac{x}{2}}\left(A\sin\left(\frac{3x}{2}\right) + B\cos\left(\frac{3x}{2}\right)\right)$

d $y = 2e^x + (Ax + B)e^{5x}$

e $y = Ae^{6x} + Be^{2x} + \cos x$

f $y = e^{\frac{3x}{5}}\left(A\sin\left(\frac{6x}{5}\right) + B\cos\left(\frac{6x}{5}\right)\right)$
$+ \frac{15}{26}\cos x + \frac{18}{13}\sin x$

2 **a** $y = (1 - 5x)e^{4x} + \frac{1}{4}e^{2x}$
b $y = 3e^{-x}\cos 2x + 2x - 1$
c $y = 2e^{x} - e^{-x} - 3\sin x + 2\cos x$
d $y = e^{-x}(2\sin x + 3\cos x) + 2\sin x - \cos x$

Chapter review

1 **a** $y = \pm\sqrt{4x^2 + 6x + c}$
b $y = A(2x - 1)^{\frac{3}{2}}$
c $y = \frac{1}{3}\ln\left|6x^2 + c\right|$ or $y = \ln\left|\sqrt[3]{6x^2 + c}\right|$
d $y = \tan^{-1}(x^2 + c)$
e $V = t^2$
f $y = \sqrt[3]{\frac{9}{4}\left(x^3 + 2x^2\right)^2}$

2 $y = Ax(x+1)$ where $A = e^c$

3 **a** $\frac{dm}{dt} = km$ general solution, $m = Ae^{kt}$
b $t = 23.3$ months (to nearest 0.1 months)

4 **a** $T = 100°C$
b $T = 4.5$ minutes (to nearest 0.1 minute)

5 **a** $\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$
b $x = \frac{Ae^{kt}}{1 - Ae^{kt}}$
c **i** $x = \frac{\frac{1}{1001}e^{0{\cdot}459t}}{1 - \frac{1}{1001}e^{0{\cdot}459t}}$, where $t = 10$
ii $t = 11.6$ days when $x = 0.25$

6 **a** $y = \frac{1}{x}(\sin x - x\cos x + c)$
b $y = x\left(\frac{2}{3}\ln|x| + c\right)$
c $y = x^2(c - \cos x)$
d $y = \frac{3x + c}{e^{\cos x}}$
e $y = \frac{e^x(x - 1) + 1}{x}$
f $y = -2xe^x$

7 **a** $y = e^x(A\sin x + B\cos x)$
b $y = Ae^{5x} + Be^{7x}$
c $y = (Ax + B)e^{-6x} + 3$
d $y = e^x(\cos 6x - \frac{7}{6}\sin 6x) + 2e^{3x}$
e $y = 3e^{2x} - 8\sin x - 3\cos x$

Chapter 6

Exercise 6A

1 **a** $x = 1$ **b** $x = -3$
c $x = 4$ **d** $x = -1, x = 1$
e $x = 2$ **f** $x = -3, x = \frac{1}{3}$
g $x = -2, x = 0, x = 1$
h $x = -1, x = 2$

2 **a** $x = 0{\cdot}21, x = 4{\cdot}79$
b $x = -1$
c $x = -6{\cdot}54, x = -0{\cdot}46$
d No vertical asymptotes

3 **a** $x = -1$ **b** $x = 2$
c No vertical asymptotes
d $x = -3$
e $x = -3, x = 3$
f No vertical asymptotes
g No vertical asymptotes
h $x = 1$

4 $x = -2$ is a repeated root of the denominator, so although $x + 2$ is a common factor of the numerator and denominator, a factor of $x + 2$ remains in the denominator after cancellation.

5 **a** $x = 2$
b No vertical asymptotes

Exercise 6B

1 **a** $y = 0$ **b** $y = 4$
c $y = 0$ **d** $y = -2$
e $y = -9$ **f** $y = 1$

2 **a** $y = 1$ **b** $y = 0$
c $y = \frac{1}{2}$
d No horizontal asymptotes

3 **a** $y = 0$ **b** $y = \frac{1}{3}$

4 **a** $9.82\,\text{ms}^{-2}$ (to 2 d.p.)
b $54\,\text{ms}^{-1}$
c $v(15) = 39.51\,\text{ms}^{-1}$, which is far less than 99% of $54\,\text{ms}^{-1}$. The model is not particularly accurate.

Exercise 6C

1 **a** $y = 2x$ **b** $y = -x$
c $y = 4x - 4$ **d** $y = \frac{x}{2}$
e $y = -2x$ **f** $y = -\frac{2x}{3} + \frac{2}{9}$

2 **a** $y = x + 1$ **b** $y = -5x - 7$
c $y = 2x + 3$ **d** $y = -\frac{x}{3} - \frac{1}{9}$
e $y = x - 9$ **f** $y = \frac{x}{4} - \frac{3}{16}$

3 **a** Vertical and horizontal
b Horizontal
c Vertical and horizontal
d Vertical and oblique
e Vertical and oblique
f No asymptotes

4 **a** $x = 1, y = 0$ **b** $x = -1, y = -1$
c $x = 0, y = 4x$ **d** $x = 1, y = 1$
e $y = -2$
f $x = \frac{-1}{\sqrt{2}}, x = \frac{1}{\sqrt{2}}, y = \frac{-x}{2}$

Exercise 6D

1 **a** local min $(0, -3)$
b local max $\left(\frac{3}{4}, \frac{49}{8}\right)$
c local max $\left(-2, \frac{22}{3}\right)$, local min $\left(2, \frac{-10}{3}\right)$
d local max $(1, 7)$, local min $(5, -25)$
e local min $\left(-2, \frac{2}{3}\right)$, local max $\left(-1, \frac{13}{12}\right)$, local min $\left(1, \frac{-19}{12}\right)$
f local min $(-0{\cdot}794, 1{\cdot}89)$

2 **a** point of horizontal inflection $(0, -2)$
b point of horizontal inflection $\left(1, \frac{1}{3}\right)$
c local max $(-2, 0{\cdot}541)$, local min $(0, 0)$

d local min (–3, –0·344), point of horizontal inflection (0, 1)

e local min (0, 0), local max (2, 0·541)

3 local maxima at $\left(\frac{3\pi}{4} + 2k\pi, \sqrt{2}\right)$

local minima at $\left(\frac{7\pi}{4} + 2k\pi, -\sqrt{2}\right)$ for

$k = \ldots, -2, -1, 0, 1, 2, \ldots$

Exercise 6E

1 a (0, –2)

b (–3, 0)

c (–3, 27), (–1, 11)

d No points of inflection

e (0, 0)

2 a (–2π, 0), (–π, 0), (0, 0), (π, 0), (2π, 0)

b $\left(\frac{-7\pi}{4}, -1\right), \left(\frac{-5\pi}{4}, -1\right), \left(\frac{-3\pi}{4}, -1\right),$
$\left(\frac{-\pi}{4}, -1\right), \left(\frac{\pi}{4}, -1\right), \left(\frac{3\pi}{4}, -1\right),$
$\left(\frac{5\pi}{4}, -1\right) \left(\frac{7\pi}{4}, -1\right)$

c $\left(\frac{-11\pi}{6}, 0\right), \left(\frac{-5\pi}{6}, 0\right), \left(\frac{\pi}{6}, 0\right), \left(\frac{7\pi}{6}, 0\right)$

d $\left(\frac{-5\pi}{4}, 0\right), \left(\frac{-\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right),$
$\left(\frac{7\pi}{6}, 0\right), \left(\frac{11\pi}{6}, 0\right)$

3 $f''(x) = \frac{-10}{(x+2)^3}$ which never equals 0, so no points of inflection.

4 Non–horizontal point of inflection at $\left(e^2, \frac{e^2}{2}\right)$

5 $(-\sqrt{6}, 2.16)$, (0, 4), $(\sqrt{6}, 5.84)$

Exercise 6F

1 a max value 5, min value –19

b max value 7, min value –114

c max value 80, min value –4·05

d max value 6, min value –2·63

e max value 4, min value 0

f max value 2·81, min value –0·25

2 a max value 1, min value 0.2

b max value 7·67, min value 2·72

c max value 1·39, min value –0·37

d max value 42·10, min value –35·89

e max value 0·5, min value –0·5

f max value 31·27, min value 0

3 a max value 6, min value 0

b max value 2, min value –4

c max value 1, min value –1

d max value 2·25, min value 0

4 $f(x)$ tends to ∞ and –∞ as x tends to 0 from above and below, respectively, so $f(x)$ has no maximum or minimum values on the given interval.

Exercise 6G

1 a $f(-x) = 5(-x)^2 = 5x^2 = f(x)$

b $g(-x) = -3 = g(x)$

c $h(-x) = 3\cos(3(-x)) = 3\cos(-3x)$
$= 3\cos(3x) = h(x)$

d $r(-t) = 6(-t)^6 - 3(-t)^4 + (-t)^2$
$= 6t^6 - 3t^4 + t^2 = r(t)$

e $s(-t) = \frac{(4(-t)^3 - 2(-t))}{6(-t)}$

$= \frac{(-4t^3 + 2t)}{(-6t)} = \frac{-(4t^3 - 2t)}{(-6t)}$

$= \frac{(4t^3 - 2t)}{6t} = s(t)$

f $d(-\theta) = 8(-\theta)\sin(2(-\theta))$
$= -8\theta(-\sin 2\theta) = 8\theta\sin 2\theta$
$= d(\theta)$

2 a $f(-x) = \frac{-(-x)^3}{2} = \frac{x^3}{2} = -f(x)$

b $q(-x) = (-x)^5 + 4(-x)^7 = -x^5 - 4x^7$
$= -(x^5 + 4x^7) = -q(x)$

c $h(-\theta) = 3\tan(-\theta) + \sin(-\theta)$
$= -3\tan\theta - \sin\theta$
$= -(3\tan\theta + \sin\theta)$
$= -h(\theta)$

d $s(-t) = (-t)^2((-t) - (-t)^3) = t^2(-t - t^3)$
$= -t^2(t - t3) = -s(t)$

e $f(-x) = \dfrac{-(5(-x)^4 - (-x)^2)}{((-x)^3 - 2(-x))} = \dfrac{-(5x^4 - x^2)}{(-x^3 + 2x)}$
$= \dfrac{-(5x^4 - x^2)}{(-(x^3 - 2x))} = \dfrac{(5x^4 - x^2)}{(x^3 - 2x)} = -f(x)$

f $v(-t) = 2(-t)^3\cos 3(-t)$
$= -2t^3\cos 3t = -v(t)$

3 a odd – sum of two odd functions

b neither – $f(0) = -2 \neq 0$ so not odd, and $f(-1) = -2 \neq 0 = f(1)$ so not even

c even – sum of two even functions (the fraction is even since it is a product of two odd functions)

d even – sum of two even functions (both $\sin^2 x$ and $\cos^2 x$ are even since they are products of two odd functions and two even functions, respectively)

e neither – $f(0) = -1 \neq 0$ so not odd, and $f(-\pi) = 1 + \pi \neq 1 - \pi = f(\pi)$ so not even

f odd – sum of two odd functions

4 $h(-x) = 3(-x)^3\cos(-x) - (-x) = -3x^3\cos x + x = -(3x^3\cos x - x) = -h(x)$

5 The constant function 0 is the only real–valued function which is both even and odd.

6 a Let $f(x)$ and $g(x)$ be even functions. Then $(f + g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f + g)(x)$, so $f + g$ is even.

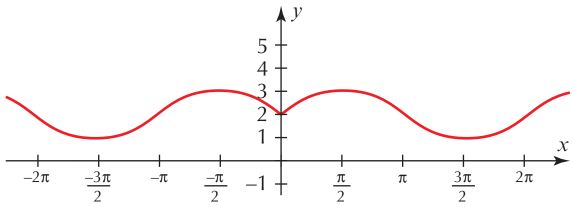
b Let $f(x)$ and $g(x)$ be odd functions. Then $(f + g)(-x) = f(-x) + g(-x) = -f(x) + (-g(x)) = -(f + g)(x)$, so $f + g$ is odd

c Let $f(x)$ and $g(x)$ be even functions. Then $(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x)$, so fg is even.

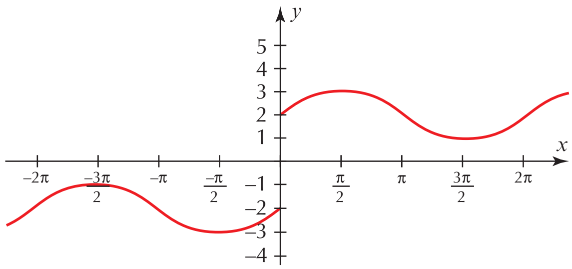
d Let $f(x)$ and $g(x)$ be odd functions. Then $(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x)$, so fg is even.

e Let $f(x)$ be an even function and $g(x)$ be an odd function. Then $(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -f(x)g(x) = (-fg)(x)$, so fg is odd.

7 a



b



Exercise 6H

1 a $x = -5$, $f(x)$ is not defined

b $x = -1$, $x = 1$, $f(x)$ not defined at either point

c $x = 1$, the limit as x tends to 1 from below is –1, but $f(1) = -2$

d $f(x)$ is continuous

e $x = 1$, the limit as x tends to 1 from below is 2, but $f(1) = 4$

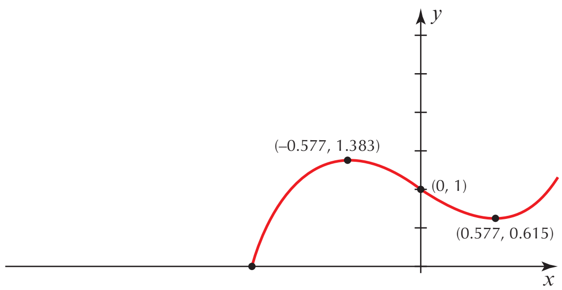
f $x = 0$, the limit as x tends to 0 from above does not exist

g $f(x)$ is continuous

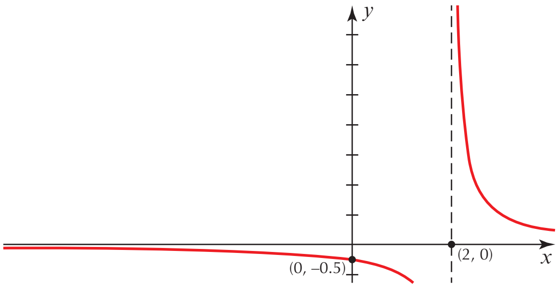
h $x = 1$, the limit as x tends to 0 from below is 1, but $f(1) = 2$

Exercise 6I

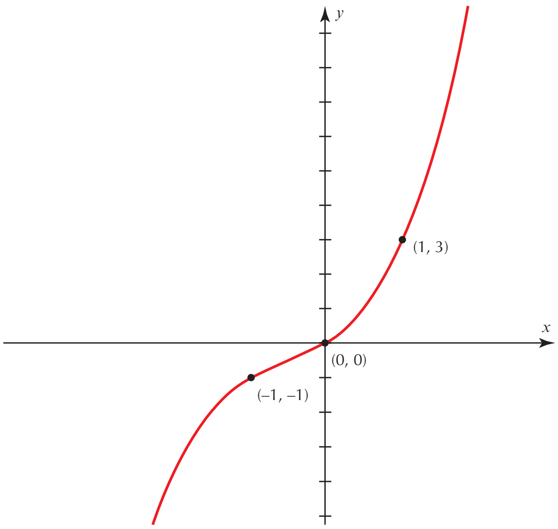
1 a



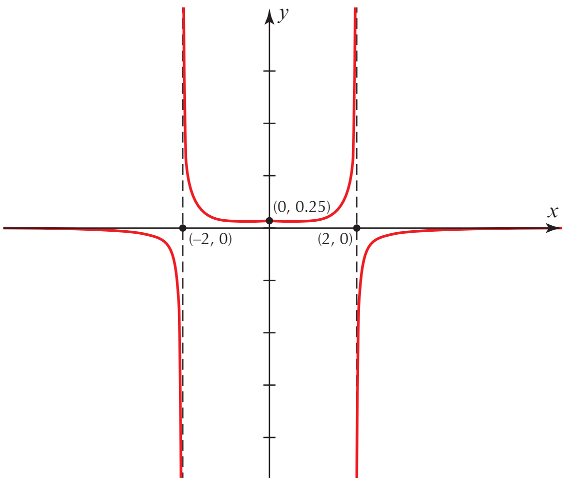
b


y
x
(2, 0)
(0, –0.5)

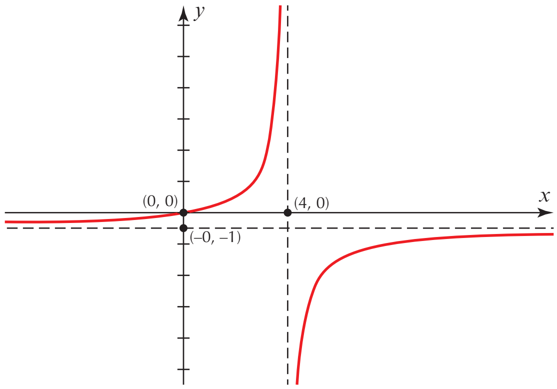
c


y
x
(1, 3)
(0, 0)
(–1, –1)

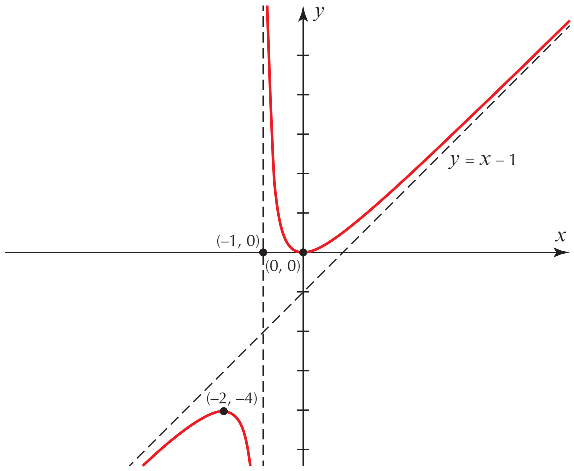
d


y
x
(0, 0.25)
(–2, 0)
(2, 0)

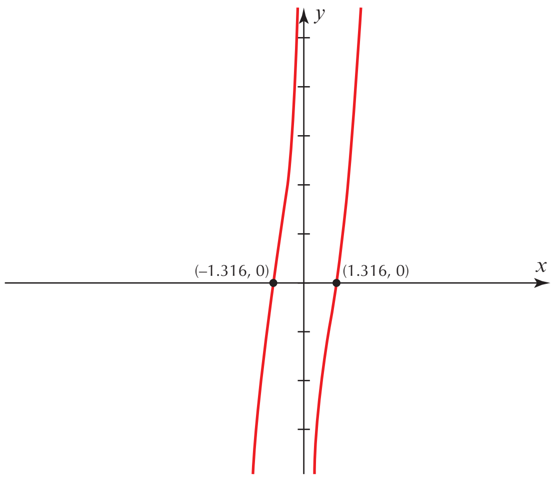
e


y
x
(0, 0)
(4, 0)
(–0, –1)

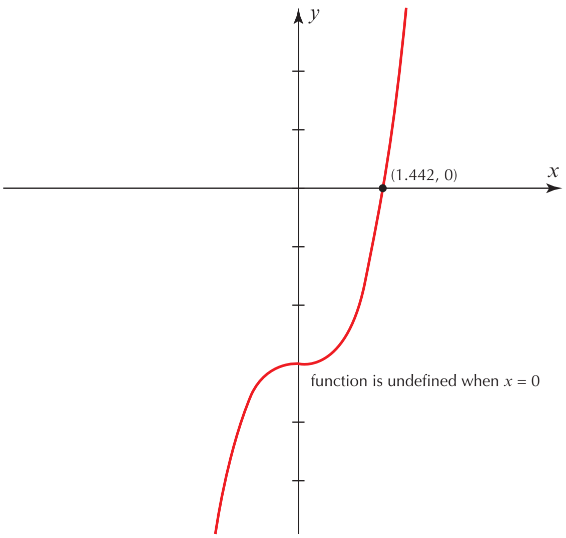
f


y
x
y = x – 1
(–1, 0)
(0, 0)
(–2, –4)

g

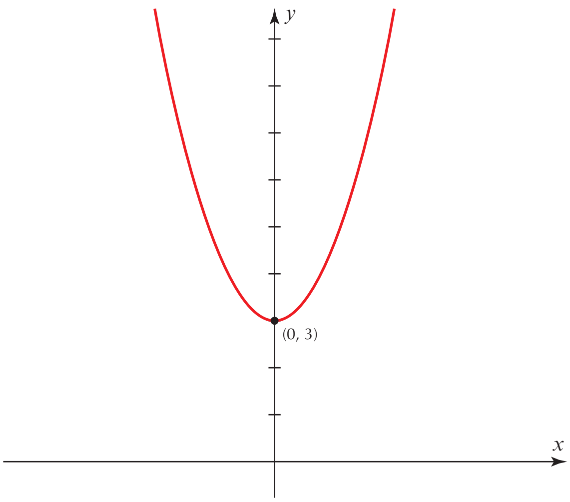

y
x
(–1.316, 0)
(1.316, 0)

h

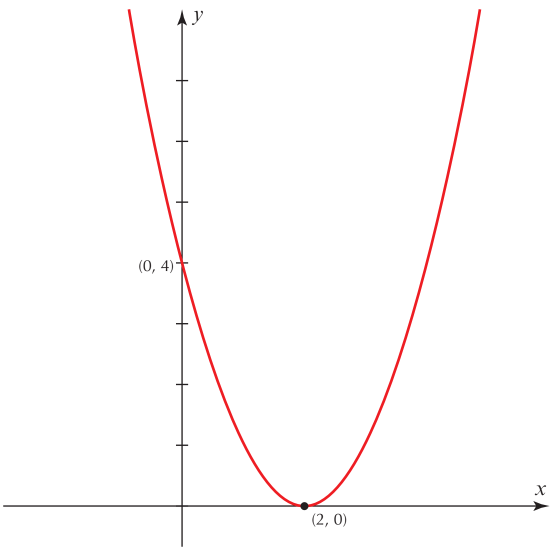


Exercise 6J

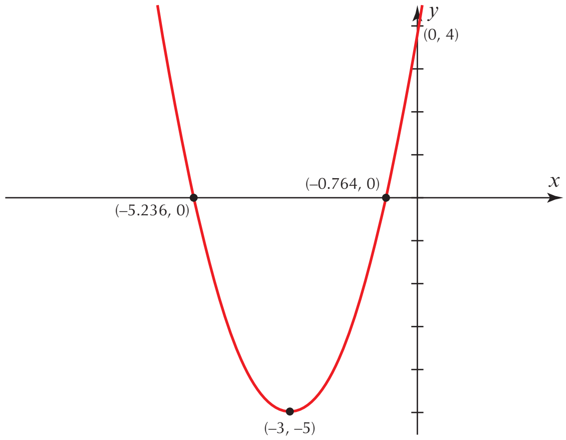
1 a Graph is shifted up 3 units.



