Vectors

AH Maths Exam Questions

Source: 2019 Specimen P2 Q13 AH Maths

(1) A line, L, has equation $\frac{x+1}{2} = \frac{y-2}{1} = \frac{z}{-1}$.

(a) Find the Cartesian equation of the plane, perpendicular to the line L, which passes through the point P(1,1,0).

(b) Find the shortest distance from P to L and explain why this is the shortest distance.

Answers:

(a)		•1 find normal vector	•1 $2\mathbf{i}+\mathbf{j}-\mathbf{k}$	3
		•2 substitute into equation of the plane	•2 $2x+y-z=d$	
		•3 find the equation of plane	•3 $2x+y-z=3$	
(b)		•4 find parametric equations for the line	•4 $x=-1+2t,\ y=2+t,\ z=-t$	7
		•5 substitute into equation of plane	•5 $2(-1+2t)+(2+t)-(-t)=3$	
		•6 solve for t	•6 $\frac{1}{2}$	
		•7 calculate coordinates	•7 $\left(0,\frac{5}{2},-\frac{1}{2}\right)$	
		•8 components of PQ	•8 $\begin{pmatrix}1\\ \frac{3}{2}\\ \frac{-1}{2}\end{pmatrix}$	
		•9 find shortest distance	•9 $\sqrt{\frac{7}{2}}$	
		•10 explanation	•10 PQ is perpendicular to L.	

Source: 2019 Q15 AH Maths	
(2)	The equations of two planes are given below. $\pi_1:\ 2x-3y-z=9$ $\pi_2:\ x+y-3z=2$ (a) Verify that the line of intersection, L_1, of these two planes has parametric equations $x=2\lambda+3$ $y=\lambda-1$ $z=\lambda$ (b) Let π_3 be the plane with equation $-2x+4y+3z=4$. Calculate the acute angle between the line L_1 and the plane π_3. (c) L_2 is the line perpendicular to π_3 passing through P$(1,3,-2)$. Determine whether or not L_1 and L_2 intersect.

Answers:

(a)		•1 verify that the line lies on one plane 1	•1 eg $2(2\lambda+3)-3(\lambda-1)-\lambda=9$
		•2 verify for other plane and state conclusion 2	•2 eg $2\lambda+3+\lambda-1-3\lambda=2$; therefore the line lies on both planes

(*Other methods valid – see Marking Scheme*)

(b)		•3 identify vectors 1	•3 $\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}-2\\4\\3\end{pmatrix}$
		•4 start to calculate angle 2,3	•4 $\cos\theta=\left(\frac{3}{\sqrt{6}\sqrt{29}}\right)$
		•5 calculate complement 2,4	•5 any answer which rounds to 0·229 or 13°
(c)		•6 parametric equations for L_2 2	•6 $x=-2\mu+1;\ y=4\mu+3;\ z=3\mu-2$
		•7 two equations for two parameters	•7 any two from $2\lambda+3=-2\mu+1$; $\lambda-1=4\mu+3$; $\lambda=3\mu-2$
		•8 solve for two possible parameters 1	•8 eg $\mu=-1;\ \lambda=0$
		•9 substitute into remaining equation and state conclusion 3	•9 eg LHS $=0$, RHS $=-5$ so lines do not intersect.

Source: 2018 Q16 AH Maths

(3)

Planes π_1, π_2 and π_3 have equations:

$$\pi_1: \quad x - 2y + z = -4$$
$$\pi_2: \quad 3x - 5y - 2z = 1$$
$$\pi_3: \quad -7x + 11y + az = -11$$

where $a \in \mathbb{R}$.

(a) Use Gaussian elimination to find the value of a such that the intersection of the planes π_1, π_2 and π_3 is a line.

(b) Find the equation of the line of intersection of the planes when a takes this value.

The plane π_4 has equation $-9x + 15y + 6z = 20$.

(c) Find the acute angle between π_1 and π_4.

(d) Describe the geometrical relationship between π_2 and π_4.

Justify your answer.

