

**X100/701**

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NATIONAL  
QUALIFICATIONS  
2007

TUESDAY, 15 MAY  
1.00 PM – 4.00 PM

MATHEMATICS  
ADVANCED HIGHER

**Read carefully**

1. Calculators may be used in this paper.
2. Candidates should answer **all** questions.
3. **Full credit will be given only where the solution contains appropriate working.**

Answer all the questions.

1. Express the binomial expansion of  $\left(x - \frac{2}{x}\right)^4$  in the form  $ax^4 + bx^2 + c + \frac{d}{x^2} + \frac{e}{x^4}$  for integers  $a, b, c, d$  and  $e$ .

2. Obtain the derivative of each of the following functions:

(a)  $f(x) = \exp(\sin 2x)$ ;

(b)  $y = 4^{(x^2 + 1)}$ .

3. Show that  $z = 3 + 3i$  is a root of the equation  $z^3 - 18z + 108 = 0$  and obtain the remaining roots of the equation.

4. Express  $\frac{2x^2 - 9x - 6}{x(x^2 - x - 6)}$  in partial fractions.

Given that

$$\int_4^6 \frac{2x^2 - 9x - 6}{x(x^2 - x - 6)} dx = \ln \frac{m}{n},$$

determine values for the integers  $m$  and  $n$ .

5. Matrices  $A$  and  $B$  are defined by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

(a) Find the product  $AB$ .

(b) Obtain the determinants of  $A$  and of  $AB$ .

Hence, or otherwise, obtain an expression for  $\det B$ .

6. Find the Maclaurin series for  $\cos x$  as far as the term in  $x^4$ .

Deduce the Maclaurin series for  $f(x) = \frac{1}{2}\cos 2x$  as far as the term in  $x^4$ .

Hence write down the first three non-zero terms of the series for  $f(3x)$ .

7. Use the Euclidean algorithm to find integers  $p$  and  $q$  such that  $599p + 53q = 1$ . 4
8. Obtain the general solution of the equation  $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = e^{2x}$ . 6
9. Show that  $\sum_{r=1}^n (4 - 6r) = n - 3n^2$ . 2
- Hence write down a formula for  $\sum_{r=1}^{2q} (4 - 6r)$ . 1
- Show that  $\sum_{r=q+1}^{2q} (4 - 6r) = q - 9q^2$ . 2
10. Use the substitution  $u = 1 + x^2$  to obtain  $\int_0^1 \frac{x^3}{(1+x^2)^4} dx$ . 5
- A solid is formed by rotating the curve  $y = \frac{x^{3/2}}{(1+x^2)^2}$  between  $x = 0$  and  $x = 1$  through  $360^\circ$  about the  $x$ -axis. Write down the volume of this solid. 1
11. Given that  $|z - 2| = |z + i|$ , where  $z = x + iy$ , show that  $ax + by + c = 0$  for suitable values of  $a$ ,  $b$  and  $c$ . 3
- Indicate on an Argand diagram the locus of complex numbers  $z$  which satisfy  $|z - 2| = |z + i|$ . 1
12. Prove by induction that for  $a > 0$ ,
- $$(1 + a)^n \geq 1 + na$$
- for all positive integers  $n$ . 5
13. A curve is defined by the parametric equations  $x = \cos 2t$ ,  $y = \sin 2t$ ,  $0 < t < \frac{\pi}{2}$ .
- (a) Use parametric differentiation to find  $\frac{dy}{dx}$ .  
Hence find the equation of the tangent when  $t = \frac{\pi}{8}$ . 5
- (b) Obtain an expression for  $\frac{d^2y}{dx^2}$  and hence show that  $\sin 2t \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = k$ ,  
where  $k$  is an integer. State the value of  $k$ . 5

[Turn over for Questions 14 to 16 on Page four

14. A garden centre advertises young plants to be used as hedging. After planting, the growth  $G$  metres (ie the increase in height) after  $t$  years is modelled by the differential equation

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

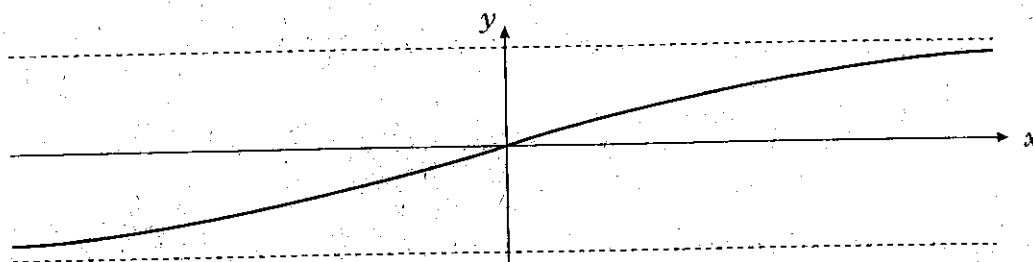
where  $k$  is a constant and  $G = 0$  when  $t = 0$ .

- (a) Express  $G$  in terms of  $t$  and  $k$ . 4
- (b) Given that a plant grows 0.6 metres by the end of 5 years, find the value of  $k$  correct to 3 decimal places. 2
- (c) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified? 2
- (d) Given that the initial height of the plants was 0.3 m, what is the likely long-term height of the plants? 2
15. Lines  $L_1$  and  $L_2$  are given by the parametric equations

$$L_1 : x = 2 + s, y = -s, z = 2 - s \quad L_2 : x = -1 - 2t, y = t, z = 2 + 3t.$$

- (a) Show that  $L_1$  and  $L_2$  do not intersect. 3
- (b) The line  $L_3$  passes through the point  $P(1, 1, 3)$  and its direction is perpendicular to the directions of both  $L_1$  and  $L_2$ . Obtain parametric equations for  $L_3$ . 3
- (c) Find the coordinates of the point  $Q$  where  $L_3$  and  $L_2$  intersect and verify that  $P$  lies on  $L_1$ . 3
- (d)  $PQ$  is the shortest distance between the lines  $L_1$  and  $L_2$ . Calculate  $PQ$ . 1

16.



- (a) The diagram shows part of the graph of  $f(x) = \tan^{-1} 2x$  and its asymptotes. State the equations of these asymptotes. 2
- (b) Use integration by parts to find the area between  $f(x)$ , the  $x$ -axis and the lines  $x = 0$ ,  $x = \frac{1}{2}$ . 5
- (c) Sketch the graph of  $y = |f(x)|$  and calculate the area between this graph, the  $x$ -axis and the lines  $x = -\frac{1}{2}$ ,  $x = \frac{1}{2}$ . 3

[END OF QUESTION PAPER]