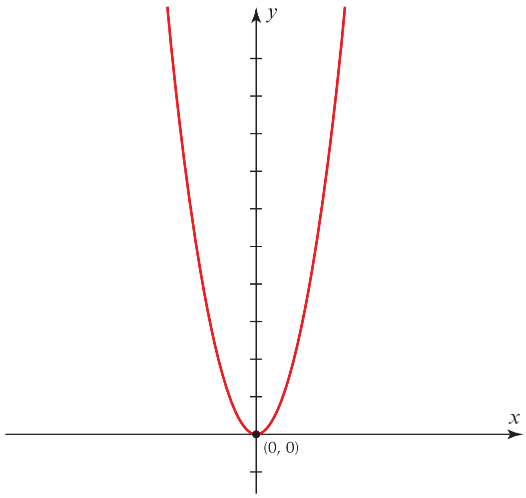
b Graph is shifted right 2 units.



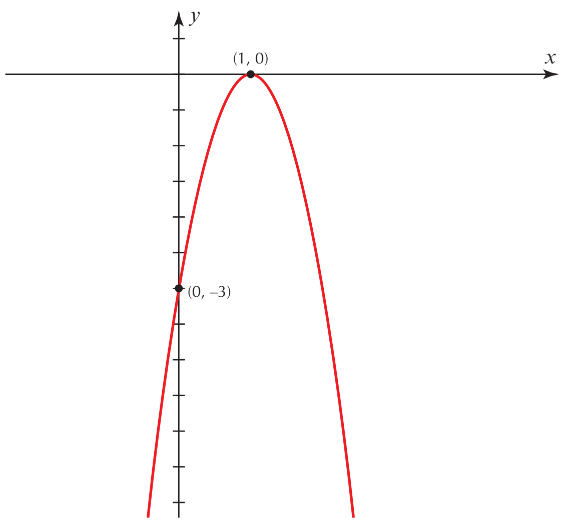
c Graph is shifted left 3 units then down 5 units.



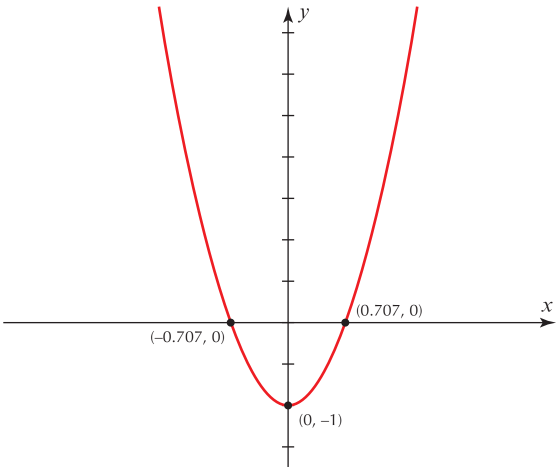
d Graph is compressed horizontally by a factor of 2.



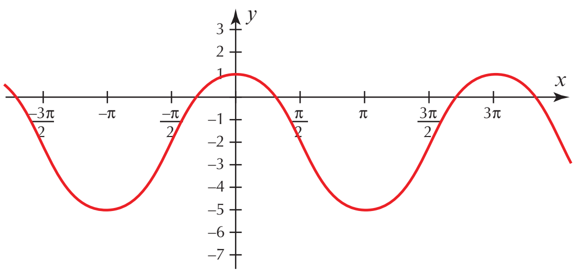
e $3f(-x + 1) = 3f(-(x - 1))$. The graph is shifted right 1 unit, then reflected in the y–axis, then reflected in the x–axis and scaled vertically by a factor of 3.



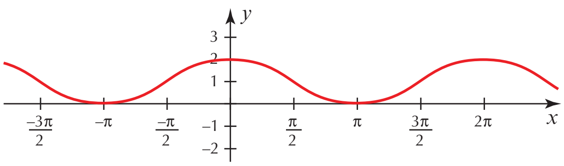
f Graph is scaled vertically by a factor of 2, then shifted down 1 unit.



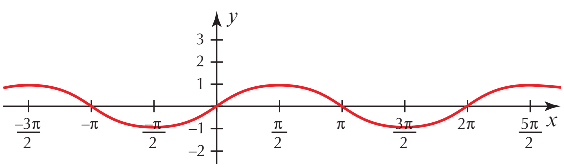
2 a Graph is scaled vertically by a factor of 3, then shifted down 2 units.



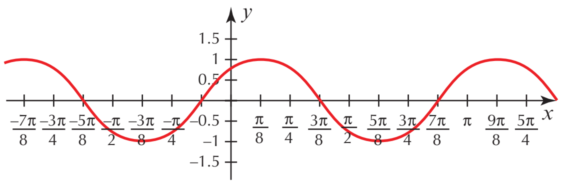
b Graph is reflected in the y–axis (so no effect since $\cos(x)$ is an even function) then shifted up 1 unit.



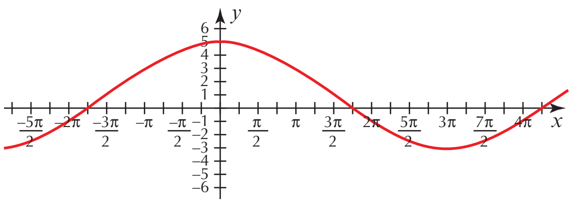
c Graph is shifted left by $\frac{\pi}{2}$ units, then reflected in the x–axis.



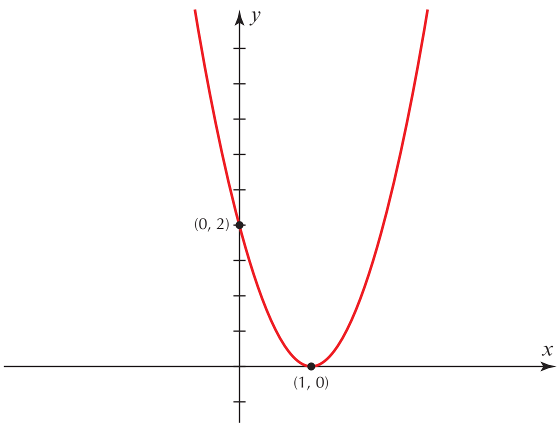
d Note that $f\left(2x - \frac{\pi}{4}\right) = f\left(2\left(x - \frac{\pi}{8}\right)\right)$, so the graph is shifted right by $\frac{\pi}{8}$ units then scaled horizontally by a factor of $\frac{1}{2}$.



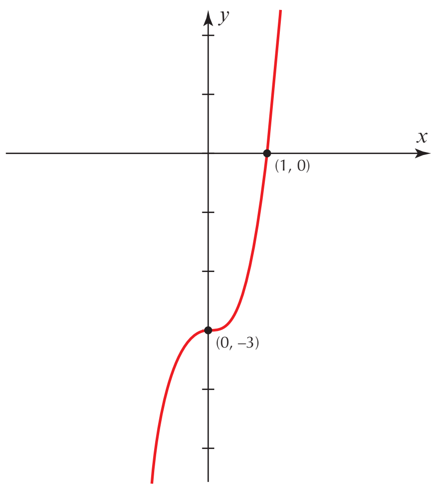
e The graph is scaled horizontally by a factor of 3, then scaled vertically by a factor of 4, then shifted up 1 unit.



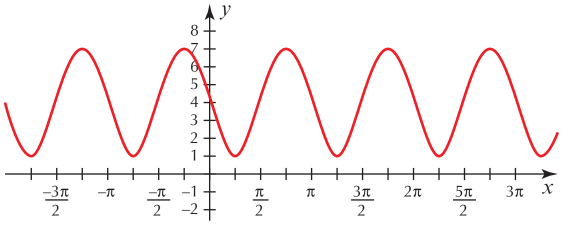
3 a



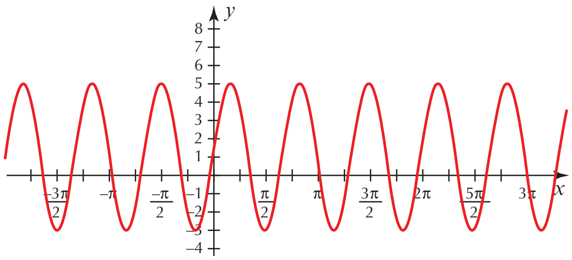
b



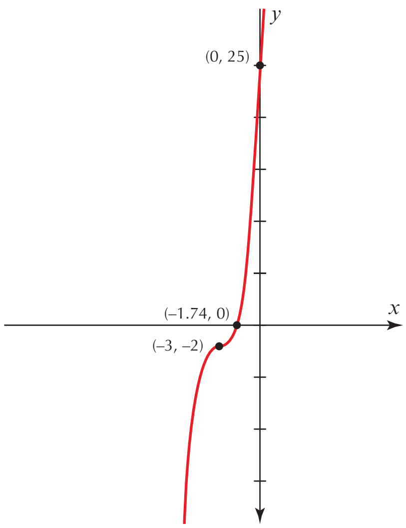
c


y
x
8
7
6
5
4
3
2
1
−1
−2
−3π/2
−π
−π/2
π/2
π
3π/2
2π
5π/2
3π

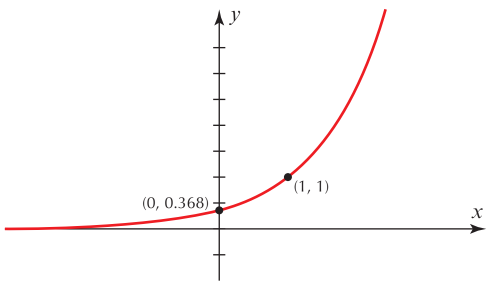
d


y
x
8
7
6
5
4
3
2
1
−1
−2
−3
−4
−3π/2
−π
−π/2
π/2
π
3π/2
2π
5π/2
3π

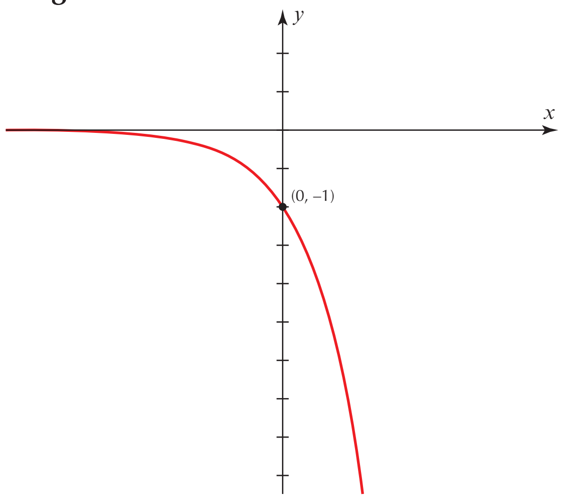
e


y
x
(0, 25)
(−1.74, 0)
(−3, −2)

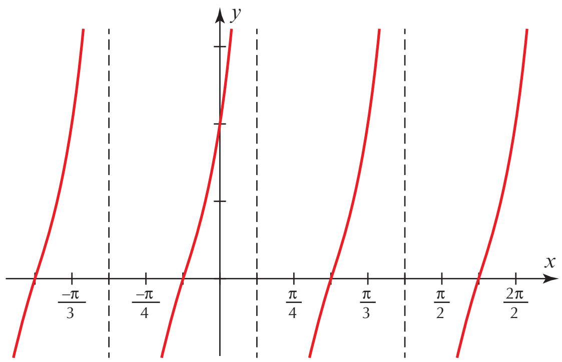
f


y
x
(0, 0.368)
(1, 1)

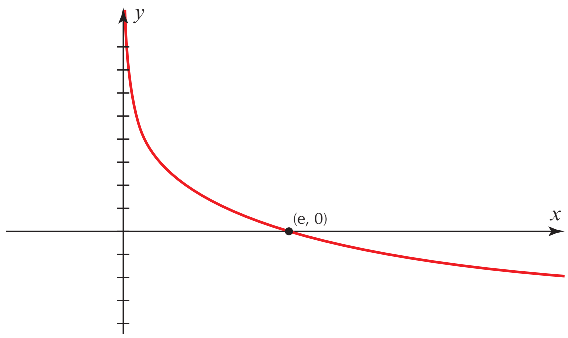
g


y
x
(0, −1)

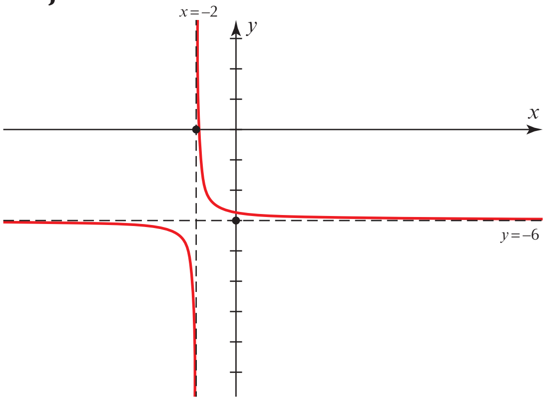
h


y
x
−π/3
−π/4
π/4
π/3
π/2
2π/2

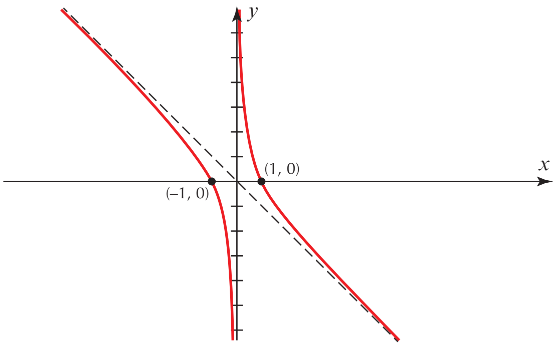
i


y
x
(e, 0)

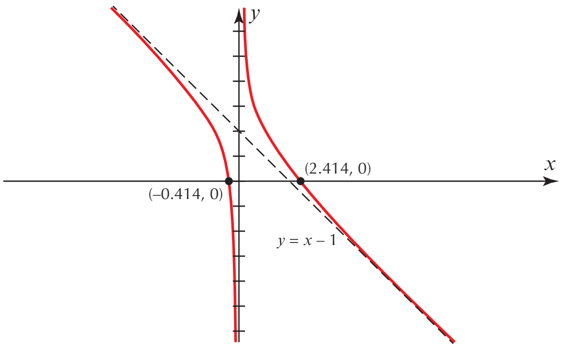
j



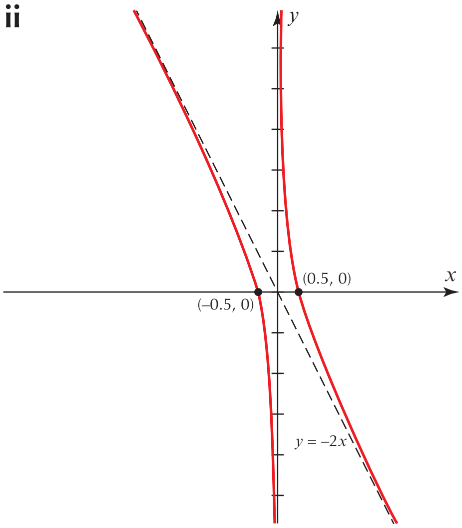
4 a



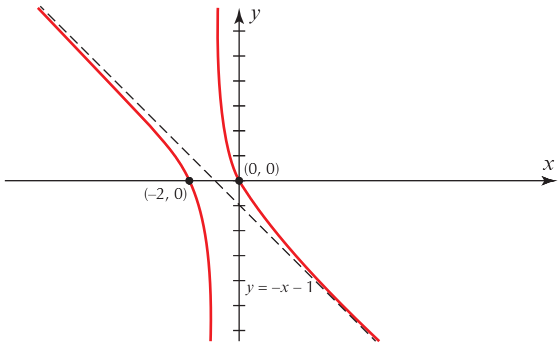
b i



ii

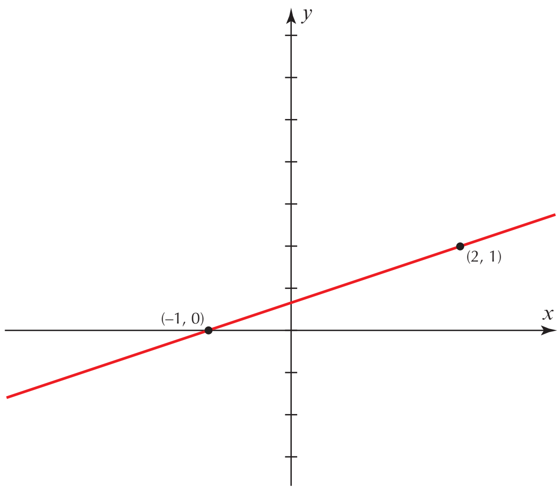


iii

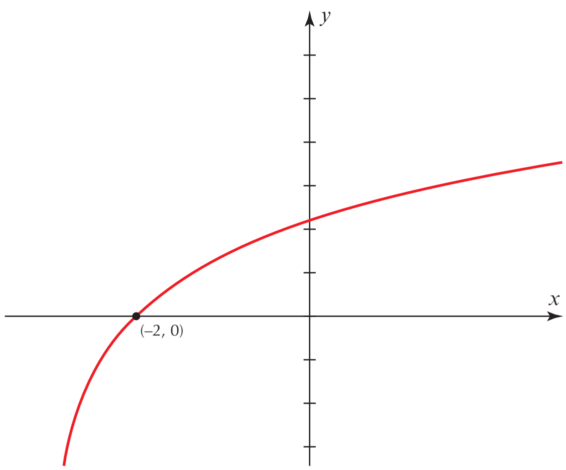


Exercise 6K

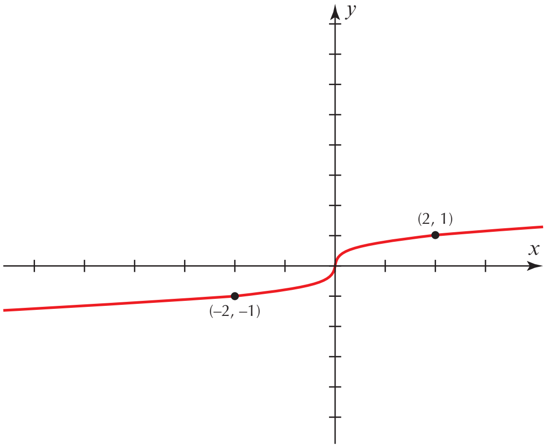
1 a



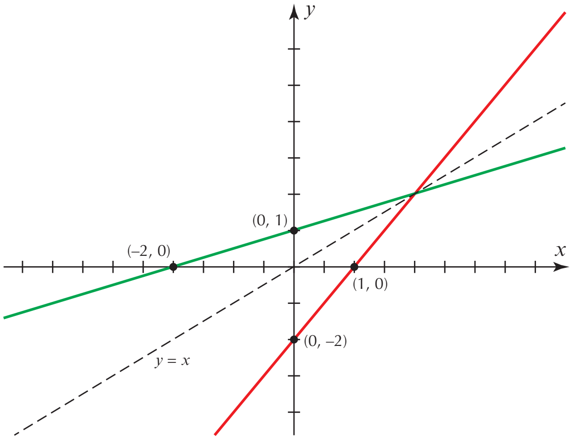
b



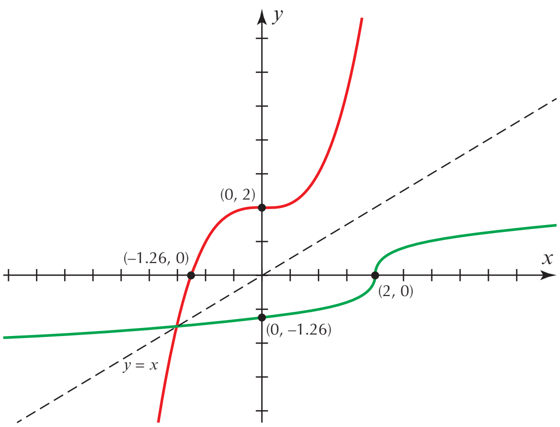
c


y
x
(2, 1)
(–2, –1)