Answers:

(a)		•1 set up augmented matrix	•1 $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 3 & -5 & -2 & 1 \\ -7 & 11 & a & -11 \end{array}\right]$
		•2 obtain two zeros [1]	•2 $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -5 & 13 \\ 0 & -3 & a+7 & -39 \end{array}\right]$
		•3 complete row operations [1,2]	•3 $\left[\begin{array}{ccc|c} 1 & -2 & 1 & -4 \\ 0 & 1 & -5 & 13 \\ 0 & 0 & a-8 & 0 \end{array}\right]$
		•4 obtain value for a [3]	•4 $a = 8$

(b)		•5 introduce parameter and substitute [1,2]	•5 $z = t$, $y - 5t = 13$
		•6 equation of line [1,3]	•6 $x = 22 + 9t$, $y = 13 + 5t$, $z = t$
(c)		•7 write down normals [1,4]	•7 $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \\ 2 \end{pmatrix}$ stated or implied
		•8 start to find angle	•8 $\cos\theta = \frac{-11}{\sqrt{38}\sqrt{6}}$ **OR** $\cos\theta = \frac{11}{\sqrt{38}\sqrt{6}}$
		•9 find acute angle [2,3,5]	•9 0·75
(d)		•10 explanation [1,2,3]	•10 Planes π_2 and π_4 are parallel because the normal of π_4 is a multiple of the normal of π_2.

Source: 2017 Q15 AH Maths

(4)

(a) A beam of light passes through the points B(7, 8, 1) and T(–3, –22, 6).

Obtain parametric equations of the line representing the beam of light.

(b) A sheet of metal is represented by a plane containing the points P(2, 1, 9), Q(1, 2, 7) and R(–3, 7, 1).

Find the Cartesian equation of the plane.

(c) The beam of light passes through a hole in the metal at point H.

Find the coordinates of H.

Answers:

(a)		•[1] obtain direction vector [1,2,4]	•[1] $\mathbf{d}=\begin{pmatrix}2\\6\\-1\end{pmatrix}$ or multiple thereof
		•[2] state parametric equations [3,4,5]	•[2] $x=2\lambda+7$ $y=6\lambda+8$ $z=-\lambda+1$ or $x=2\lambda-3$ $y=6\lambda-22$ $z=-\lambda+6$ Or equivalent

(b)		•[3] identify vectors	•[3] any two from $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} -5 \\ 6 \\ -8 \end{pmatrix}$, $\overrightarrow{QR} = \begin{pmatrix} -4 \\ 5 \\ -6 \end{pmatrix}$ or equivalent
		•[4] evidence of strategy for finding normal 1	•[4] $\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -2 \\ -5 & 6 & -8 \end{vmatrix}$ or equivalent
		•[5] calculate normal	•[5] $\mathbf{n} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$
		•[6] obtain equation	•[6] $4x + 2y - z = 1$

(c)		•[7] substitute into equation of plane	•[7] $4(2\lambda + 7) + 2(6\lambda + 8) - (-\lambda + 1) = 1$
		•[8] find λ	•[8] $\lambda = -2$
		•[9] determine coordinates of H 1	•[9] $\mathrm{H}(3, -4, 3)$

Source: 2016 Q14 AH Maths

(5)

Two lines L_1 and L_2 are given by the equations:

$$L_1: \quad x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = -7\lambda$$

$$L_2: \quad \frac{x-3}{-2} = \frac{y-8}{1} = \frac{z+1}{3}$$

(a) Show that the lines L_1 and L_2 intersect and find the point of intersection.

(b) Calculate the obtuse angle between the lines L_1 and L_2.

Answers:

(a)		•[1] convert any two components of L_2 to parametric form [1]	•[1] two from $x = 3 - 2\mu$, $y = 8 + \mu$, $z = -1 + 3\mu$
		•[2] two linear equations involving two distinct parameters	•[2] two from $4 + 3\lambda = 3 - 2\mu$, $2 + 4\lambda = 8 + \mu$, $-7\lambda = -1 + 3\mu$
		•[3] find parameter values	•[3] $\lambda = 1, \ \mu = -2$
		•[4] verify third component in **both** equations or equivalent	•[4] eg $z_1 = -7 \times 1$ and $z_2 = 3(-2) - 1$ therefore the lines intersect
		•[5] find point of intersection	•[5] $(7, 6, -7)$