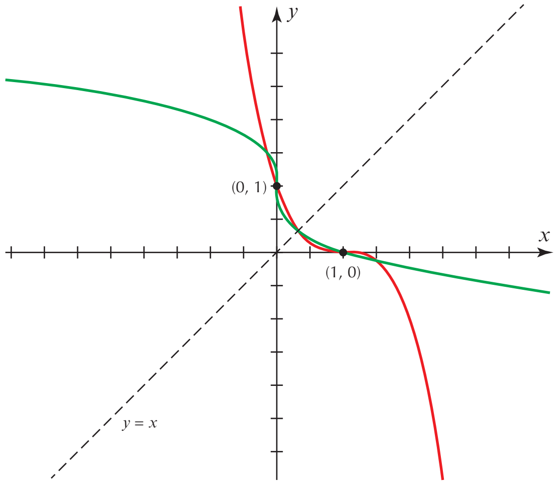
2 a


y
x
(0, 1)
(–2, 0)
(1, 0)
(0, –2)
y = x

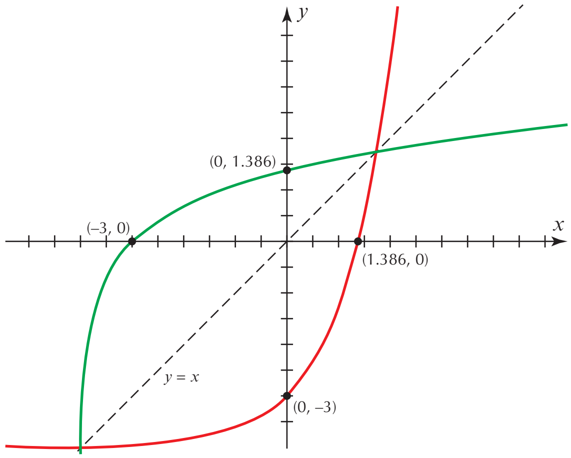
b


y
x
(0, 2)
(–1.26, 0)
(2, 0)
(0, –1.26)
y = x

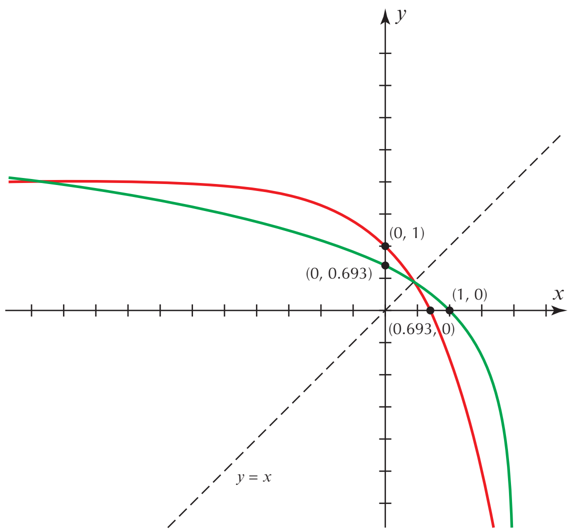
c


y
x
(0, 1)
(1, 0)
y = x

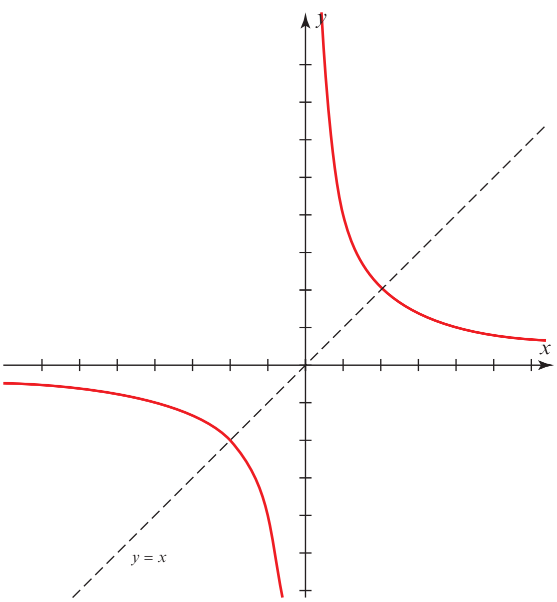
d


y
x
(0, 1.386)
(–3, 0)
(1.386, 0)
(0, –3)
y = x

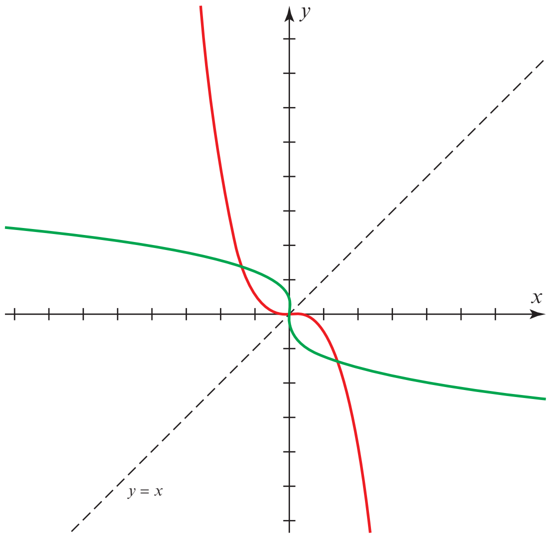
e


y
x
(0, 1)
(0, 0.693)
(1, 0)
(0.693, 0)
y = x

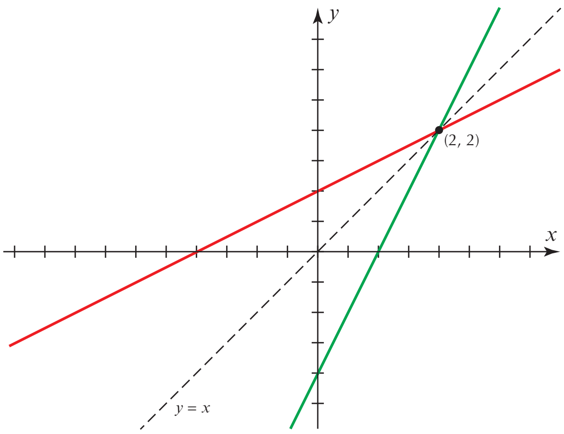
f



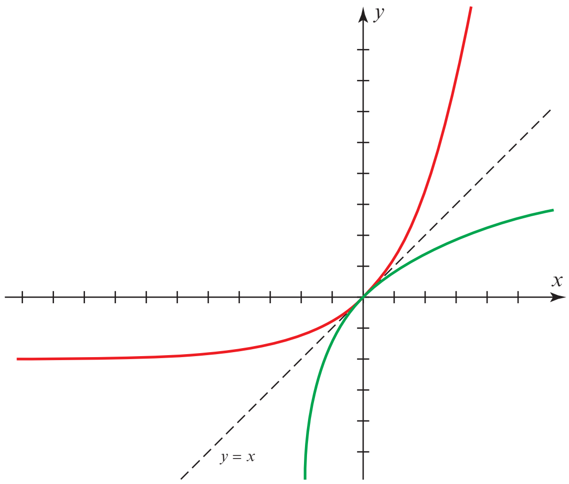
c



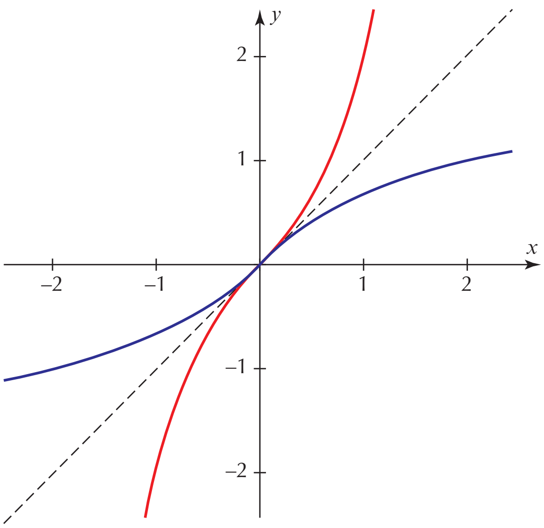
3 a



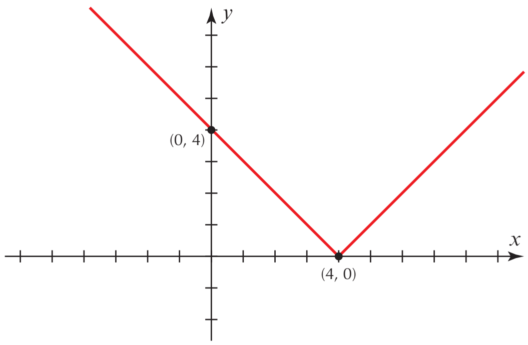
d



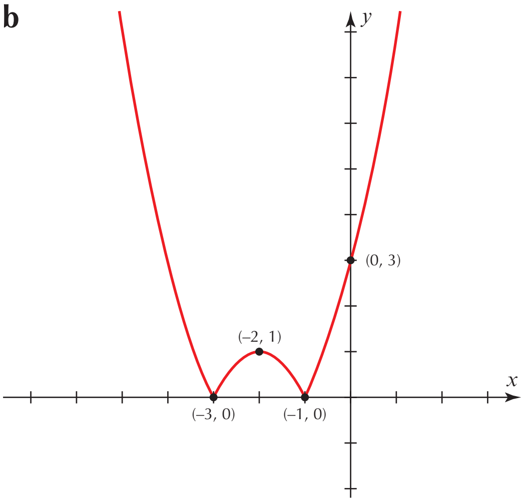
b



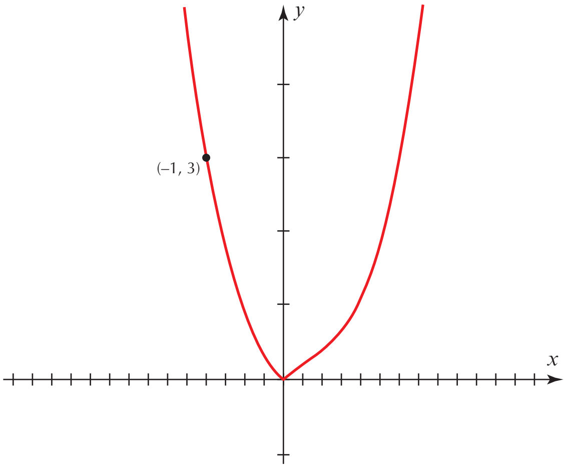
4 a



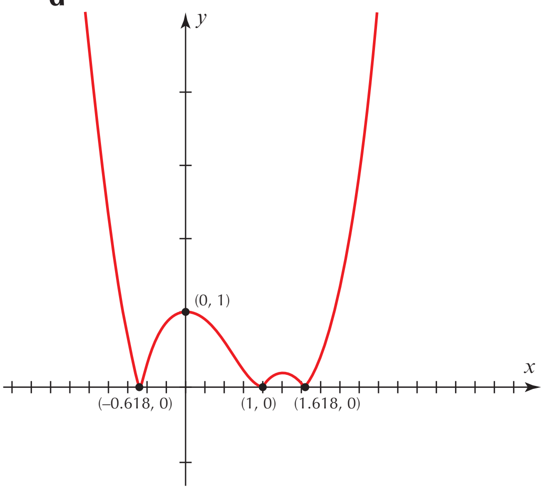
b


y
x
(0, 3)
(–2, 1)
(–3, 0)
(–1, 0)

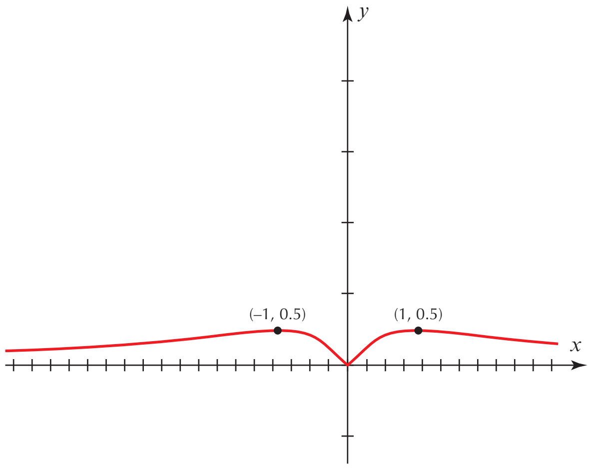
c


y
x
(–1, 3)

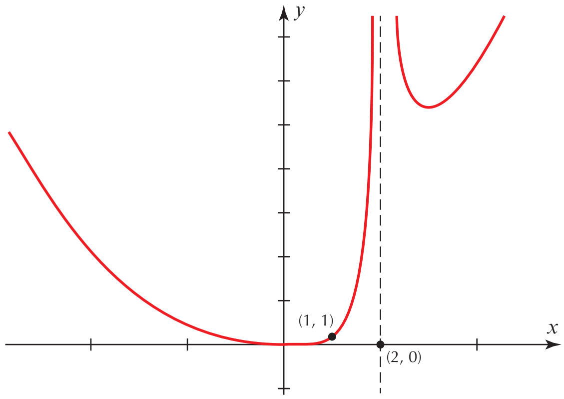
d


y
x
(0, 1)
(–0.618, 0)
(1, 0)
(1.618, 0)

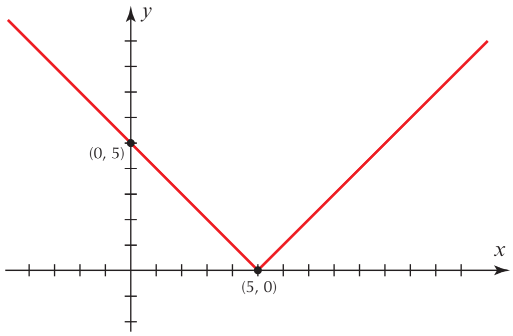
e


y
x
(–1, 0.5)
(1, 0.5)

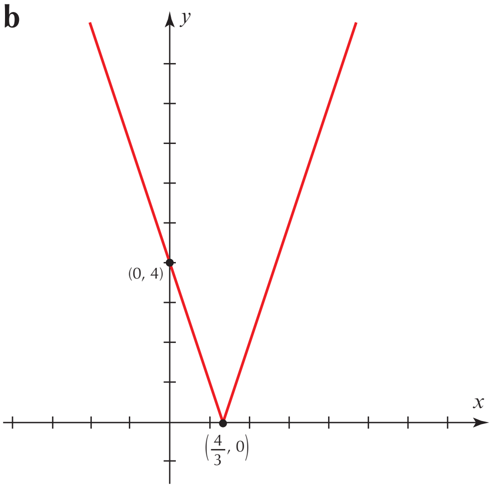
f


y
x
(1, 1)
(2, 0)

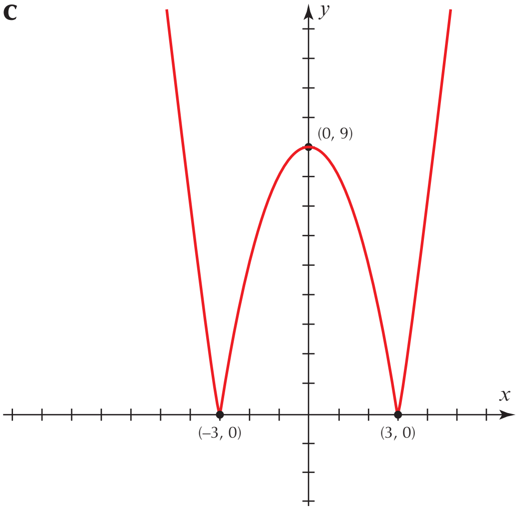
5 a


y
x
(0, 5)
(5, 0)