(b)		•[6] identify first direction vector [1,2,3]	•[6] $\mathbf{d}_1 = 3\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}$
		•[7] identify second direction vector [1,2,3]	•[7] $\mathbf{d}_2 = -2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$
		•[8] calculate magnitudes and scalar product	•[8] $\lvert\mathbf{d}_1\rvert = \sqrt{74}, \quad \lvert\mathbf{d}_2\rvert = \sqrt{14}$ and $\mathbf{d}_1 \cdot \mathbf{d}_2 = -6 + 4 - 21 = -23$
		•[9] calculate obtuse angle [4,5]	•[9] $\cos^{-1}\left(\frac{-23}{\sqrt{74}\sqrt{14}}\right) \approx 135 \cdot 6°$

Source: 2015 Q15 AH Maths	
(6)	A line, L_1, passes through the point P(2, 4, 1) and is parallel to $$\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$ and a second line, L_2, passes through Q(−5, 2, 5) and is parallel to $$\mathbf{u}_2 = -4\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$ (*a*) Write down the vector equations for L_1 and L_2. (*b*) Show that the lines L_1 and L_2 intersect and find the point of intersection. (*c*) Determine the equation of the plane containing L_1 and L_2.

Answers:

a		$u_1 = i + 2j - k$ direction vector $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ $u_2 = -4i + 4j + k$ direction vector $\begin{pmatrix}-4\\4\\1\end{pmatrix}$ $v_1 = \begin{pmatrix}2\\4\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}, \quad v_2 = \begin{pmatrix}-5\\2\\5\end{pmatrix} + \mu\begin{pmatrix}-4\\4\\1\end{pmatrix}$	**2**	•[1] & •[2] for vector equations[1,2,3,6,8].

Question	Expected response	Max mark	Illustrations of evidence
b	If they intersect $2+\lambda=-5-4\mu \qquad 4\mu+\lambda=-7$ $4+2\lambda=2+4\mu \qquad \underline{4\mu-2\lambda=2}$ $1-\lambda=5+\mu \qquad \lambda=-3$ $\mu=-1$ $z_1=1-(-3) \qquad z_2=5+(-1)$ $=4 \qquad =4$ Since $z_1=z_2$, the lines intersect at $(-1,-2,4)$	4	•[3] two equations for two parameters •[4] two parameter solutions •[5] for checking third component **in both equations.** •[6] point of intersection[4,].
(c)	$u_1 \times u_2$ to get normal $\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -4 & 4 & 1 \end{vmatrix}$ or $\begin{vmatrix} i & j & k & i & j \\ 1 & 2 & -1 & 1 & 2 \\ -4 & 4 & 1 & -4 & 4 \end{vmatrix}$ $=\boldsymbol{i}(2+4)-\boldsymbol{j}(1-4)+\boldsymbol{k}(4+8)$ $=6\boldsymbol{i}+3\boldsymbol{j}+12\boldsymbol{k}$ $6x+3y+12z=\begin{pmatrix}6\\3\\12\end{pmatrix}.\begin{pmatrix}\text{Point of}\\\text{intersection}\end{pmatrix}$ $=36$ So equation of plane is $6x+3y+12z=36$ **OR** $2x+y+4z=12$	4	•[7] correct strategy to find normal. •[8] correct processing to obtain vector. •[9] substituting normal vector into an equation of a plane. May also use either of the given points. •[10] *finding correct value for constant and* correct equation.

Source: 2014 Q5 AH Maths

(7) Three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ are given by $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ where

$$\boldsymbol{u} = 5\boldsymbol{i} + 13\boldsymbol{j},\ \ \boldsymbol{v} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k},\ \ \boldsymbol{w} = \boldsymbol{i} + 4\boldsymbol{j} - \boldsymbol{k}.$$

Calculate $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w})$.

Interpret your result geometrically.

Answers:

$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$	**3**	•[1] Setting up cross product correctly.[1,4]
$= -13\boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k}$		•[2] Correctly evaluates cross product.
$\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w}) = \begin{pmatrix} 5 \\ 13 \\ 0 \end{pmatrix} . \begin{pmatrix} -13 \\ 5 \\ 7 \end{pmatrix} = 0$		•[3] Correctly evaluates dot product with $\boldsymbol{u}$ and vector from answer •[2].
$\boldsymbol{u}$ lies in the same plane as the one containing both $\boldsymbol{v}$ and $\boldsymbol{w}$. **OR** $\boldsymbol{u}$ is parallel to the plane containing $\boldsymbol{v}$ and $\boldsymbol{w}$. **OR** $\boldsymbol{u}$ is perpendicular to the normal of $\boldsymbol{v}$ and $\boldsymbol{w}$. **OR** All **4** points lie in the same plane. **OR** $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v} \times \boldsymbol{w}$. **OR** Volume of parallelepiped is zero. **OR** $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ are coplanar/linearly dependent.	**1**	•[4] Any one correct statement.[2,3,6]
OR $\boldsymbol{u}.(\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} 5 & 13 & 0 \\ 2 & 1 & 3 \\ 1 & 4 & -1 \end{vmatrix}$		•[1] Setting up combined product correctly.[4]
$= 5\begin{vmatrix} 1 & 3 \\ 4 & -1 \end{vmatrix} - 13\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix}$		•[2] Correctly processes determinant.[5]
$= 0$		•[3] Correctly evaluates determinant to reach 0.

Source: 2013 Q15 AH Maths

(8)

(*a*) Find an equation of the plane π_1, through the points $A(0, -1, 3)$, $B(1, 0, 3)$ and $C(0, 0, 5)$.

(*b*) π_2 is the plane through A with normal in the direction $-\mathbf{j} + \mathbf{k}$.

Find an equation of the plane π_2.

(*c*) Determine the acute angle between planes π_1 and π_2.

Answers:

a		$\overrightarrow{AB} = \begin{pmatrix}1\\1\\0\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}0\\1\\2\end{pmatrix}$ **OR** $\overrightarrow{BC} = \begin{pmatrix}-1\\0\\2\end{pmatrix}$		•[1] Any two correct[1] vectors.[2]
		$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix}$ or equivalent		•[2] Evidence of appropriate method.[3]
		$= 2\mathbf{i} - 2\mathbf{j} + \mathbf{k}$		•[3] Obtains vector product (any form).
		$2x - 2y + z = 2 \times 0 - 2 \times -1 + 1 \times 3$		
		$\pi_1 : \mathbf{2x - 2y + z = 5}$		•[4] Obtains constant *and* states equation of plane.
		OR $\mathrm{r} = \begin{pmatrix}0\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\2\end{pmatrix}$ or equivalent		

b		$0 \times 0 + (-1) \times (-1) + 1 \times 3 = 4$		•[5] Evidence of appropriate method.[4]
		$\pi_2 : \mathbf{-y + z = 4}$		•[6] Processes to obtain equation of second plane.

c		Normal vectors: $\boldsymbol{n}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\boldsymbol{n}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\lvert\boldsymbol{n}_1\rvert = \sqrt{9} = 3, \lvert\boldsymbol{n}_2\rvert = \sqrt{2}$ cos (angle between normals) = $\frac{\boldsymbol{n}_1.\boldsymbol{n}_2}{\lvert\boldsymbol{n}_1\rvert\lvert\boldsymbol{n}_2\rvert} = \frac{2\times 0 - 2\times -1 + 1\times 1}{3\sqrt{2}} = \frac{3}{3\sqrt{2}} = \frac{1}{\sqrt{2}}$ Angle $= 45^\circ$ acute angle between planes is $45^\circ \left(\text{or } \frac{\pi}{4}\right)$.		•[7] Obtains two correct lengths. •[8] Evidence knows how to use formula. •[9] Processes to statement of answer.[5]
		OR $= 2i - 2j + k$, so $\lvert 2i - 2j + k \rvert = 3$ and $\lvert -j + k \rvert = \sqrt{2}$ $3 = \lvert n_1 \rvert . \lvert n_2 \rvert .\cos\theta = 3\sqrt{2} .\cos\theta$ $\cos\theta = \frac{1}{\sqrt{2}}$ so $\theta = \frac{\pi}{4}$ (or 45°)		•[7] States vector *and* obtains moduli. •[8] Evidence knows how to use formula. •[9] Processes to statement of answer.

Source: 2012 Q5 AH Maths

(9) Obtain an equation for the plane passing through the points $P\,(-2, 1, -1)$, $Q\,(1, 2, 3)$ and $R\,(3, 0, 1)$.