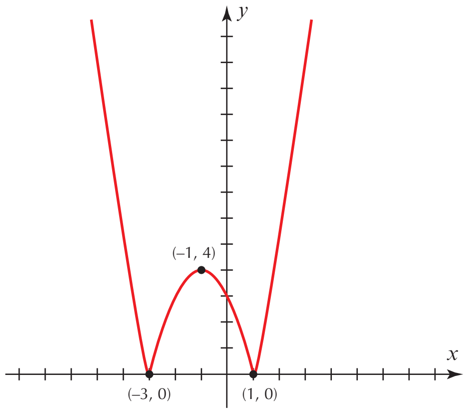
b



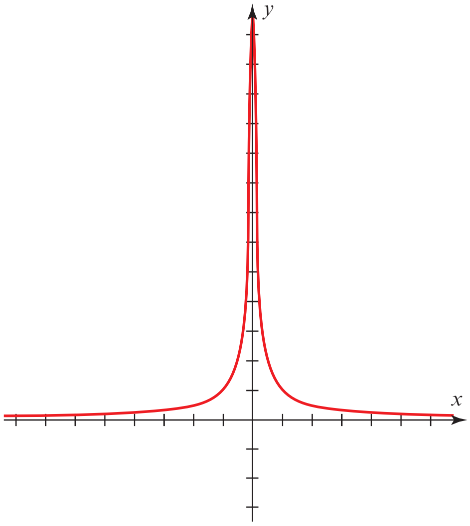
c



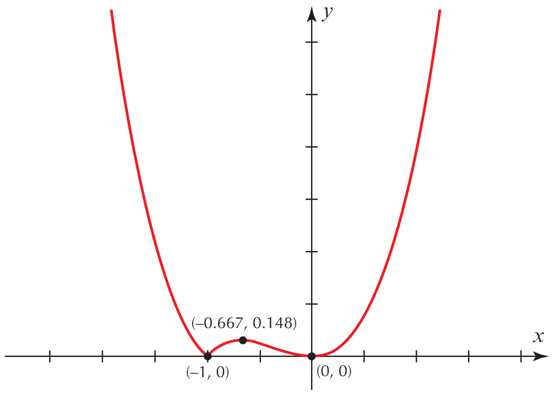
d



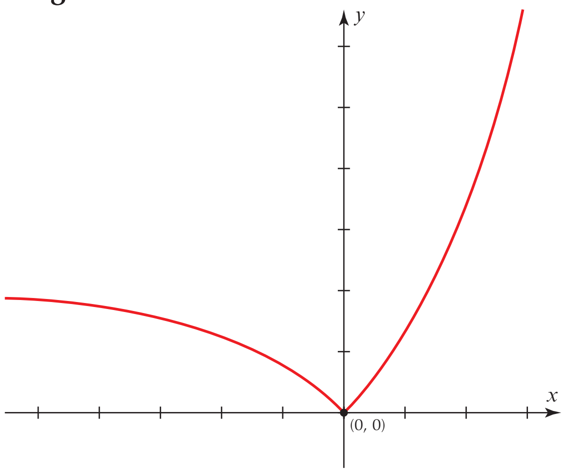
e



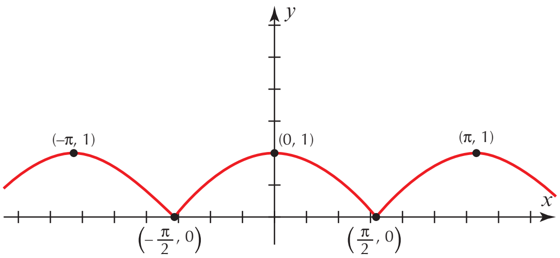
f



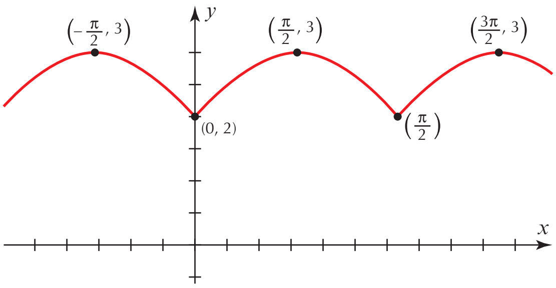
g



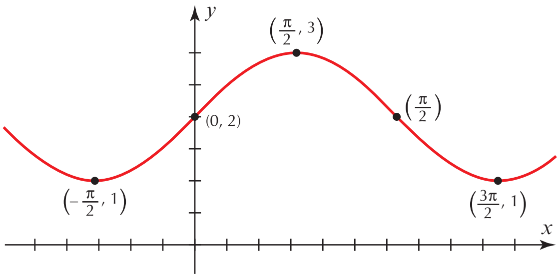
h



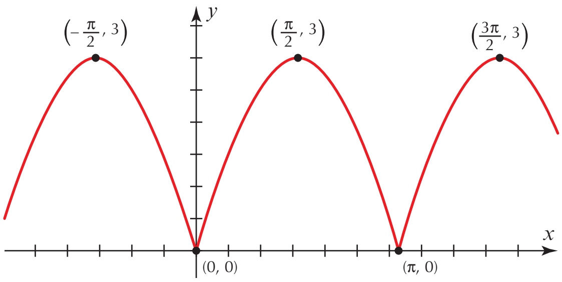
6 a



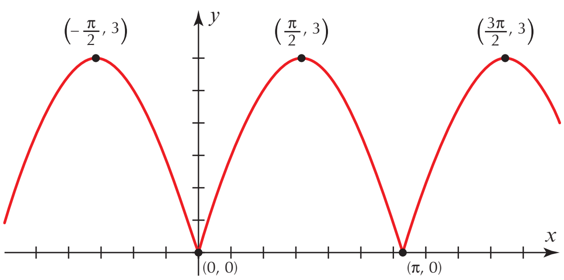
b



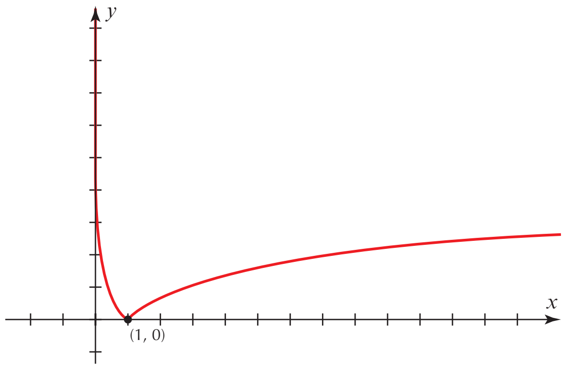
c



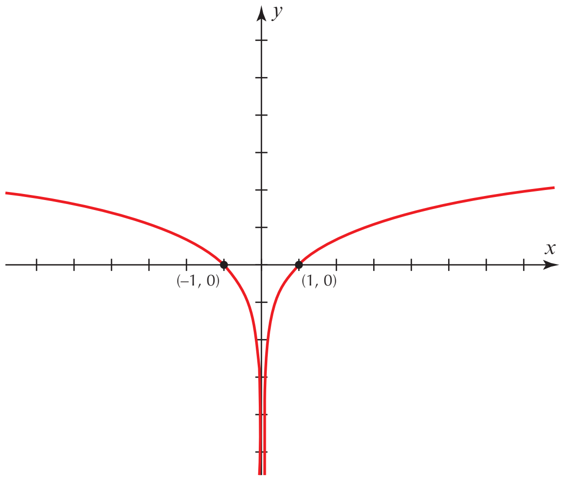
d



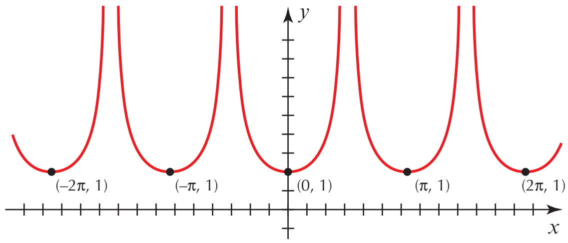
7 a



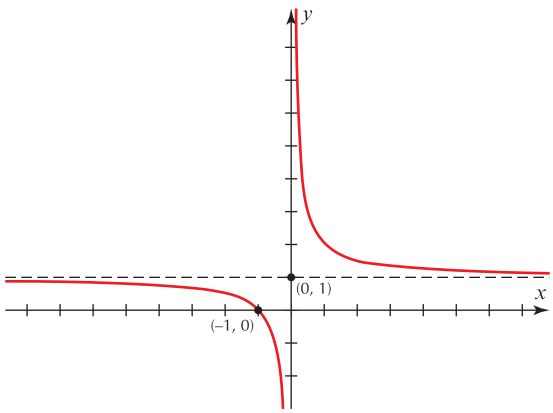
b



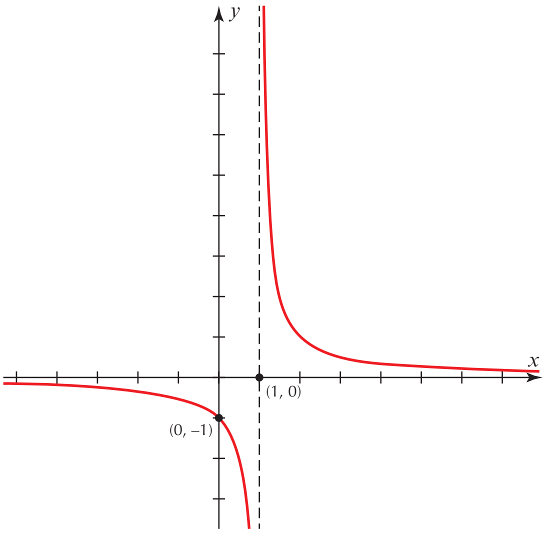
8



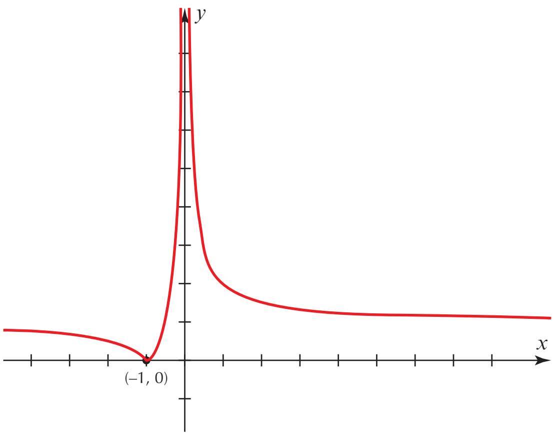
9 a



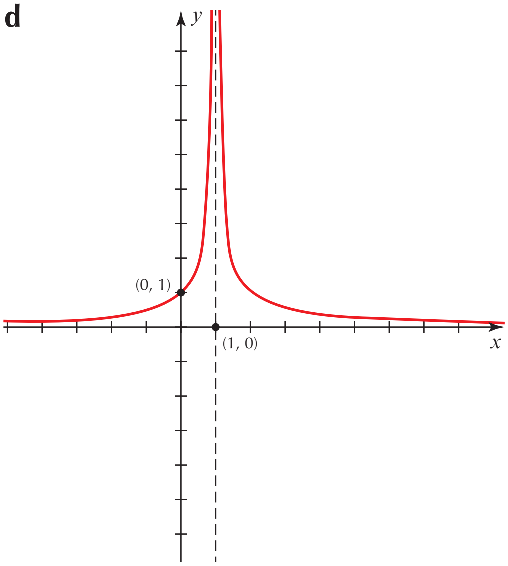
b



c



d



e For each value of y, there are two possible values of x (i.e. $|f(x)|$ is not an injective function).

Chapter review

1 **a** $x = -2, x = 2$

b $x = -1$

c $x = -1, x = 0$

2 **a** $y = 0$

b $y = -2$

c $y = -2$

d $y = \frac{3}{4}$

3 **a** $y = -3x + 3$

b $y = v + 4$

c $y = \frac{2t}{3} + \frac{10}{3}$

4 $x = 2$ (vertical), $y = x + 7$ (oblique)

5 local min (–3, –6·75), point of horizontal inflection (0, 0)

6 **a** local max (–1, 1·667), local min (3, –9)

b point of inflection $\left(1, \frac{-11}{3}\right)$

7 **a** max value 144, min value 13.5

b max value 6·15, min value –3·1

c max value –1, min value –5

8 **a** Even – sum of two even functions

b Neither – $g(-\pi) = \cos(-\pi) - \frac{1}{\pi}$

$= -1 + \frac{1}{\pi}$, but $g(\pi) = -1 - \frac{1}{\pi}$ and $-g(\pi)$

$= 1 + \frac{1}{\pi}$

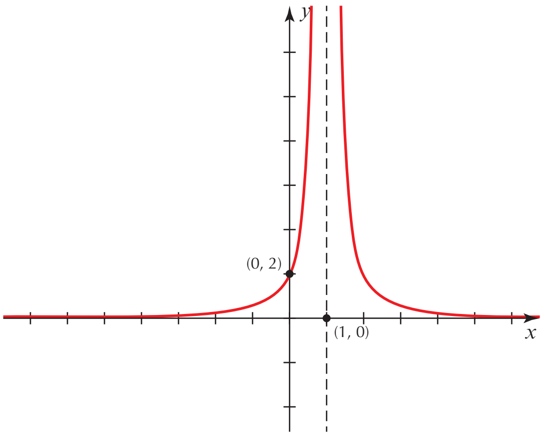
c Even – product of two odd functions

9 **a** $x = -1, x = 5$

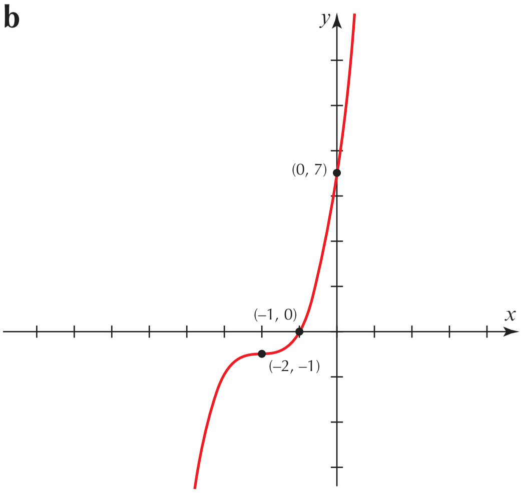
b $x = 1$

c $x = 2$

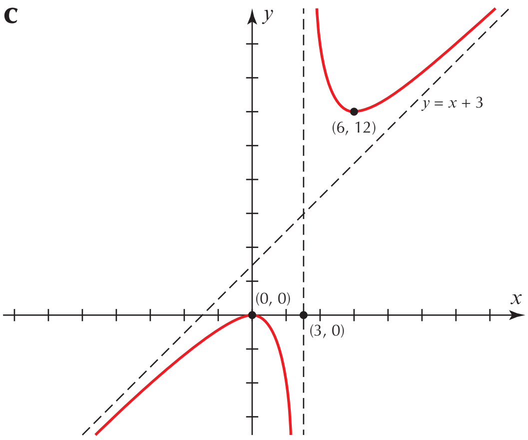
10 a



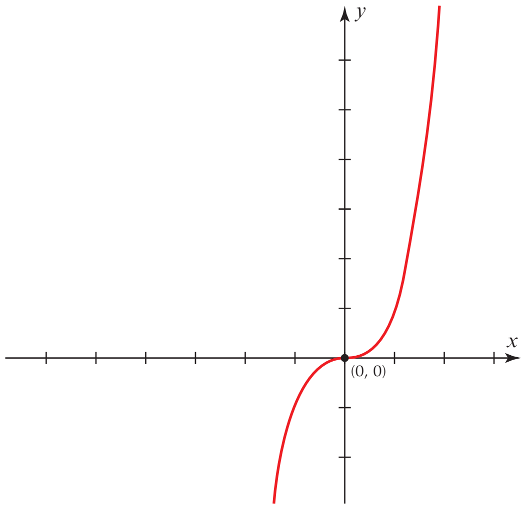
b



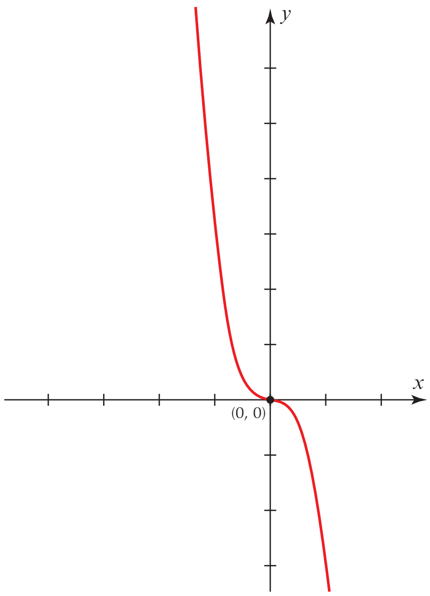
c



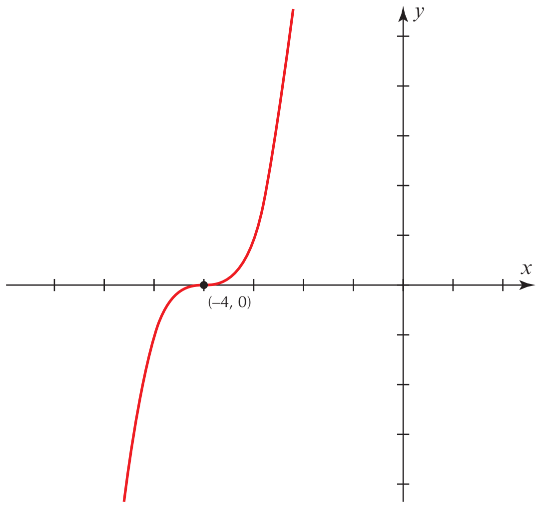
11 a



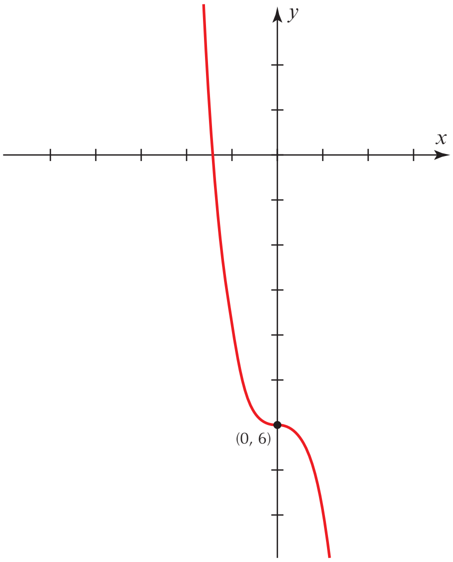
b i Graph is reflected in the x–axis and scaled vertically by a factor of 3.



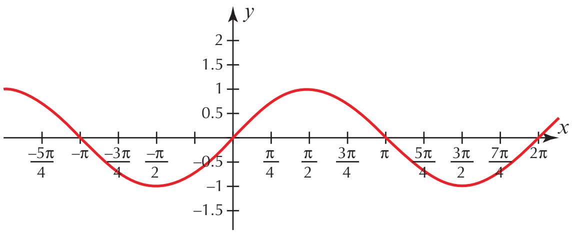
ii Graph is shifted left 4 units.



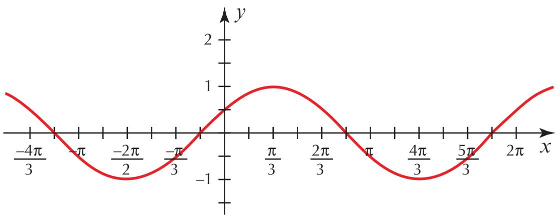
iii Graph is reflected in the *y*–axis, then scaled vertically by a factor of 2, then shifted down 6 units.



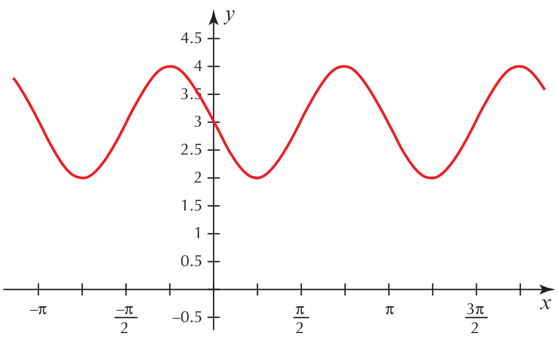
12 a



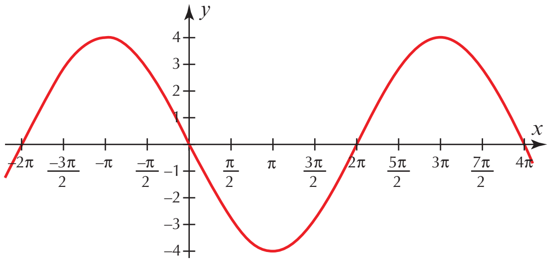
b **i** Graph is shifted left by $\frac{\pi}{6}$ units.



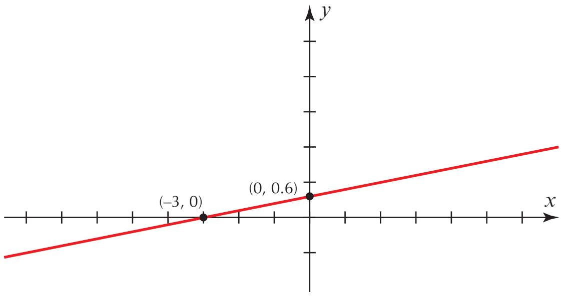
ii Period is halved, then graph is reflected in the *x*–axis, then shifted up by 3 units.



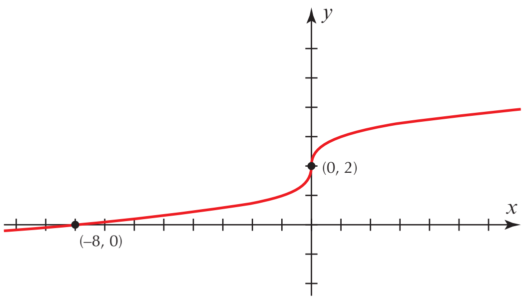
c Period is doubled, graph is reflected in the *y*–axis, then scaled vertically by a factor of 4.



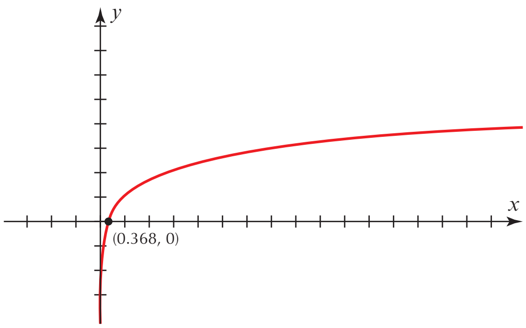
13 a



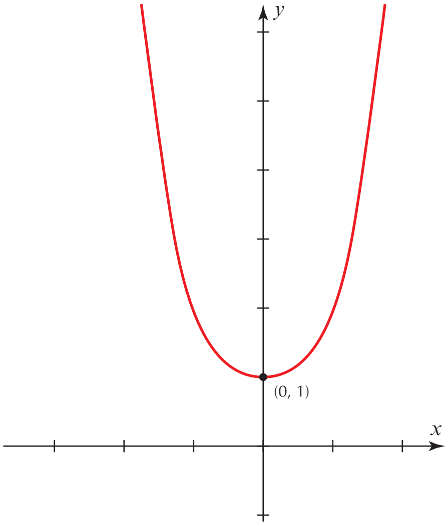
b



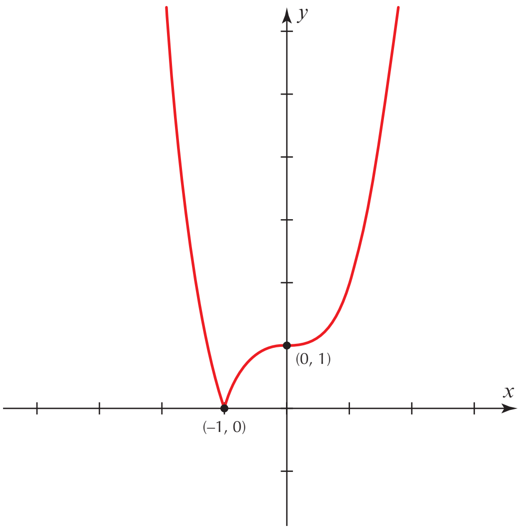
c



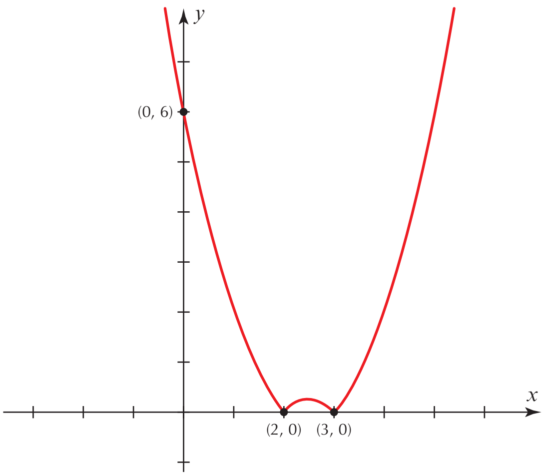
14 a


y
x
(0, 1)

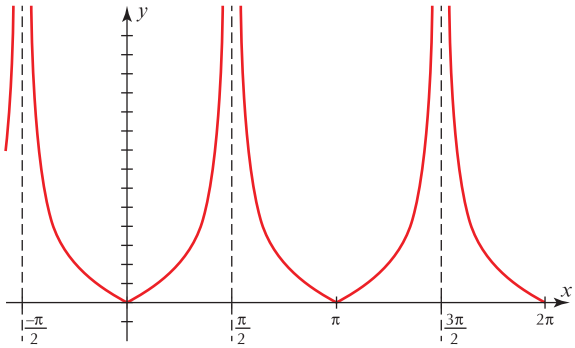
b


y
x
(0, 1)
(–1, 0)

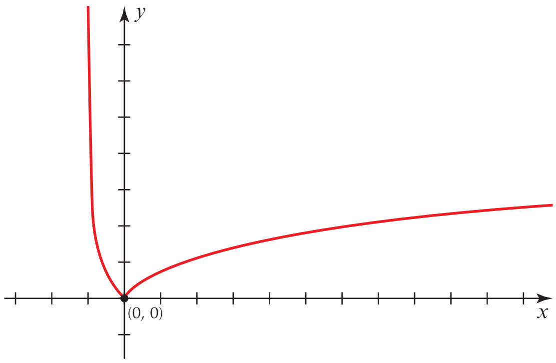
c


y
x
(0, 6)
(2, 0)
(3, 0)