Answer:

Method 1

$\overrightarrow{PQ} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{QR} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$	1	$\overrightarrow{PR}$ could be used
A normal to the plane:		
$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 2 & -2 & -2 \end{vmatrix}$	1M	
$= \mathbf{i}\begin{vmatrix} 1 & 4 \\ -2 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$		
$= 6\mathbf{i} + 14\mathbf{j} - 8\mathbf{k}$	1	
Hence the equation has the form:		
$6x + 14y - 8z = d.$	1	
The plane passes through $P(-2,\ 1,\ -1)$ so		
$d = -12 + 14 + 8 = 10$		
which gives an equation $6x + 14y - 8z = 10$	1	
i.e. $3x + 7y - 4z = 5.$		

Method 2

A plane has an equation of the form $ax + by + cz = d$. Using the points P, Q, R we get		
$-2a + b - c = d$ $a + 2b + 3c = d$ $3a + c = d$	1M	
Using Gaussian elimination to solve these we have		or other valid method
$\begin{vmatrix} -2 & 1 & -1 & d \\ 1 & 2 & 3 & d \\ 3 & 0 & 1 & d \end{vmatrix} \Rightarrow \begin{vmatrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \\ 0 & 6 & 8 & 2d \end{vmatrix}$	1	
$\Rightarrow \begin{vmatrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \\ 0 & 0 & 2 & -\frac{8}{5}d \end{vmatrix}$	1	
$\Rightarrow c = -\frac{4}{5}d, \quad b = \frac{7}{5}d, \quad a = \frac{3}{5}d$	1	
These give the equation		
$(\frac{3}{5}d)x + (\frac{7}{5}d)y + (-\frac{4}{5}d)z = d$		
i.e. $3x + 7y - 4z = 5$	1	

Source: 2011 Q15 AH Maths

(10) The lines L_1 and L_2 are given by the equations

$$\frac{x-1}{k} = \frac{y}{-1} = \frac{z+3}{1} \text{ and } \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2},$$

respectively.

Find:

(*a*) the value of k for which L_1 and L_2 intersect and the point of intersection;

(*b*) the acute angle between L_1 and L_2.

Answers:

(a) In terms of a parameter s, L_1 is given by $x = 1 + ks,\ y = -s,\ z = -3 + s$	**1**	
In terms of a parameter t, L_2 is given by $x = 4 + t, y = -3 + t, z = -3 + 2t$	**1**	
Equating the y coordinates and equating the z coordinates: $\left.\begin{aligned} -s &= -3 + t \\ -3 + s &= -3 + 2t \end{aligned}\right\}$	**1**	
Adding these $-3 = -6 + 3t$ $\Rightarrow t = 1 \Rightarrow s = 2.$	**1**	
From the x coordinates $1 + ks = 4 + t$ Using the values of s and t $1 + 2k = 5 \Rightarrow k = 2$	**1**	
The point of intersection is: $(5, -2, -1)$.	**1**	
(b) L_1 has direction $2\mathbf{i} - \mathbf{j} + \mathbf{k}$. L_2 has direction $\mathbf{i} + \mathbf{j} + 2\mathbf{k}$.	**1**	For both directions.
Let the angle between L_1 and L_2 be θ, then $\cos\theta = \frac{(2\mathbf{i} - \mathbf{j} + \mathbf{k}).(\mathbf{i} + \mathbf{j} + 2\mathbf{k})}{\lvert 2\mathbf{i} - \mathbf{j} + \mathbf{k}\rvert \lvert \mathbf{i} + \mathbf{j} + 2\mathbf{k}\rvert}$	**1**	
$= \frac{2 - 1 + 2}{\sqrt{6}\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$	**1**	
$\theta = 60°$	**1**	
The angle between L_1 and L_2 is 60°.		

Source: 2010 Q6 AH Maths	
(11)	Given $\mathbf{u} = -2\mathbf{i} + 5\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{w} = -\mathbf{i} + \mathbf{j} + 4\mathbf{k}$. Calculate $\mathbf{u}.(\mathbf{v} \times \mathbf{w})$.