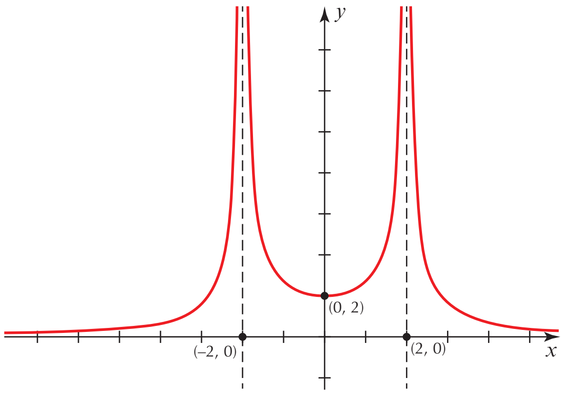
d


y
x
$\frac{-\pi}{2}$
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

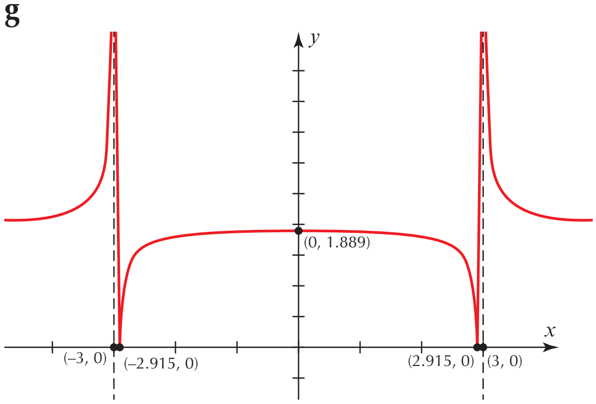
e


y
x
(0, 0)

f


y
x
(0, 2)
(–2, 0)
(2, 0)

g


y
x
(0, 1.889)
(–3, 0)
(–2.915, 0)
(2.915, 0)
(3, 0)

Chapter 7

Exercise 7a

1 a $a = 8, d = 3$
b Not an arithmetic sequence
c $a = 21, d = -2$
d Not an arithmetic sequence
e $a = 5, d = -0{\cdot}3$
f $a = \frac{7}{6}, d = -\frac{1}{6}$
g $a = -3, d = -4$
h Not an arithmetic sequence

2 a $3n + 2$ **b** $4n - 13$
c $-2n + 12$ **d** $-90n + 330$
e $3xn - 2x$ **f** $-4xn + 10x$

3 a 22 **b** −3
c 9 **d** −5·2

4 a 30 **b** 36
c 201 **d** 23

5 a $a = 3, d = 4$ **b** $a = 20, d = -7$
c $a = 24, d = -2$ **d** $a = 5, d = 4$
e $a = 27{\cdot}5, d = -2{\cdot}4$

6 137

7 −42

8 a $a = 41, d = -4$ **b** 12th term

9 a −248·5 **b** 29th term

10 $x = -1, 3$
−3, −1, 1 or −3, 3, 9

11 256 pence

Exercise 7B

1 a 400 **b** 525
c −1340 **d** −1088
e 1421 **f** 2173·5
g $210x + 250$ **h** $805x - 483$

2 a 25 **b** 17
c 33 **d** 34

3 a 10 **b** 5
c 7 **d** 5

4 a $a = 7, d = 5$
b $a = -8, d = 3$
c $a = 1, d = -\frac{2}{7}$

5 6318

6 17

7 15

8 10 000

9 4950

10 $n + n^2$

11 1012_8

12 $u_n = 9 - 2n$

13 a $S_n = 10n - n^2$ **b** $n = 3$ or $n = 7$

14 £68·90

15 a £650 **b** $S_n = 25n(15 + n)$
c 11 years

16 a 13 years **b** £774 600

17 8740 points

18 a 440 straws **b** 30 triangles

Exercise 7C

1 a $a = 1, r = 2$
b Not a geometric sequence (it is an arithmetic sequence)
c $a = 256, r = \frac{1}{2}$
d $a = 1, r = -3$
e $a = \frac{1}{3}, r = \frac{1}{2}$
f $a = 125, r = -\frac{2}{5}$
g Not a geometric sequence

2 a 2187 **b** $\frac{1}{512}$
c $\frac{1}{2048}$ **d** $\frac{5}{2187}$
e 46 875 **f** 24 576

3 a 2 **b** 800
c 512 **d** $\frac{1}{2}$

4 **a** 10 **b** 14
c 10 **d** 9

5 **a** $a = 2$, $r = 3$
b $a = 625$, $r = -\frac{1}{5}$
c $a = 4$, $r = \frac{3}{2}$
d $a = -8$, $r = 2$

6 $x = 20$

7 $x = 3$

8 21st term

9 12th term

10 After 18 years

11 19 years

12 Day 25 of training

Exercise 7D

1 **a** 29 524 **b** $\frac{32767}{16}$
c −13 107 **d** 2604·2 (5sf)
e 2 391 484 **f** $\frac{16383}{8}$
g −209 715 **h** 2604·2 (5sf)

2 **a** 10 **b** 8
c 8 **d** 12

3 **a** 3 **b** 2000
c 1 **d** 2

4 $-\frac{5}{2}, \frac{3}{2}$

5 14 762

6 19

7 20

8 5 cm

9 $1{\cdot}84 \times 10^{19}$ grains (3 s.f.)

10 36·112 metres

Exercise 7E

1 **a** S_∞ does not exist
b 16
c $\frac{729}{4}$
d S_∞ does not exist
e 50
f S_∞ does not exist

2 **a** 160 **b** 90
c 20 **d** −3125

3 **a** 40 **b** $\frac{1}{3}$
c $r = \frac{3}{4}$, $S_6 = \frac{50505}{128}$
d $r = \frac{1}{2}$, $u_6 = \frac{1}{16}$
e $\frac{24}{25}$

4 $a = 2$, $r = \frac{1}{2}$

5 $9, 6, 4, \frac{8}{3}, \ldots$ or $45, -30, 20 - \frac{40}{3}, \ldots$

6 **a** $\frac{45}{99}$ **b** $\frac{27}{99}$
c $\frac{123}{999}$ **d** $\frac{19}{990}$
e $\frac{83}{198}$ **f** $\frac{1721}{990}$

Exercise 7F

1 **a** $1 + 2x + 4x^2 + 8x^3 + \ldots$
b $1 - 4x + 16x^2 - 64x^3 + \ldots$
c $1 - 10x + 100x^2 - 1000x^3 + \ldots$
d $1 - \frac{1}{3}x + \frac{1}{9}x^2 - \frac{1}{27}x^3 + \ldots$

2 **a** $\frac{1}{3} - \frac{1}{9}x + \frac{1}{27}x^2 - \frac{1}{81}x^3 + \ldots$
b $\frac{1}{3} + \frac{2}{9}x + \frac{4}{27}x^2 + \frac{8}{81}x^3 + \ldots$
c $\frac{1}{2} - x + 2x^2 - 4x^3 + \ldots$

d $\frac{1}{5}-\frac{2}{25}x+\frac{4}{125}x^2-\frac{8}{625}x^3+\ldots$

3 **a** $|x|<3$ **b** $|x|<\frac{3}{2}$

c $|x|<\frac{1}{2}$ **d** $|x|<\frac{5}{2}$

4 **a** $5+10x+20x^2+40x^3+\ldots$

b $x-3x^2+9x^3-27x^4+\ldots$

c $-1+7x-35x^2+175x^3-\ldots$

5 **a** $\frac{1}{6}x+\frac{7}{36}x^2+\frac{37}{216}x^3$

b $-\frac{1}{4}-\frac{1}{16}x-\frac{3}{64}x^2-\frac{5}{256}x^3\ldots$

6 $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3\ldots$; 0.949

7 $-\sec x-\sec x\tan x-\sec x\tan^2 x$
$-\sec x\tan^3 x,\ \left(|x|<\frac{\pi}{4}\right)$

Exercise 7G

1 **a** $1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\ldots$

b $x-\frac{1}{6}x^3+\frac{1}{120}x^5-\frac{1}{5040}x^7+\ldots$

c $1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+\ldots$

d $1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6+\ldots$

e $-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\ldots$

f $1+x+x^2+x^3+\ldots$

g $x-\frac{1}{3}x^3+\frac{1}{5}x^5-\frac{1}{7}x^7+\ldots$

2 **a** $1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\ldots$

b $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$

c $2x-\frac{4}{3}x^3+\frac{4}{15}x^5-\frac{8}{315}x^7+\ldots$

d $1-\frac{9}{2}x^2+\frac{27}{8}x^4-\frac{81}{80}x^6+\ldots$

e $1+x-\frac{1}{3}x^3-\frac{1}{6}x^4+\ldots$

f $x^2-\frac{1}{6}x^4+\frac{1}{120}x^6-\frac{1}{5040}x^8+\ldots$

g $1+x+\frac{1}{2}x^2-\frac{1}{8}x^4+\ldots$

h $2x+2x^2-\frac{1}{3}x^3-x^4+\ldots$

3 **a** $x^2-\frac{1}{3}x^4+\frac{2}{45}x^6-\frac{1}{315}x^8+\ldots$

b $1-x^2+\frac{1}{3}x^4-\frac{2}{45}x^6+\ldots$

Exercise 7H

1 **a** $1+3x+\frac{9}{2}x^2+\frac{9}{2}x^3+\ldots$

b $1-2x+2x^2-\frac{4}{3}x^3+\ldots$

c $1+x^2+\frac{1}{2}x^4+\frac{1}{6}x^6+\ldots$

d $3x-\frac{9}{2}x^3+\frac{81}{40}x^5-\frac{243}{560}x^7+\ldots$

e $\frac{1}{2}x-\frac{1}{48}x^3+\frac{1}{3840}x^5$
$-\frac{1}{645120}x^7+\ldots$

f $1-\frac{1}{2}x^4+\frac{1}{24}x^8-\frac{1}{720}x^{12}+\ldots$

2 $1-x^2+\frac{1}{3}x^4-\frac{2}{45}x^6+\ldots$

3 **a** $1+2x+2x^2-2x^4+\ldots$

b $2x+4x^2+\frac{8}{3}x^3+\ldots$

c $4+4x+3x^2+\frac{5}{3}x^3+\frac{3}{4}x^4+\ldots$

d $\ln 2+\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\ldots$

e $\ln 2-\frac{1}{4}x^2-\frac{1}{96}x^4+\ldots$

f $4x+\frac{4}{3}x^3+\ldots$

Exercise 7I

1 **a** $3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 80$

b $4 + 32 + 108 + 256 + 500 + 864 = 1764$

c $0 + 8 \times 1 + 11 \times 2 + 14 \times 3 + 17 \times 4 + 20 \times 5 + 23 \times 6 = 378$

d $\frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} = \frac{28009}{45045}$

$= 0{\cdot}622(3\text{s.f.})$

e $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = \frac{63}{64}$

f $-2 + 4 - 6 + 8 - 10 + 12 - 14 + 16 = 8$

g $26 + 29 + 32 + 35 = 122$

h $3 \times 4 + 5 \times 7 + 7 \times 10 + 9 \times 13 + 11 \times 16 = 410$

2 **a** $\sum_{r=1}^{n} 2r + 1$ **b** $\sum_{r=1}^{n} 3r^3$

c $\sum_{r=1}^{n} \frac{1}{2^r}$ **d** $\sum_{r=1}^{20} r^2$

e $\sum_{r=1}^{14} 56 - 4r$ **f** $\sum_{r=1}^{24} (r + 1)$

g $\sum_{r=1}^{18} r^2(r + 2)$ **h** $\sum_{r=1}^{10} (-1)^r\, 3r$

3 **a** 16 **b** 9 **c** 20

Exercise 7J

1 **a** $\frac{3n}{2}(n + 1)$ **b** $2n$

c $\frac{1}{2}n(5n + 17)$ **d** $n - 2n^2$

e $\frac{1}{4}n(29 + n)$

2 **a** $\frac{1}{2}n(3n + 5)$; 506

b $n + 2n^2$; 559

c $n^2 - 2n$; 9477

d $\frac{1}{2}n(11 - n)$; -42

e $p^2 + 2p - 24$

3 **a** $n^3 + 3n^2 + 3n$

b $n^2 + 4n$

c $\cos(n + 1) - \cos 1$

d $8n^3 + 24n^2 + 24n - 19$

e $-\frac{n}{n + 1}$

4 $\sum_{r=1}^{n} 2r + 1 = \sum_{r=1}^{n} (n + 1)^2 - n^2$

$= (n + 1)^2 - 1^2$

$= n^2 + 2n$

$= n(n + 2)$

5, 6 Similar steps to Q4

Exercise 7K

1 **a** $\frac{5}{6}n(n + 1)(2n + 1)$

b $\frac{1}{2}n(2n^2 + 3n + 5)$

c $\frac{1}{3}n(n^2 + 6n + 11)$

d $\frac{1}{2}n(2n^2 + n - 3)$

e $-\frac{1}{3}n(5n^2 + 12n + 1)$

f $\frac{1}{6}n(4n^2 - 9n - 31)$

2 **a** $\frac{n^2}{2}(n + 1)^2$

b $\frac{n^2}{2}(n + 1)^2 + 3n$

c $\frac{n}{4}(n + 1)^2(n + 4)$

d $\frac{n}{12}(n + 1)(3n^2 + 7n + 2)$

e $\frac{n}{4}(n^3 + 10n^2 + 37n + 60)$

3 **a** 1155 **b** 450
c 852 **d** 98
e 2638 **f** 13 920

4 **a** 1420 **b** 2210
c 12 150 **d** 1 588 440

5 $\frac{1}{3}n(n+1)(n+2)$; 8120

6 $\frac{1}{4}n(n+1)(n+2)(n+3)$; 53 130

Chapter review

1 **a** $u_n = 3n + 1$ **b** 61
c 111 **d** 167th term

2 **a** 510 **b** 1390
c 10 **d** $a = -15$; $d = 6$

3 **a** 512 **b** $a = 3$; $r = 4$

4 **a** 14 348 906
b 5115
c 16

5 **a** $r = \frac{1}{2}$ so S_∞ exists since $|r| < 1$; 64
b 125

6 **a** $1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6$
b $1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3$
c $3x + 6x^2 + \frac{3}{2}x^3 - 5x^4$
d $-2x - \frac{2}{3}x^3 - \frac{2}{5}x^5 - \frac{2}{7}x^7$

7 **a** 50 **b** 6455
c 37 170 **d** $\frac{8}{9}$

8 **a** $\frac{1}{3}n(n+1)(n+2)$
b $\frac{1}{6}n(n+1)(n+2)(3n-1)$

Chapter 8 Matrices

Exercise 8A

1 **a** 4×2 **b** 1×3
c 3×3 **d** 3×1

2 **a** $x = 4, y = 3$ **b** $x = 3, y = -1$
c $x = 2, y = -3$ **d** $x = -2, y = 5$

3 **a** $\begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & 2 & 5 \\ 6 & 7 & 1 \end{pmatrix}$

c $\begin{pmatrix} 2 & -5 \\ 4 & 0 \\ 1 & 1 \\ -2 & -3 \end{pmatrix}$

4 **a** $\begin{pmatrix} 25 & 12 \\ -11 & 9 \end{pmatrix}$ **b** $\begin{pmatrix} -25 & 16 \\ 19 & -8 \end{pmatrix}$

c $\begin{pmatrix} 2 & 20 \\ -6 & -3 \end{pmatrix}$

5 **a** **i** $\begin{pmatrix} 8 & 5 \\ -3 & 3 \end{pmatrix}$ **ii** $\begin{pmatrix} 8 & 5 \\ -3 & 3 \end{pmatrix}$

b **i** $\begin{pmatrix} 8 & -3 \\ 5 & 3 \end{pmatrix}$ **ii** $\begin{pmatrix} 8 & -3 \\ 5 & 3 \end{pmatrix}$

c **i** $\begin{pmatrix} -10 & 1 \\ 7 & -3 \end{pmatrix}$ **ii** $\begin{pmatrix} 10 & -1 \\ -7 & 3 \end{pmatrix}$

d $\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}$

e **i** $\begin{pmatrix} -3 & 6 \\ 9 & 0 \end{pmatrix}$ **ii** $\begin{pmatrix} -3 & 6 \\ 9 & 0 \end{pmatrix}$

f **i** $\begin{pmatrix} 5 & 13 \\ -2 & 1 \end{pmatrix}$ **ii** $\begin{pmatrix} 5 & 13 \\ -2 & 1 \end{pmatrix}$

iii $\begin{pmatrix} 5 & 13 \\ -2 & 1 \end{pmatrix}$

g **i** $\begin{pmatrix} 16 & 10 \\ -6 & 6 \end{pmatrix}$ **ii** $\begin{pmatrix} 16 & 10 \\ -6 & 6 \end{pmatrix}$

6 **a** $\begin{pmatrix} 6 + 2\sqrt{3} & \frac{29}{3} \\ -1 & 15 \end{pmatrix}$

b $\begin{pmatrix} 1 - \frac{\sqrt{3}}{3} & \frac{25}{18} \\ \frac{7}{6} & \frac{5}{2} \end{pmatrix}$

c $\begin{pmatrix} \frac{3\sqrt{3}}{5} - 4 & \frac{-29}{5} \\ \frac{-16}{5} & -10 \end{pmatrix}$

7 Proof (student's own answers)

8 $x = 2$

9 $a = 5, b = 2$

10 $s = -2, t = -2$

Exercise 8B

1 **a** (37) **b** (7)

c $(3x + 2y + 5z)$ **d** $\begin{pmatrix} 10 \\ 0 \end{pmatrix}$

e $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ **f** $\begin{pmatrix} 5 & 5 \\ 5 & -5 \end{pmatrix}$

g $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$ **h** $\begin{pmatrix} 4 & 9 \\ 1 & 3 \end{pmatrix}$

i $\begin{pmatrix} -5 \\ -10 \\ 3 \end{pmatrix}$

j $\begin{pmatrix} \frac{10 + 4\sqrt{3}}{5} & \frac{13}{6} \\ \frac{15\sqrt{5} - 4}{5} & \frac{2\sqrt{5} - \sqrt{3}}{2} \end{pmatrix}$

k $\begin{pmatrix} \sin 2\theta \\ 1 \end{pmatrix}$ **l** $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

2 **a** **i** $\begin{pmatrix} 8 & 9 \\ 12 & 11 \end{pmatrix}$

ii $\begin{pmatrix} 14 & 6 \\ 15 & 5 \end{pmatrix}$ $AB \neq BA$

b **i** $\begin{pmatrix} 11 \\ 9 \end{pmatrix}$

ii $\begin{pmatrix} 11 \\ 9 \end{pmatrix}$ $A(BC) = (AB)C$

c **i** $\begin{pmatrix} 8 & 12 \\ 9 & 11 \end{pmatrix}$

ii $\begin{pmatrix} 8 & 12 \\ 9 & 11 \end{pmatrix}$ $(AB)' = B'A'$

3 **a** $\begin{pmatrix} 7 & -4 \\ 8 & -1 \end{pmatrix}$ **b** $\begin{pmatrix} 13 & -11 \\ 22 & -9 \end{pmatrix}$

4 **a** $x = 4, y = -4$ **b** $x = 1, y = -2$

5 **a** $\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 3 & -2 \\ 5 & 1 \end{pmatrix}$

c $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$ **d** $\begin{pmatrix} p & q \\ r & s \end{pmatrix}$

6 **a** Proof (student's own answer)

b $A^3 = 4A - 15I$

7 **a** $A^2 = 5A - 6I$ **b** $A^4 = 65A - 114I$

8 $AB = \begin{pmatrix} -1 & 6 & 3 \\ 8 & 0 & 4 \\ -1 & 10 & 5 \end{pmatrix}$

$BA = \begin{pmatrix} -1 & 4 & 3 \\ 12 & 0 & 7 \\ 3 & 4 & 5 \end{pmatrix}$ $AB \neq BA$

9 **a** $\begin{pmatrix} 4 + 3x & 6x \\ 18 & 3x + 16 \end{pmatrix}$

b $x = -1, \; y = 8$

10 $x = 1$ or $x = -1$

11 $x = 2$ or $x = -\frac{1}{2}$

Exercise 8C

1 **a** 11 **b** –10 **c** –2 **d** 0

2 **a** $\det(AB) = \det A \det B = -77$

b $\det(BA) = \det(AB) = -77$

c $\det A' = \det A = 11$

3 **a** –4 **b** –2 **c** –48

4 **a** $\det(AB) = \det A \det B = -540$

b $\det(BA) = \det(AB) = -540$

c $\det A' = \det A = 27$

Exercise 8D

1 **a** $\begin{pmatrix} \frac{3}{10} & \frac{2}{5} \\ \frac{-1}{10} & \frac{1}{5} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{7}{62} & \frac{-1}{31} \\ \frac{3}{62} & \frac{4}{31} \end{pmatrix}$

c $\begin{pmatrix} \frac{1}{17} & \frac{2}{17} \\ \frac{3}{17} & \frac{-11}{17} \end{pmatrix}$ **d** singular matrix

e $\begin{pmatrix} \frac{7}{43} & \frac{-4}{43} \\ \frac{2}{43} & \frac{5}{43} \end{pmatrix}$ **f** $\begin{pmatrix} \frac{-8}{33} & \frac{1}{33} \\ \frac{3}{11} & \frac{1}{11} \end{pmatrix}$

2 **a** $\begin{pmatrix} \frac{4}{11} & \frac{1}{11} \\ \frac{-3}{11} & \frac{2}{11} \end{pmatrix}$ **b** $\begin{pmatrix} \frac{-1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{1}{7} \end{pmatrix}$

c $\begin{pmatrix} \frac{-13}{77} & \frac{5}{77} \\ \frac{5}{77} & \frac{4}{77} \end{pmatrix}$ **d** $\begin{pmatrix} \frac{-13}{77} & \frac{5}{77} \\ \frac{5}{77} & \frac{4}{77} \end{pmatrix}$

e $\begin{pmatrix} \frac{-2}{77} & \frac{13}{77} \\ \frac{1}{11} & \frac{-1}{11} \end{pmatrix}$ **f** $\begin{pmatrix} \frac{-2}{77} & \frac{13}{77} \\ \frac{1}{11} & \frac{-1}{11} \end{pmatrix}$

3 Proof (student's own answers)

4 **a** $x = 4,\ y = 2$ **b** $x = -1,\ y = 3$

c $x = -\frac{1}{2},\ y = 2$

5 **a** $\frac{1}{3t+12}\begin{pmatrix} 3 & -2 \\ 6 & t \end{pmatrix} t = -4$

b $\frac{1}{5-6t}\begin{pmatrix} 5 & -2t \\ -3 & 1 \end{pmatrix} t = \frac{5}{6}$

c $\frac{1}{2t^2-32}\begin{pmatrix} t & -4 \\ -8 & 2t \end{pmatrix} t = \pm 4$

d $\frac{1}{t^2+t-20}\begin{pmatrix} t-2 & -2 \\ -7 & t+3 \end{pmatrix} t = -5, t = 4$

6 **a**, **b** Proof (student's own answers)

c $A^4 = 105A - 46I$

7 **a** $x = \pm 6$

b $A^2 = 12A,\ A^4 = 1728A$

8 **a** $m = -1,\ n = 2$

b $A^{-1} = \frac{1}{2}A + \frac{1}{2}I$

9 $A^2 = \begin{pmatrix} p^2 & 0 \\ 1-p^2 & 1 \end{pmatrix}$

$A^3 = \begin{pmatrix} p^3 & 0 \\ 1-p^3 & 1 \end{pmatrix}$

$A^n = \begin{pmatrix} p^n & 0 \\ 1-p^n & 1 \end{pmatrix}$

Exercise 8E

1 **a** $\begin{pmatrix} \frac{1}{3} & \frac{4}{21} & \frac{2}{7} \\ 0 & \frac{1}{7} & \frac{-2}{7} \\ \frac{-1}{3} & \frac{2}{21} & \frac{1}{7} \end{pmatrix}$

b $\begin{pmatrix} \frac{-6}{5} & -1 & \frac{8}{5} \\ \frac{-3}{5} & -1 & \frac{4}{5} \\ 1 & 1 & -1 \end{pmatrix}$

c $\begin{pmatrix} \frac{-7}{5} & \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{-1}{5} & \frac{1}{5} \\ \frac{11}{5} & \frac{-1}{5} & \frac{-4}{5} \end{pmatrix}$

d singular matrix

e $\begin{pmatrix} \frac{6}{13} & \frac{-2}{13} & \frac{-5}{13} \\ \frac{-5}{13} & \frac{6}{13} & \frac{2}{13} \\ \frac{3}{13} & \frac{-1}{13} & \frac{4}{13} \end{pmatrix}$

f $\begin{pmatrix} 2 & -1 & -1 \\ -7 & 3 & 5 \\ 3 & -1 & -2 \end{pmatrix}$

2 $k = \frac{-15}{6}$

3 **a** $AB = 8I$

b $x = 5,\ y = -2,\ z = 3$

4 $\begin{pmatrix} \frac{1}{4} & \frac{-1}{4} & \frac{1}{4} \\ \frac{1}{20} & \frac{7}{20} & \frac{-3}{20} \\ \frac{-7}{20} & \frac{11}{20} & \frac{1}{20} \end{pmatrix} x = 1,\ y = -2,\ z = 3$

Exercise 8F

1 $A'(12, -3), B'(19, -2), C'(6, 3), D'(4, -5)$

2 **a** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

c $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ **d** $\begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

3 **a** $\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$

b $\begin{pmatrix} 0 & -0\cdot5 \\ -0\cdot5 & 0 \end{pmatrix}$

c $\begin{pmatrix} 2 & 2\sqrt{3} \\ -2\sqrt{3} & 2 \end{pmatrix}$ **d** $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

4 Proof (student's own answer)

5 $k = -\sqrt{3}$

6 **a** 1 and 30° **b** (0, 1) **c** 60°

d $\begin{pmatrix} 0\cdot5 & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & 0\cdot5 \end{pmatrix}$

Exercise 8G

1 **a** $x = 1\ y = 2\ z = -3$

b $x = -2\ y = 3\ z = 4$

c $x = 3\ y = -2\ z = 1$

d $x = 5\ y = 0\ z = -2$

e $x = \frac{1}{2},\ y = \frac{-3}{2},\ z = 4$

f $x = \frac{5}{2},\ y = \frac{-1}{2},\ z = 3$

2 $y = 2x^2 - 3x + 5$

3 $z = z \quad y = \frac{11}{7} + \frac{5}{7}z \quad x = \frac{-z}{7} - \frac{5}{7}$

4 $a = 4$

5 $a = 2$

6 **i** $s = -2,\ t \neq -11$

ii $s = -2,\ t = -11$

iii $s \neq 3$

7 $x = \frac{27}{2} \quad y = \frac{-161}{20}$ The system is ill-conditioned as a small change in coefficients give a very big change in solution.

8 B and C

Chapter review

1 **a** $\begin{pmatrix} 7 - 2\sqrt{2} & 4 \\ -13 & -1 \end{pmatrix}$

b $\begin{pmatrix} 2\sqrt{2} + 4 & 5 \\ -3\sqrt{2} & -3 \end{pmatrix}$

c $\begin{pmatrix} 6 & \sqrt{2} + 3 \\ 4\sqrt{2} + 12 & 13 \end{pmatrix}$

d $\begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & -4 \\ -2 & 1 & 3 \end{pmatrix}$

e -7 **f** 42

g $\frac{1}{3\sqrt{2} - 4}\begin{pmatrix} 3 & -1 \\ -4 & \sqrt{2} \end{pmatrix}$

h $\begin{pmatrix} -3 & 50 & 83 \\ 5 & 17 & 33 \\ 10 & 61 & 36 \end{pmatrix}$

i $\begin{pmatrix} \frac{-7}{9} & \frac{-5}{9} & \frac{11}{9} \\ \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{-2}{9} & \frac{-4}{9} & \frac{7}{9} \end{pmatrix}$

2 $M = \begin{pmatrix} \frac{-4}{3} & -1 \\ \sqrt{2} + \frac{8}{3} & 3 \end{pmatrix}$

3 **a** Proof (student's own answers)

b $P^4 = 40P - 39I$

4 **a** $p = 5$ and $q = -12$

b $x = \frac{-1}{4} \quad y = \frac{3}{4}$

5 **a** $\frac{1}{12 + 4a}\begin{pmatrix} 6 & a \\ -4 & 2 \end{pmatrix}$

b $a = -3$

6 $\begin{pmatrix} 5 & -2 \\ 2 & 1 \end{pmatrix}$ $(-1, 14)$

7 **a** $\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$

b $\begin{pmatrix} 0\cdot5 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -0\cdot5 \end{pmatrix}$

8 $z = \dfrac{-2}{\lambda - 3}, y = \dfrac{26 - 12\lambda}{7(\lambda - 3)}, x = \dfrac{6\lambda - 34}{7(\lambda - 3)}$

when $\lambda = 3$ the system is inconsistent and there are no solutions.

9 $x = \dfrac{3}{5}, y = \dfrac{1}{5}, z = 0$ when $t = -1$ the system is redundant and the general solution is $z = z, y = \dfrac{7z + 1}{5}, x = \dfrac{3 - 4z}{5}$

10 Solution to $\begin{matrix} x + y = 3 \\ x + 0\cdot99y = 2 \end{matrix}$ is $x = -97$, $y = 100$

Solution to $\begin{matrix} x + 0\cdot9y = 3 \\ x + 0\cdot99y = 2 \end{matrix}$ is

$x = 13, y = \dfrac{-100}{9}$

This shows the system is ill-conditioned as a small change in y coefficient leads to a large change in solution.

Chapter 9

Exercise 9A

1 **a** 2 : 3 : −4 **b** −1 : 8 : −5 **c** −5 : 1 : 6

2 **a** $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

b $\frac{3}{\sqrt{38}}, \frac{-5}{\sqrt{38}}, \frac{2}{\sqrt{38}}$

c $\frac{-3}{\sqrt{62}}, \frac{2}{\sqrt{62}}, \frac{7}{\sqrt{62}}$

3 Proof (student's own answers)

Exercise 9B

1 **a** $\begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix}$ **b** $\begin{pmatrix} -5 \\ -1 \\ 2 \end{pmatrix}$ **c** 0

d $4\begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ **e** $2\begin{pmatrix} 1 \\ -7 \\ 2 \end{pmatrix}$

2 $-4(\mathbf{a}\times\mathbf{b})$

3 **a** $\frac{\sqrt{257}}{2}$ square units

b $\frac{\sqrt{65}}{2}$ square units

4 **a** $\begin{pmatrix} -22 \\ -1 \\ -13 \end{pmatrix}$

b $\mathbf{u} = \pm\sqrt{654}(-22\mathbf{i} - \mathbf{j} - 13\mathbf{k})$

Exercise 9C

1 **a** −20 **b** 0 **c** 0

2 $V = 46$

3 $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 7$

Exercise 9D

1 **a** $\frac{x-3}{2} = \frac{y+2}{1} = \frac{z-4}{-3}$

b $\text{r} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

2 $\text{r} = \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

3 **a** $\frac{x+1}{1} = \frac{y-3}{2} = \frac{z-6}{5}$

b **i** No **ii** Yes **iii** Yes

4 $\text{r} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$

5 **a** $\frac{x-1}{2} = \frac{y}{7} = \frac{z-2}{-4}$

b $\frac{x-4}{-4} = \frac{y+1}{3} = \frac{z-5}{-6}$

c $\frac{x+2}{-1} = \frac{y-6}{-8} = \frac{z+1}{0}$

d $\frac{x}{2} = \frac{y}{0} = \frac{z-4}{-7}$

Exercise 9E

1 (9, 3, 0)

2 28·6° or 0·498 radians

3 **a** POI (−4, 1, −3); angle 22·5° or 0·393 radians

b do not intersect

c POI (4, 6, 1); angle 170° or 2·97 radians

4 **a** (1, 4, 7) **b** (−4, 4, −3)

c $5\sqrt{5}$

5 (−1, −6, 8); angle 10·89° or 0·19 radians

Exercise 9F

1 $x-2y+2z=-4$

2 **a** $\mathbf{r} = 2\mathbf{i} + 5\mathbf{j} + \mathbf{k} + s(-3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) + t(2\mathbf{i} - 7\mathbf{j} + 2\mathbf{k})$

b $x = 2-3s + 2t$, $y = 5 - 2s - 7t$, $z = 1 + 4s+2t$

c $x = -5$, $y = -8$, $z = 15$

3 It lies on the plane with $s = 2$ and $t = -1$

4 Proof (student's own answers)

5 **a** $x - y - 2z = -9$

b $x - 4y + 3z = 5$

6 $7x + 17y + 15z = 82$

7 $d = \frac{-5}{\sqrt{69}}$

8 **a** $d = \frac{-12}{\sqrt{65}}$

b $6x - 2y - 5z = 3$

c $d = \frac{15}{\sqrt{65}}$

Exercise 9G

1 **a** **i** (−1·5, −10, 2) **ii** 12·17°

b **i** (5, 0, 5) **ii** 47·37°

c **i** (−3, −3, −1) **ii** 20·92°

d **i** (−1, 4, 0) **ii** 33·06°

e **i** (3, 19, 12) **ii** 3·78°

f **i** (2, −8, −1) **ii** 22·51°

2 No intersection so parallel

3 t can be any value so the line lies on the plane

4 $\mathbf{n}_2 = -3\mathbf{n}_1 \Rightarrow$ planes are parallel

5 **a** **i** $\frac{x-1}{8} = \frac{y+1}{-5} = \frac{z}{-7}$

ii 71·2°

b **i** $\frac{x+2}{5} = \frac{y-1}{7} = \frac{z}{4}$

ii 72·45°

c **i** $\frac{x-3}{1} = \frac{y+2}{-2} = \frac{z}{-1}$

ii 80·73°

6 (3, 2, −1)

7 **a** $z = 4t,\ y = 1 + 7t,\ x = -2 + 5t$

b Planes 1 and 2 intersect along $x = 1 - 8t,\ y = -1 + 5t,\ z = 7t$.

Planes 1 and 3 intersect along $x = \frac{11}{14} - 8t,\ y = -\frac{13}{14} + 5t,\ z = 7t$

Planes 2 and 3 are parallel.

Chapter review

1 $\begin{pmatrix} 15 \\ 12 \\ 39 \end{pmatrix}$

2 −146

3 $\frac{x-1}{1} = \frac{y+2}{6} = \frac{z-4}{-2}$

4 $\frac{x-2}{1} = \frac{y+5}{2} = \frac{z-3}{3}$

5 $52x + 23y - 7z = -57$

6 $14x - 5y + 11z = 64$

7 **a** $x = 2t + 1,\ y = -3t + 2,\ z = 5t + 5$

b $3x - y + z = -22$

c **i** (−3, 8, −5) **ii** 43·22°

8 No solutions for $a \neq 3$ and infinite solutions for $a = 3$, the line would lie on the plane.

9 **a** $x - 3y + 5z = 13$

b $\frac{x+29}{13} = \frac{y+14}{6} = z$

c 21·2°

10 **a** $b = 1$, POI (1, 5, −1)

b 68·99°

11 (1, 0, 4)

Chapter 10

Exercise 10A

1 **a** quotient = 13, remainder = 1
b $q = 53$, $r = 0$
c $q = -14$, $r = 1$
d $q = -7$, $r = 7$
e $q = 4$, $r = 3$
f $q = 25$, $r = 0$

Exercise 10B

1 **a** 2_{10} **b** 4_{10}
c 10_{10} **d** 15_{10}
e 44_{10} **f** 146_{10}

2 **a** 31_{10} **b** 59_{10}
c 182_{10} **d** 5946_{10}
e $216\,761_{10}$ **f** $263\,650_{10}$

3 **a** 19_{10} **b** 25_{10}
c 62_{10} **d** 157_{10}
e 6383_{10} **f** $15\,659_{10}$

Exercise 10C

1 **a** 10_2 **b** 111_2
c 11010_2 **d** 100101_2
e 10010111_2 **f** 10101100000_2

2 **a** 22_8 **b** 140_8
c 151_8 **d** 4512_8
e 12763_8 **f** 31311_8

3 **a** 1211000_3 **b** 110223_4
c 20243_5 **d** 10043_6
e 3600_7 **f** 1730_9

4 **a** 100_3 **b** 62_9
c 3356_7 **d** 3066_8

5 **a** **i** 1101_2
ii 011110_2
iii 101111_2

b The results should be the same for each method (after excluding any redundant zeroes from the start of the final binary number).

c Both methods give 201021_3.

d No.

e The method works when the initial base is the square of the base you are changing to.

Let x_4 be a number expressed in base 4, and write $x_4 = (a_n a_{n-1} \dots a_1 a_0)_4$, where the a_i are the digits of x_4. So each a_i is an integer between 0 and 3, inclusive.

Replacing each digit a_i by its binary conversion gives a binary expression $q_k r_k q_{k-1} r_{k-1} \dots q_1 r_1 q_0 r_0$, where q_i and r_i are the quotient and remainder, respectively, when a_i is divided by 2.

We need to show that $(q_n r n q_{n-1} r_{n-1} \dots q_1 r_1 q_0 r_0)_2 = x_2$ (where x_2 is the binary expression for x_4).

Since $a_i \leq 3$, by Euclidean division $a_i = 2q_i + r_i$ for all i, where $0 \leq q_i, r_i \leq 1$. Therefore, using base 10,

$$
\begin{aligned}
x_{10} &= 4^n(2q_n + r_n) + \dots + 4(2q_1 + r_1) \\
&\quad + (2q_0 + r_0) \\
&= 2^{2n}(2q_n + r_n) + \dots + 2^2(2q_1 + r_1) \\
&\quad + (2q_0 + r_0) \\
&= 2^{2n+1}q_n + 2^{2n}r_n + \dots + 2^{2+1}q_1 \\
&\quad + 2^2 r_1 + 2q_0 + r_0 \\
&= (q_n r_n \dots q_1 r_1 q_0 r_0)_2 \quad \text{as required.}
\end{aligned}
$$

f The method doesn't work, but will work if we use three binary digits when converting each octal digit. This works since $8 = 2^3$.

Exercise 10D

1 **a** 26_{10} **b** 194_{10}
c 130_{10} **d** 478_{10}
e 2748_{10} **f** 4095_{10}
g $19\,310_{10}$ **h** $13\,226_{10}$
i $735\,903_{10}$

2 **a** 35_{16} **b** $6D_{16}$
c $E4_{16}$ **d** 317_{16}
e $3F6_{16}$ **f** $199B_{16}$

g 2711_{16} **h** $5D66_{16}$
i $12DC5_{16}$

3 **a** 41422_5 **b** 23313_4
c 295_{16} **d** FF_{16}
e 256120_7 **f** 10101100_2

Exercise 10E

1 **a** 15 **b** 1
c 1 **d** 1
e 8 **f** 1

2 **a** 8 **b** 1
c 1 **d** 2
e 1 **f** 1

3 **a** No (gcd(1155, 2695) = 385).
b No (gcd(121, 2695) = 11).
c Yes.

4 3 is the largest integer which divides every number in the given set.

5 **a** $\frac{9}{19}$ **b** $\frac{24}{37}$
c $\frac{63}{4}$ **d** $\frac{64}{177}$
e $\frac{166}{77}$ **f** $\frac{7373}{10025}$

6 **a** The gcd is 1 in all cases.

b m divides a and m divides $a + b$, so we may write $a = rm$ and $a + b = sm$ for some integers r and s. Then $b = (a + b) - a = sm - rm = (s - r)m$, and since $s - r$ is an integer this implies that m divides b.

c Use proof by induction. When $k = 1$, $\gcd(u_1, u_2) = \gcd(1, 1) = 1$. Assume true for $k = r$, so $\gcd(u_r, u_{r+1}) = 1$. Let $d = \gcd(u_{r+1}, u_{r+2})$. We must show that $d = 1$. Clearly d divides u_{r+1}, and since $u_{r+2} = u_{r+1} + u_r$, d divides $u_{r+1} + u_r$. By part **b**, d must also divide u_r. Since d divides both u_r and u_{r+1}, d must divide $\gcd(u_r, u_{r+1})$ which equals 1 by the induction hypothesis. Thus $d = 1$.

So if the statement holds for $k = r$ it holds for $k = r + 1$, and since it holds for $k = 1$, by induction it holds for all $k \geq 1$.

d Conjecture: $\gcd(u_k, u_{k+2}) = 1$ for all $k \geq 1$. Use proof by induction.

When $k = 1$, $\gcd(u_1, u_3) = \gcd(1, 2) = 1$. Assume true for $k = r$, so $\gcd(u_r, u_{r+2}) = 1$. Let $d = \gcd(u_{r+1}, u_{r+3})$. We must show that $d = 1$. Clearly d divides u_{r+1}, and since $u_{r+3} = u_{r+2} + u_{r+1}$, d divides $u_{r+2} + u_{r+1}$. By part b, d must also divide u_{r+1}. Since d divides both u_{r+2} and u_{r+1}, d must divide $\gcd(u_{r+2}, u_{r+1}) = 1$ (by part c). Thus $d = 1$.

So if the statement holds for $k = r$ it holds for $k = r + 1$, and since it holds for $k = 1$, by induction it holds for all $k \geq 1$.

Exercise 10F

1 **a** Since m divides a, $a = rm$ for some integer r. Therefore $ka = k(rm) = (kr)m$, and since kr is an integer this shows that m divides ka.

b Since m divides both a and b, we may write $a = rm$ and $b = sm$ for integers r and s. Then $a + b = rm + sm = (r + s)m$, and since $r + s$ is an integer this shows that m divides $a + b$.

c Since m divides both a and b, we may write $a = rm$ and $b = sm$ for integers r and s. Then $a - b = rm - sm = (r - s)m$, and since $r - s$ is an integer this shows that m divides $a - b$.

Exercise 10G

1 **a** $p = -2, q = 1$ **b** $p = -1, q = 3$
c $p = 1, q = -2$ **d** $p = 1, q = 4$
e $p = -1, q = -6$ **f** $p = 5, q = -26$

2 **a** $9 = 1 \times 135 - 2 \times 63$
b $9 = -62 \times 819 + 27 \times 1881$
c $6 = -4 \times 228 + 9 \times 102$
d $20 = -3 \times 360 + 5 \times 220$

e $143 = 6 \times 1573 - 5 \times 1859$

f $25 = 344 \times 13\,225 - 251 \times 18125$

3 a Since $d = \gcd(a, b)$, d must divide both a and b. Thus we may write $a = md$ and $b = nd$ for integers m and n. Hence $c = ax + by = mdx + ndy = d(mx + ny)$. Since $mx + ny$ is an integer, this shows that d divides c.

b Since $d = \gcd(a, b)$, by the extended Euclidean algorithm there exist integers m and n such that $d = am + bn$. Suppose that $c = kd$ for some integer k. Then $c = kd = k(am + bn) = (km)a + (kn)b$, and taking $x = km$ and $y = kn$ gives the required integer solutions.

c A linear Diophantine equation $ax + by = c$ has integer solutions x and y if and only if c is a multiple of $\gcd(a, b)$.

4 a $x = 72$, $y = -46$

b $x = -12$, $y = 3$

c $x = -8$, $y = 38$

Exercise 10H

1 a gcd = 195, lcm = 2925

b gcd = 28, lcm = 901 208

c gcd = 288, lcm = 20 160

d gcd = 1, lcm = 10 658 609

e gcd = 20, lcm = 2000

f gcd = 25, lcm = 149 175

2 b i 6 **ii** 6 **iii** 6

3 Using the given roots we have $f(x) = (x + 1)(x - 2)(x - 3)(x - 6)$. If a is an integer such that $f(a) = 5$, then since $a + 1$, $a - 2$, $a - 3$ and $a - 6$ are distinct integers this implies that 5 factorises as a product of four distinct integers. However, by the Fundamental Theorem of Arithmetic, the only factorisations of 5 into distinct integers are 1×5 and $-1 \times (-5)$. This contradiction shows that there is no integer a such that $f(a) = 5$.

4 a The equation $ax + 7y = 1$ has integer solutions x and y if and only if a is coprime to 7. Since 7 is prime, a is coprime to 7 if and only if a is not a multiple of 7. There are fourteen multiples of 7 between 1 and 100. Therefore there are $100 - 14 = 86$ values of a for which the equation has integer solutions.

b Arguing as in part **a**, but excluding multiples of 2 and 3 (since $6 = 2 \times 3$), there are $100 - 67 = 33$ values of b for which the equation has integer solutions.

Chapter review

1 a 1734_9 **b** 20302_5

c 10100101111_2 **d** 2457_8

e $52F_{16}$

2 a 96_{10} **b** 595_{10}

c 3890_{10} **d** 146133_{10}

3 a 505_8 **b** 10221012_3

c 131_9 **d** 10111101_2

4 gcd(8155, 3110) = 5

5 a 1, 1, 3, 5, 11, 21, 43, 85, 171, 341

b When $k = 1$ we have $u_1 = 1$, which is odd. Assume true for $k = r$, so u_r is odd. Then $u_{r+1} = u_r + 2u_{r-1}$. u_r is odd and $2u_{r-1}$ is even, and the sum of an odd integer with an even integer is odd. Therefore u_{r+1} is odd. So if the statement holds for $k = r$ it holds for $k = r + 1$, and since it holds for $k = 1$ by induction it holds for all $k \geq 1$.

c When $k = 1$, $\gcd(u_1, u_2) = \gcd(1, 1) = 1$. Assume true for $k = r$, so $\gcd(u_r, u_{r+1}) = 1$. Let $d = \gcd(u_{r+1}, u_{r+2})$. We must show that $d = 1$. Clearly d divides u_{r+1}, and since $u_{r+2} = u_{r+1} + 2u_r$, d divides $u_{r+1} + 2u_r$. Hence d must also divide $2u_r$. By part b, u_{r+1} and u_{r+2} are odd, so d must be odd (since d divides both). Therefore if d divides $2u_r$ it must divide u_r. Since d divides both u_r and u_{r+1}, d must divide gcd(u_r, u_{r+1}) which equals 1 by the induction hypothesis. Thus $d = 1$.

So if the statement holds for $k = r$ it holds for $k = r + 1$, and since it holds for $k = 1$ by induction it holds for all $k \geq 1$.

d gcd(u_k, u_{k+2}) = 1 for $1 \leq k \leq 10$.

When $k = 1$, gcd(u_1, u_3) = gcd(1, 3) = 1. Assume true for $k = r$, so gcd(u_r, u_{r+2}) = 1. Let d = gcd(u_{r+1}, u_{r+3}). We must show that $d = 1$. Clearly d divides u_{r+1}, and since $u_{r+3} = u_{r+2} + 2u_{r+1}$, d divides $u_{r+2} + 2u_{r+1}$. Hence d must also divide u_{r+2}. By part c, gcd(u_{r+1}, u_{r+2}) = 1, so d must divide 1, and hence $d = 1$.

So if the statement holds for $k = r$ it holds for $k = r + 1$, and since it holds for $k = 1$ by induction it holds for all $k \geq 1$.

e No. As a counter-example, $u_3 = 3$, $u_6 = 21$, and gcd(3, 21) = 3 ≠ 1.

6 $\dfrac{499}{600}$

7 17 = 2 × 323 – 1 × 629

8 gcd(102, 796) = 2 = –39 × 102 + 5 × 796

9
a $a = 2^7$, $b = 2^2 \times 3^2 \times 11$, gcd = 4, lcm = 12 672

b $a = 3 \times 5 \times 7^2$, $b = 5 \times 7 \times 11$, gcd = 35, lcm = 8085

c $a = 2^2 \times 3^3$, $b = 2 \times 3^2 \times 5$, gcd = 18, lcm = 540

d $a = 11^2 \times 13$, $b = 3^2 \times 11 \times 13$, gcd = 143, lcm = 14 157

e $a = 2^4 \times 5 \times 11$, $b = 2^3 \times 11 \times 17$, gcd = 88, lcm = 14 960

f $a = 5^3 \times 17$, $b = 5^3 \times 19$, gcd = 125, lcm = 40 375

Chapter 11

Exercise 11A

1 a Existential b Universal
c Universal d Existential
e Universal

2 a $\exists\, a \in \mathbb{Z} : a > 5$
b $\forall\, a \in \mathbb{Z},\ a > 5$
c $\forall\, x \in \mathbb{R},\ x - 0.3 < x$
d $\exists\, A \in M_2(\mathbb{R}) : \det(A) = 3$
e $\exists\, q \in \mathbb{Q} : 4q = 3$

Exercise 11B

1 a $x > 15 \Rightarrow x > 4$
b $a = 8 \Rightarrow ab = 16$
c $n \in \mathbb{N} \Rightarrow n \in \mathbb{Z}$
d $f(x) = 4x^2 \Rightarrow f'(x) = 8x$
e $c = 4 \Rightarrow c^2 = 16$
f $y = 3 \Leftrightarrow y^3 = 27$
g $\varphi = \pi \Rightarrow \cos\varphi = 1$
h $\varphi = 0 \Rightarrow \tan\varphi = 0$

2 a $P \Rightarrow Q$
b $P \Rightarrow Q$
c $P \Leftrightarrow Q$
d No implications.

3 a $P \Leftarrow Q$ b $P \Rightarrow Q$
c $P \Leftarrow Q$ d $P \Leftrightarrow Q$
e $P \Rightarrow Q$

4 a No implications
b $Q \Rightarrow P$
c $P \Leftrightarrow Q$
d $P \Rightarrow Q$

5 $Q \Rightarrow P$

6 a If a is an integer, then $a = 5$
b For a 3×3 matrix A, if A is invertible then $\det A \neq 0$.
c 3 divides y if 3 divides x.
d 2 divides y only if 2 divides x.
e If B is a square matrix, then B^{-1} exists.
f $f'(x) = 0$ if sufficient for $f(x) = 3$.
g $a = 0$ implies $ax = 0$ for all $x \in \mathbb{R}$.
h There exists a rational number which divides 5 only if there exists an integer which divides 5.

7 a David has travelled to Europe if David has travelled to Italy.
b The food being fruit is necessary for the food to be a banana.
c Laura plays badminton only if Laura plays a sport.
d Hameed having a son is sufficient for Hameed to be a father.
e Brian having a job implies Brian works for the local council.

Exercise 11C

1 a $x \geqslant 3$
b $y < 2.743$
c $x^2 - 2x + 4 = 6$
d $\exists\, z \in S : z \geqslant 3$
e $\forall\, r \in \mathbb{Q},\ rt \neq \frac{1}{2}$
f $\exists\, A \in M_2(\mathbb{R}) : A = 0$
g $\forall\, x \in \mathbb{C}, \exists\, y \in \mathbb{C} : \frac{y}{x} \neq y$
h $\exists\, p \in \mathbb{Z} : \forall\, q \in \mathbb{Z},\ pq \neq 30$

2 a $x \geqslant 4$ or $x \leqslant 2$
b $x \geqslant 3$ and $x \leqslant 5$
c $ab \neq 1$ and $a \neq 0$
d $\exists\, p \in \mathbb{Q} : pq \neq 0$ or $r \leqslant q$
e $x \geqslant 3.165$ or ($x \leqslant 1.212$ and $x \neq 0$)
f (y is rational or $y \leqslant \pi$) and (y is rational or $y \geqslant -\pi$)

Exercise 11D

1 a $0^2 = 0$ which is not positive
b $0{\cdot}5^2 = 0{\cdot}25 < 0{\cdot}5$
c $2 \times (-1) = -2 < -1$
d There are no real numbers y such that $0y = 1$.

2 **a** $n = 11$ $(11 - 11^2 + 100 = -10)$

b $x = 6$ $(6^3 - 6^2 - 10 \times 6 = 120)$

3 The function $f(x) = x^2 - x + 5$ is an example with no real roots.

4 Let $g(x) = x^2$. Then $g(1) = 1^2 = (-1)^2 = g(-1)$, but $1 \neq -1$.

5 The matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is non-zero but not invertible.

Exercise 11E

1 When $n = 1$ we have $1 = 1^2$ so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$1 + 3 + \ldots + (2k - 1) + (2(k + 1) - 1)$
$= k^2 + (2(k + 1) - 1)$ (by ind hyp)

$= k^2 + 2k + 1$

$= (k + 1)^2$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

2 When $n = 3$ we have $3^3 = 27 > 12 = 4 \times 3$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$3^{k+1} = 3 \times 3^k$

$> 3 \times 4k$ (by ind hyp)

$= 4 \times 3k$

$> 4(k + 1)$ (since $3k > k + 1$ for all $k \geqslant 1$),

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all integers greater than 2.

3 When $n = 1$ we have $3 \times 1^5 + 7 \times 1 = 10 = 2 \times 5$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$3(k + 1)^5 + 7(k + 1) = 3(k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1) + 7k + 7$

$= (3k^5 + 7k) + 15k^4 + 30k^3 + 30k^2 + 15k + 10$

The bracketed term is a multiple of 5 by the inductive hypothesis, and all the other terms in the sum are clearly multiples of 5. Therefore the whole right-hand side is a multiple of 5, and the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

4 When $n = 1$ we have $3^3 + 2^0 = 28 = 4 \times 7$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$3^{2(k+1)+1} + 2^{(k+1)-1}$

$= 3^{2k+3} + 2^k$

$= 3^2(3^{2k+1} + 2^{k-1}) - 3^2 2^{k-1} + 2^k$

$= 9(3^{2k+1} + 2^{k-1}) - 2^{k-1}(9 - 2)$

$= 9(3^{2k+1} + 2^{k-1}) - 2^{k-1} \times 7$

Since $3^{2k+1} + 2^{k-1}$ is a multiple of 7 by the inductive hypothesis, the whole right-hand side is a multiple of 7, and the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

5 When $n = 1$ we have $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k$

$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ (by ind hyp)

$= \begin{pmatrix} 1+0 & k+1 \\ 0+0 & 0+1 \end{pmatrix}$

$= \begin{pmatrix} 1 & k+1 \\ 0 & 1 \end{pmatrix}$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

6 When $n = 1$ we have $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 2^1 & a(2^1 - 1) \\ 0 & 1 \end{pmatrix}$, so the result holds.

Assume true for $n = k$. When $n = k + 1$ we have:

$$\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}^k$$

$$= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2^k & a(2^k - 1) \\ 0 & 1 \end{pmatrix} \text{ (by ind hyp)}$$

$$= \begin{pmatrix} 2 \times 2^k + a \times 0 & 2 \times a(2^k - 1) + a \times 1 \\ 0 \times 2^k + 1 \times 0 & 0 \times a(2^k - 1) + 1 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a(2(2^k - 1) + 1) \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{k+1} & a(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

7 When $n = 1$ we have $(\cos\theta + i\sin\theta)^1 = \cos(1 \times \theta) + i\sin(1 \times \theta)$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$(\cos\theta + i\sin\theta)^{k+1}$
$= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k$
$= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$ (by ind hyp)
$= \cos\theta\cos k\theta - \sin\theta\sin k\theta + i(\cos\theta\sin k\theta + \sin\theta\cos k\theta)$
$= \cos((k + 1)\theta) + i\sin((k + 1)\theta)$ (using multiple angle identities for cos and sin),

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

8 a When $n = 1$ we have $(e^x)^1 = e^x = e^{1x}$ so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$(e^x)^{k+1} = e^x(e^x)^k$
$= e^x e^{kx}$ (by ind hyp)
$= e^{x+kx}$
$= e^{(k+1)x}$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

b When $n = 1$ we have $1\ln x = \ln x = \ln x^1$ so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$(k + 1)\ln x = k\ln x + \ln x$
$= \ln x^k + \ln x$ (by ind hyp)
$= \ln x^k x$
$= \ln x^{k+1}$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

9 When $n = 1$ we have $a_1 = 2a_0 + 1 = 2 \times 0 + 1 = 1 = 2^1 - 1$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$a_{k+1} = 2a_k + 1$
$= 2(2^k - 1) + 1$ (by ind hyp)
$= 2^{k+1} - 2 + 1$
$= 2^{k+1} - 1$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

10 When $n = 1$ we have $u_1 = \dfrac{u_0}{(u_0 + 1)} = \dfrac{1}{(1 + 1)} = \dfrac{1}{2} = \dfrac{1}{(1 + 1)}$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$$u_{k+1} = \frac{u_k}{(u_k + 1)}$$

$$= \frac{\frac{1}{(k + 1)}}{\frac{1}{(k + 1)} + 1}$$

$$= \frac{1}{(1+k+1)}$$

$$= \frac{1}{((k+1)+1)}$$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

11 a $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

b Conjecture: $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{(n+1)}$ for all $n \in \mathbb{N}$.

When $n = 1$ the result holds. Assume true for $n = m$. When $n = m + 1$ we have:

$$\sum_{k=1}^{m+1} \frac{1}{k(k+1)} = \sum_{k=1}^{m} \frac{1}{k(k+1)} + \frac{1}{(m+1)(m+2)}$$

$$= \frac{m}{m+1} + \frac{1}{(m+1)(m+2)}$$

(by ind hyp)

$$= \frac{m(m+2)+1}{(m+1)(m+2)}$$

$$= \frac{m^2+2m+1}{(m+1)(m+2)}$$

$$= \frac{(m+1)(m+1)}{(m+1)(m+2)}$$

$$= \frac{m+1}{m+2}$$

$$= \frac{m+1}{(m+1)+1}$$

so the result holds for $n = m + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

12 a $0, \frac{1}{2}, \frac{5}{6}, \frac{23}{24}, \frac{119}{120}$

b Conjecture: $\sum_{k=1}^{n} \frac{k-1}{k!} = \frac{(n!-1)}{n!}$ for all $n \in \mathbb{N}$.

When $n = 1$ the result holds. Assume true for $n = m$. When $n = m + 1$ we have:

$$\sum_{k=1}^{m+1} \frac{k-1}{k!} = \sum_{k=1}^{m} \frac{k-1}{k!} + \frac{(m+1)-1}{(m+1)!}$$

$$= \frac{m!-1}{m!} + \frac{m}{(m+1)!} \text{ (by ind hyp)}$$

$$= \frac{(m!-1)(m+1)}{(m+1)!} + \frac{m}{(m+1)!}$$

$$= \frac{(m+1)! - (m+1) + m}{(m+1)!}$$

$$= \frac{(m+1)!-1}{(m+1)!}$$

so the result holds for $n = m + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

13 a When $n = 1$ the left-hand side is $1 \times (3 \times 1 - 1) = 2$, while the right-hand side is $1^2 \times (1 + 1) = 2$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$$\sum_{r=1}^{k+1} r(3r-1)$$

$$= \sum_{r=1}^{k} r(3r-1) + (k+1)(3(k+1)-1)$$

$$= k^2(k+1) + (k+1)(3k+3-1)$$

(by ind hyp)

$$= k^3 + 4k^2 + 5k + 2$$

$$= (k^2 + 2k + 1)(k + 2)$$

$= (k+1)^2((k+1)+1)$ so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

b When $n = 1$ the left-hand side is $4 \times 13 + 3 \times 12 + 1 = 8$, while the right-hand side is $1 \times (1 + 1)3 = 8$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$$\sum_{r=1}^{k+1} (4r^3 + 3r^2 + r) =$$

$$\sum_{r=1}^{k} (4r^3 + 3r^2 + r) + 4(k+1)^3 + 3(k+1)^2 + (k+1)$$

$= k(k+1)^3 + 4(k+1)^3 + 3(k+1)^2 + (k+1)$

(by ind hyp)

$= (k+1)(k(k+1)^2 + 4(k+1)^2 + 3(k+1) + 1)$

$= (k+1)(k^3 + 6k^2 + 12k + 8)$

$= (k+1)(k+2)^3 = (k+1)((k+1)+1)^3$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

c When $n = 1$ the left-hand side is $(-1)1 \times 12 = -1$, while the right-hand side is $\frac{(-1) \times 1 \times (1+1)}{2}$ so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$$\sum_{r=1}^{k+1}(-1)^r r^2 = \sum_{r=1}^{k}(-1)^r r^2 + (-1)^{k+1}(k+1)^2$$

$$= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2$$

(by ind hyp)

$$= \frac{(-1)^k k(k+1)}{2} + \frac{2(-1)^{k+1}(k+1)^2}{2}$$

$$= \frac{(-1)^{k+1}(-k(k+1) + 2(k+1)^2)}{2}$$

$$= \frac{(-1)^{k+1}((k+1)(2(k+1) - k)}{2}$$

$$= \frac{(-1)^{k+1}((k+1)(k+2)}{2}$$

$$= \frac{(-1)^{k+1}((k+1)((k+1)+1)}{2}$$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

14 When $n = 1$ both sides equal 1, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have:

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$$= \left(\sum_{i=1}^{k} i\right)^2 + (k+1)^3 \text{ (by ind hyp)}$$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2 + 4(k+1))}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{2^2}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2$$

$$= \left(\sum_{i=1}^{k+1} i\right)^2$$

so the result holds for $n = k + 1$. Since it holds for $n = 1$, by induction the result holds for all $n \in \mathbb{N}$.

Exercise 11F

1 a $3 \times 0 - 4 = -4 < 9$

$3 \times 1 - 4 = -1 < 9$

$3 \times 2 - 4 = 2 < 9$

$3 \times 3 - 4 = 5 < 9$

$3 \times 4 - 4 = 8 < 9$

Therefore $3x - 4 < 9$ for $x \in \{0, 1, 2, 3, 4\}$.

b $(-2)^2 - (-2) = 6 \geqslant 0$

$(-1)^2 - (-1) = 2 \geqslant 0$

$0^2 - 0 = 0 \geqslant 0$

$1^2 - 1 = 0 \geqslant 0$

$2^2 - 2 = 2 \geqslant 0$

Therefore $x^2 - x < 9$ for $x \in \{-2, -1, 0, 1, 2\}$.

c $1^3 = 1$ which is odd

$3^3 = 27$ which is odd

$5^3 = 125$ which is odd

$7^3 = 343$ which is odd

Therefore if y is an odd integer in the set $\{1, 2, 3, 4, 5, 6, 7\}$ then y^3 is an odd integer.

d $\frac{0}{2} = \frac{0}{3} = \frac{0}{4} = 0 \leqslant 1$

$\frac{1}{2} \leqslant 1$

$\frac{1}{3} \leqslant 1$

$\frac{1}{4} \leqslant 1$

$\frac{2}{2} \leqslant 1$

$\frac{2}{3} \leqslant 1$

$\frac{2}{4} \leqslant 1$

Therefore if $z = \frac{x}{y}$ is a rational number with $x \in \{0, 1, 2\}$ and $y \in \{2, 3, 4\}$, then $z \leqslant 1$.

2 a Let $a = 2r + 1$ and $b = 2s$ for some integers r and s. Then

$a + b = (2r + 1) + 2s = 2(r + s) + 1$, so $a + b$ is odd.

b Let $a = 2r + 1$ and $b = 2s + 1$ for some integers r and s. Then

$a + b = (2r + 1) + (2s + 1) = 2r + 2s + 2 = 2(r + s + 1)$, so $a + b$ is even.

c Let $a = 2r$ and $b = 2s$ for some integers r and s. Then

$ab = (2r)(2s) = 2(2rs)$, so ab is even.

d Let $a = 2r$ and $b = 2s + 1$ for some integers r and s. Then

$ab = (2r)(2s + 1) = 2(2rs + r)$, so ab is even.

e Let $a = 2r + 1$ and $b = 2s + 1$ for some integers r and s. Then

$ab = (2r + 1)(2s + 1) = (2r)(2s) + 2r + 2s + 1 = 2(2rs + r + s) + 1$, so ab is odd.

3 a True

$a + b > a + c$

$\Leftrightarrow (a + b) - a > (a + c) - a$

$\Leftrightarrow (a - a) + b > (a - a) + c$

$\Leftrightarrow b > c$

b True

$a - b < a - c$

$\Leftrightarrow (a - b) - a < (a - c) - a$

$\Leftrightarrow (a - a) - b < (a - a) - c$

$\Leftrightarrow -b < -c$

$\Leftrightarrow b > c$

c True

Note that since a is a natural number, $a > 0$.

$ab > ac \Leftrightarrow ab - ac > 0$

$\Leftrightarrow a(b - c) > 0$

$\Leftrightarrow b - c > 0$

$\Leftrightarrow b > c$

d False. A counter-example is $a = -1$, $b = 2$, $c = 1$.

4 Let n be an odd natural number, and write $n = 2k + 1$ for some integer k. Then

$n^2 = (2k + 1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1.$

Since $2k^2 + 2k$ is a natural number, this shows that n^2 is odd.

5 Let $m, n \in \mathbb{N}$. Then $(m + n)^2 = m^2 + 2mn + n^2 > m^2 + n^2$ since $2mn$ is positive.

6 $m^2 = (3k + 2)^2$

$= 9k^2 + 6k + 4$

$= 9k^2 + 6k + 3 + 1$

$= 3(3k^2 + 2k + 1) + 1,$

so m^2 is equal to an integer which is a multiple of 3 plus 1, hence is not divisible by 3.

7 *Let* $m \in \mathbb{N}$ and consider the product $m(m + 1)(m + 2)$. Every second natural number is even, so at least one of m, $m + 1$ and $m + 2$ must be even, so divisible by 2, so the product must also be divisible by 2. Moreover, every third natural number is divisible by 3, so one of m, $m + 1$ and $m + 2$ is divisible by 3, and so the product must also be divisible by 3. If a natural number is divisible by both 2 and 3, it is divisible by $2 \times 3 = 6$.

Hence the product of any three consecutive natural numbers is divisible by 6.

8 Let r and s be consecutive odd integers, with $r < s$. Then we may write $r = 2k - 1$ and $s = 2k + 1$ for some integer k.
We have

$$s^2 - r^2 = (2k + 1)^2 - (2k - 1)^2$$
$$= 4k^2 + 4k + 1 - (4k^2 - 4k + 1)$$
$$= 8k$$

and so $s^2 - r^2$ is a multiple of 8, as required. Also, $r^2 - s^2 = -(s^2 - r^2) = -8k$ which is also a multiple of 8.

9 If $x = y$, then clearly $2f(x) = 2f(y)$. Now assume that $2f(x) = 2f(y)$. Dividing both sides of the equation by 2 we get $f(x) = f(y)$, and since f is an injective function this implies that $x = y$. Hence $2f$ is also an injective function.

10 a True. Write $n = 4k$ for some integer k. Then

$$n^2 = (4k)^2 = 16k^2$$
$$= 4(4k^2), \text{ so } n^2 \text{ is divisible by 4.}$$

b False. $2^2 = 4$ is a multiple of 4, but 2 is not a multiple of 4.

11 a True. Suppose that $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and suppose without loss of generality that a ≠ 0 (the proof is essentially choosing b, c or d to be non-zero).

Then $kA = k\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$, and since $k \neq 0$ we must have $ka \neq 0$, so kA is a non-zero matrix.

b False. Taking $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ provides a counter-example.

c False. Taking $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ provides a counter-example.

d False. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then $\det(A) = 1 \times 0 - 0 \times 0 = 0$.

e False. Taking A = $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ gives a counter-example.

12 Let $S_n = 1 + 2 + 3 + \ldots + n$. Consider $2S_n$. Adding S_n to itself but with the order of the second sum reversed, we have:

$$2S_n$$
$$= 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n$$
$$+\ n + (n-1) + (n-2) + \ldots + 3 + 2 + 1.$$

Written this way, the sum consists of n columns, with the sum of the two entries in each column equal to $n + 1$. Hence $2S_n = n(n + 1)$, and so $S_n = \dfrac{n(n+1)}{2}$, as required.

Exercise 11G

1 Suppose that m is not odd, so m is even. Then we may write $m = 2k$ for some integer k. We have $m^2 = (2k)^2 = 4k^2 = 2(2k^2)$ which is even, so m^2 is not odd.

2 Suppose that x is not even, so x is odd. Then we may write $x = 2k + 1$ for some integer k. We have

$$x^2 - 2x + 4 = (2k + 1)^2 - 2(2k + 1) + 4$$
$$= 4k^2 + 4k + 1 - 2k - 2 + 4$$
$$= 4k^2 + 2k + 2 + 1$$
$$= 2(2k^2 + k + 1) + 1$$

which shows that $x^2 - 2x + 4$ is odd, so not even.

3 In each case, assume that the statement "at least one of x and y is irrational" is false, i.e. assume that both x and y are rational numbers, and write $x = \frac{m}{n}$ and $y = \frac{r}{s}$ where $m, n, r, s \in \mathbb{Z}$ with $n, r, s \neq 0$.

a $x - 2y = \frac{m}{n} - 2\left(\frac{r}{s}\right)$

$$= \frac{m}{n} - 2\frac{r}{s}$$

$$= \frac{ms}{ns} - 2\frac{nr}{ns}$$

$$= \frac{(ms - 2nr)}{ns}$$

Both $ms - 2nr \in \mathbb{Z}$ and $ns \in \mathbb{Z}$, and since $n, s \neq 0$, $ns \neq 0$, so $x - 2y$ is a rational number, so is not irrational.

b $3xy = 3\left(\frac{m}{n}\right)\left(\frac{r}{s}\right)$

$$= \frac{3mr}{ns}$$

Both $3mr \in \mathbb{Z}$ and $ns \in \mathbb{Z}$, and since $n, s \neq 0$, $ns \neq 0$, so $3xy$ is a rational number, so is not irrational.

c $\frac{x}{y} = \frac{\left(\frac{m}{n}\right)}{\left(\frac{r}{s}\right)}$

$$= \frac{ms}{nr}$$

Both $ms \in \mathbb{Z}$ and $nr \in \mathbb{Z}$, and since $n, r \neq 0$, $nr \neq 0$, so $\frac{x}{y}$ is a rational number, so is not irrational.

4 Suppose that both x and y are less than or equal to 8. Then $xy \leqslant 8 \times 8 = 64$, so xy is not greater than 64.

5 Suppose that both x and y lie outside $(-0{\cdot}5, 0{\cdot}5)$. Then $|x| > 0{\cdot}5$ and $|y| > 0{\cdot}5$, so $|xy| = |x||y| > 0{\cdot}5 \times 0{\cdot}5 = 0{\cdot}25$. Hence xy cannot lie in the interval $(-0{\cdot}25, 0{\cdot}25)$.

6 Suppose that $2y$ is not a transcendental number. Then $2y$ is algebraic, so is the root of some polynomial, say $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Therefore

$$0 = a_n(2y)^n + a_{n-1}(2y)^{n-1} + \ldots + a_1(2y) + a_0$$
$$= (2^n a_n)y^n + (2^{n-1}a_{n-1})y^{n-1} + \ldots + (2a_1)y + a_0$$

which shows that y is a root of the polynomial $(2^n a_n)x^n + (2^{n-1}a_{n-1})x^{n-1} + \ldots + (2a_1)x + a_0$. Since all the coefficients of this polynomial are integers, this shows that y is an algebraic number, so is not transcendental.

Exercise 11H

1 Assume for contradiction that m is not divisible by 3. Then we may write $m = 3k + 1$ or $m = 3k + 2$ for some integer k. We showed in Example 11.18 that if $m = 3k + 1$ then m^2 is not divisible by 3, while in Exercise 11F, Q5 we showed that if $m = 3k + 2$ then m^2 is not divisible by 3. Therefore if m is not divisible by 3, then m^2 is not divisible by 3. However, this contradicts the fact that m^2 *is* divisible by 3. Thus our initial assumption was false, and m is divisible by 3.

2 Suppose for contradiction that $\sqrt{2} + \sqrt{3}$ is not irrational, i.e. is a rational number. Then we may write $\sqrt{2} + \sqrt{3} = \frac{m}{n}$ for some integers m and n, with $n \neq 0$. We have

$$(\sqrt{2} + \sqrt{3})^2 = 2 + 2\sqrt{2}\sqrt{3} + 3$$
$$= 5 + 2\sqrt{6}$$

and also $(\sqrt{2} + \sqrt{3})^2 = \left(\frac{m}{n}\right)^2$

$$= \frac{m^2}{n^2}$$

Therefore

$$5 + 2\sqrt{6} = \frac{m^2}{n^2}$$

$$\Rightarrow 2\sqrt{6} = \frac{m^2}{n^2} - 5$$

$$\Rightarrow 2\sqrt{6} = \frac{m^2 - 5n^2}{n^2}$$

$$\Rightarrow \sqrt{6} = \frac{m^2 - 5n^2}{2n^2}$$

which implies that $\sqrt{6}$ is rational, contradicting the fact that $\sqrt{6}$ is irrational. Therefore our initial assumption was false, and $\sqrt{2} + \sqrt{3}$ is irrational.

3 Suppose for contradiction that $\sqrt{3}$ is rational, and write $\sqrt{3} = \frac{m}{n}$ where m and n are integers with no common factors. We have

$$3 = \left(\frac{m}{n}\right)^2 = \frac{m^2}{n^2}$$

and so $m^2 = 3n^2$. Therefore m^2 is divisible by 3, and we showed in Q1 that this implies m must also be divisible by 3. Hence we may write $m = 3k$ for some integer k, and we have $3n^2 = m^2 = (3k)^2 = 9k^2$, which implies that $n^2 = 3k^2$. Thus n^2 is divisible by 3, and so n is also divisible by 3. Therefore both m and n are divisible by 3. However, we assumed that m and n had no common factors – this is a contradiction. Our initial assumption must therefore be false, and $\sqrt{3}$ is irrational.

4 Assume for contradiction that $\log_2(3)$ is rational, and write $\log_2(3) = \frac{m}{n}$ for some integers m and n, with $n \neq 0$. Then

$$\log_2(3) = \frac{m}{n}$$
$$\Rightarrow 3 = 2^{\frac{m}{n}}$$
$$\Rightarrow 3^n = 2^m$$

However, the left-hand side is odd, while the right-hand side is even, which is a contradiction. Therefore $\log_2(3)$ is irrational.

5 Suppose for contradiction that $\sqrt{2} + \sqrt{5} \geqslant \sqrt{15}$. Then $(\sqrt{2} + \sqrt{5})^2 \geqslant 15$. We have $(\sqrt{2} + \sqrt{5})^2 = 2 + 2\sqrt{2}\sqrt{5} + 5 = 7 + 2\sqrt{10}$, and so $7 + 2\sqrt{10} \geqslant 15$ which implies $2\sqrt{10} \geqslant 8$ and then $\sqrt{10} \geqslant 4$. In turn this implies that $10 \geqslant 4^2 = 16$, a contradiction. Hence our initial assumption was false, and $\sqrt{2} + \sqrt{5} < \sqrt{15}$.

6 Suppose for contradiction that 3π is not transcendental, so 3π is algebraic. Then 3π is the root of some polynomial, say $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$. Therefore

$$0 = a_n(3\pi)^n + a_{n-1}(3\pi)^{n-1} + \ldots + a_1(3\pi) + a_0$$
$$= (3^n a_n)\pi^n + (3^{n-1}a_{n-1})\pi^{n-1} + \ldots + (3a_1)\pi + a_0$$

which shows that π is a root of the polynomial $(3^n a_n)x^n + (3^{n-1}a_{n-1})x^{n-1} + \ldots + (3a_1)x + a_0$. Since all the coefficients of this polynomial are integers, this shows that π is an algebraic number, contradicting the fact that π is transcendental. Hence our initial assumption was false, and 3π is transcendental.

7 Suppose for contradiction there are a finite number of primes of the form $4n - 1$, and let $p_1, p_2, \ldots, p_k$ be a list of these primes. Define $x = 4p_1p_2\ldots p_k - 1$ (so in particular x is an integer of the form $4s - 1$).

Suppose first that x is divisible by a prime of the form $4n - 1$, and without loss of generality assume this prime is p_1. Then we may write $x = p_1 y$ for some integer y, and we have $p_1 y = 4p_1p_2\ldots p_k - 1$ which rearranges to give $-1 = p_1 y - 4p_1p_2\ldots p_k = p_1(y - 4p_2p_3\ldots p_k)$. This implies that -1 is divisible by p_1, contradicting the fact that -1 is not divisible by any prime.

The other possibility is that x is only divisible by primes of the form $4n + 1$. However, when expanding a product of this form we see that

$$x = (4n_1 + 1)(4n_2 + 1)\ldots(4n_s + 1)$$
$$= 4r + 1$$

where r is some integer (this can be proved formally using induction, for example). This contradicts x being an integer of the form $4s - 1$.

Since we have arrived at a contradiction in each possible case, our initial assumption must be false and there are an infinite number of primes of the form $4n - 1$.

8 Suppose for contradiction that the person is an inhabitant of the island, so is either a Knight or a Knave.

First assume that the person is a Knight, so always tells the truth. Then the statement "I am a liar" must be true, so the Knight is a liar. This contradicts the fact that Knights always tell the truth.

Now assume the person is a Knave, so always lies. Then the statement "I am a liar" must be false, so the Knave is *not* a liar. This contradicts the fact that Knaves are liars.

Since both possibilities lead to contradictions, the original assumption must be false, and the person is not an inhabitant of the island.

9 **a** Let $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k}$ be the prime decomposition of m.

Then $m^2 = p_1^{2\alpha_1} p_2^{2\alpha_2} \ldots p_k^{2\alpha_k}$, and we see that if a prime p appears in the prime decomposition of m^2 it must also appear in the prime decomposition of m, so must divide m.

b When using proof by contrapositive we assumed m was not divisible by 2, respectively 3, and checked for each case that m^2 was not divisible by 2, respectively 3. For $p = 2$ there was one case to consider, for $p = 3$ there were two cases to consider. For an arbitrary prime p we cannot use this method, as the number of situations to consider is arbitrarily large.

c Suppose for contradiction that $\sqrt{p}$ is rational, and write $\sqrt{p} = \frac{m}{n}$ where m and n are integers with no common factors. We have

$$p = \left(\frac{m}{n}\right)^2$$

$$= \frac{m^2}{n^2}$$

and so $m^2 = pn^2$. Therefore m^2 is divisible by p, and by part a m is also divisible by p. Hence we may write $m = pk$ for some integer k, and we have $pn^2 = m^2 = (pk)^2 = p^2k^2$, which implies that $n^2 = pk^2$. Thus n^2 is divisible by p, and so n is also divisible by p. Therefore both m and n are divisible by p. However, we assumed that m and n had no common factors – this is a contradiction. Our initial assumption must therefore be false, and $\sqrt{p}$ is irrational.

d If a is a square number which divides an integer m^2, it need not be the case that a divides m. For example, 4 divides 2^2 but 4 does not divide 2.

Chapter review questions

1 **a** $\forall\, x \in \mathbb{R}, f(x) > 0$

b $\exists\, a \in \mathbb{Z} : 9a \neq 0$

c b divides 16 $\Rightarrow 2b$ divides 32

d $\frac{2}{r} \neq 0 \Leftrightarrow r \neq 0$

2 **a** $x \neq -4$

b $\exists\, x \in \mathbb{R} : x \neq -4$

c $\forall\, x \in \mathbb{R}, x \neq -4$

d $x \geqslant -1{\cdot}8$ and $x \leqslant 3{\cdot}55$

e $y \neq 0$ and $(\forall\, z \in \mathbb{Q}, yz \neq 1)$

3 **a** The domain A contains exactly k elements, and since f is injective the image of each element is distinct. Hence the range of f contains exactly k elements. Since B contains exactly k elements, it must be the case that the range of f is equal to B. Hence B is surjective.

b If f is surjective, then f is injective.

We can prove the contrapositive statement. Suppose that f is not injective. Then there exist $x, y \in$ A such that $x \neq y$ but $f(x) = f(y)$. This implies that the range of f contains fewer than k elements, so cannot equal the whole of B. Thus f is not surjective.

c If $f : A \to B$ is a function, with A and B containing exactly k elements where k is a natural number, then f being injective is equivalent to f being surjective.

4 We use proof by induction. When $n = 4$ we have $2^4 = 16 > 12 = 3 \times 4$, so the result holds. Assume true for $n = k$. When $n = k + 1$ we have

$$\begin{aligned} 2^{k+1} &= 2 \times 2^k \\ &> 2 \times 3k \text{ (by ind hyp)} \\ &= 3 \times 2k \\ &> 3(k+1) \text{ since } 2k > k + 1 \text{ for } k \geqslant 4. \end{aligned}$$

Hence the result holds for $n = k + 1$, and since it holds for $n = 4$, by induction the result holds for all $n \geqslant 4$.

5 When $n = 1$ we have $\sum_{i=1}^{1}(3i - 2) = 1 = \frac{1(3 \times 1 - 1)}{2}$, so the result holds.

Assume true for $n = k$. When $n = k + 1$ we have

$$\begin{aligned} \sum_{i=1}^{k+1}(3i-2) &= \sum_{i=1}^{k}(3i-2) + 3(k+1) - 2 \\ &= \frac{k(3k-1)}{2} + 3(k+1) - 2 \\ &\quad \text{(by ind hyp)} \\ &= \frac{k(3k-1)}{2} + \frac{6(k+1)}{2} - \frac{4}{2} \\ &= \frac{(3k^2 + 5k + 2)}{2} \\ &= \frac{(k+1)(3k+2)}{2} \\ &= \frac{(k+1)(3(k+1)-1)}{2} \end{aligned}$$

Hence the result holds for $n = k + 1$, and since it holds for $n = 1$, by induction the result holds for all natural numbers n.

6 Use proof by induction. When $n = 1$ we have $f^1(x) = f(x) = x^2 = x^{2^1}$ so the result holds. Assume true for $n = k$. When $n = k + 1$ we have

$$\begin{aligned} f^{k+1}(x) &= f(f^k(x)) \\ &= f(x^{2^k}) \text{ (by ind hyp)} \\ &= (x^{2^k})^2 \\ &= x^{2 \times 2^k} \\ &= x^{2^{k+1}} \end{aligned}$$

Hence the result holds for $n = k + 1$, and since it holds for $n = 1$, by induction the result holds for all natural numbers n.

7 a Since 5 divides x we may write $x = 5k$ for some integer k. Then $3x = 3 \times 5k = 5 \times 3k$, and so 5 divides $3x$.

b Since x is even we may write $x = 2k$ for some integer k. Then

$$\begin{aligned} x^2 + 3x - 1 &= (2k)^2 + 3(2k) - 1 \\ &= 4k^2 + 6k - 1 \\ &= 2(2k^2 + 3k) - 1 \end{aligned}$$

which shows that $x^2 + 3x - 1$ is odd.

c

$$\begin{aligned} & 2a - b > 2a - c \\ \Leftrightarrow\ & 2a - b - 2a > 2a - c - 2a \\ \Leftrightarrow\ & -b > -c \\ \Leftrightarrow\ & b < c \end{aligned}$$

8 a False. Taking $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ provides a counter-example.

b True. Using proof by contrapositive, if $A = 0$ then

$$kA = k\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} k \times 0 & k \times 0 \\ k \times 0 & k \times 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \text{ so } kA = 0.$$

9 a Assume that both x and y are rational, so we may write $x = \frac{m}{n}$ and

$y = \frac{r}{s}$ where $m, n, r, s \in \mathbb{Z}$ and n, $s \neq 0$. We have

$$x + y = 3\left(\frac{m}{n}\right) + \frac{r}{s}$$
$$= 3\left(\frac{m}{n}\right) + \frac{r}{s}$$
$$= 3\left(\frac{ms}{ns}\right) + \frac{nr}{ns}$$
$$= \frac{(3ms + nr)}{ns}$$

Both $3ms + nr \in \mathbb{Z}$ and $ns \in \mathbb{Z}$, and since $n, s \neq 0$, $ns \neq 0$, so $3x + y$ is a rational number, so is not irrational

b Assume both x and y are less than or equal to 11 (but greater than 0). Then $xy \leqslant 11 \times 11 = 121$, so xy is not greater than 121.

10 Assume for contradiction that $\sqrt{6}$ is a rational number, and write $\sqrt{6} = \frac{m}{n}$ where m and n are integers with no common factors. Then $6 = \frac{m^2}{n^2}$ which implies that $m^2 = 6n^2$. Therefore 6 divides m^2, and in particular both 2 and 3 divide m^2. By results established previously in this chapter (see Example 11.19 and Exercise 11H, Q1), this implies that both 2 and 3 divide m, and so 6 must also divide m. Writing $m = 6k$ for some integer k, we now have $6n^2 = m^2 = (6k)^2 = 36k^2$, and so $n^2 = 6k^2$ and 6 divides n^2. Therefore 6 must also divide n, contradicting the fact that m and n have no common factors. Our original assumption was therefore false, and $\sqrt{6}$ is irrational.

11 For contradiction assume that we can write $\log_4(5) = \frac{m}{n}$ where m and n are integers with $n \neq 0$. Then

$$\log_4(5) = \frac{m}{n}$$
$$\Rightarrow 5 = 4^{\frac{m}{n}}$$
$$\Rightarrow 5^n = 4^m$$

which is a contradiction as 5^n is odd but 4^m is even. Therefore $\log_4(5)$ is irrational.