Answer:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ -1 & 1 & 4 \end{vmatrix}$$ **1M** a valid approach

$$= \mathbf{i}\begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & -1 \\ -1 & 4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix}$$ **1**

$$= 9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k}$$ **1**

$$\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = (-2\mathbf{i} + 0\mathbf{j} + 5\mathbf{k}).(9\mathbf{i} - 11\mathbf{j} + 5\mathbf{k})$$

$$= -18 + 0 + 25$$

$$= 7.$$ **1**

Source: 2009 Q16 AH Maths

(12)

(*a*) Use Gaussian elimination to solve the following system of equations

$$\begin{aligned} x + y - z &= 6 \\ 2x - 3y + 2z &= 2 \\ -5x + 2y - 4z &= 1. \end{aligned}$$

(*b*) Show that the line of intersection, L, of the planes $x + y - z = 6$ and $2x - 3y + 2z = 2$ has parametric equations

$$\begin{aligned} x &= \lambda \\ y &= 4\lambda - 14 \\ z &= 5\lambda - 20. \end{aligned}$$

(*c*) Find the acute angle between line L and the plane $-5x + 2y - 4z = 1$.

Answers:

(a)

$$\begin{aligned} x + y - z &= 6 \\ 2x - 3y + 2z &= 2 \\ -5x + 2y - 4z &= 1 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 2 & -3 & 2 & 2 \\ -5 & 2 & -4 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 7 & -9 & 31 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & -5 & 4 & -10 \\ 0 & 0 & -\frac{17}{5} & 17 \end{array}\right)$$

1,1,1

$$z = 17 \div \left(\frac{-17}{5}\right) = -5$$ **1**

$$-5y - 20 = -10 \Rightarrow y = -2$$

$$x - 2 + 5 = 6 \Rightarrow x = 3$$ **1**

(b) Let $x = \lambda$.

Method 1

In first plane: $x + y - z = 6$.

$\lambda + (4\lambda - 14) - (5\lambda - 20) = 5\lambda - 5\lambda + 6 = 6$. **1**

In the second plane:

$2x - 3y + 2z = 2\lambda - 3(4\lambda - 14) + 2(5\lambda - 20) = 5\lambda - 5\lambda + 2 = 2$. **1**

Method 2

$$y - z = 6 - \lambda \Rightarrow y = 6 + z - \lambda$$

$$-3y + 2z = 2 - 2\lambda$$ **1**

$$(-18 - 3z + 3\lambda) + 2z = 2 - 2\lambda$$

$$-z = 20 - 5\lambda \Rightarrow z = 5\lambda - 20$$ **1**

$$\text{and } y = 4\lambda - 14$$

Method 2

$$\begin{array}{rrrrlll} x & + y & - z & = & 6 & (1) & \\ 2x & - 3y & + 2z & = & 2 & (2) & \\ 5x & & - z & = & 20 & (2) + 3(1) & \\ 4x & - y & & = & 14 & (2) + 2(1) & \mathbf{1} \end{array}$$

$$y = 4x - 14$$

$$z = 5x - 20$$

$$x = \lambda, y = 4\lambda - 14, z = 5\lambda - 20$$ **1**

(c) Direction of L is $\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$, direction of normal to the plane is $-5\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$. Letting θ be the angle between these then

$$\cos\theta = \frac{-5 + 8 - 20}{\sqrt{42}\sqrt{45}}$$ **1M,1**

$$= \frac{-17}{3\sqrt{210}}$$

This gives a value of 113.0° which leads to the angle $113.0° - 90° = 23.0°$. **1,1**

Source: 2008 Q14 AH Maths	
(13)	(*a*) Find an equation of the plane π_1 through the points $A(1, 1, 1)$, $B(2, -1, 1)$ and $C(0, 3, 3)$. (*b*) The plane π_2 has equation $x + 3y - z = 2$. Given that the point $(0, a, b)$ lies on both the planes π_1 and π_2, find the values of a and b. Hence find an equation of the line of intersection of the planes π_1 and π_2. (*c*) Find the size of the acute angle between the planes π_1 and π_2.

Answers:

(a)

$$\overrightarrow{AB} = \mathbf{i} - 2\mathbf{j} \qquad \overrightarrow{AC} = -\mathbf{i} + 2\mathbf{j} + 2\mathbf{k} \qquad \mathbf{1}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ -1 & 2 & 2 \end{vmatrix} = (-4 - 0)\mathbf{i} - (2 - 0)\mathbf{j} + (2 - 2)\mathbf{k} \qquad \mathbf{1}$$

$$= -4\mathbf{i} - 2\mathbf{j}$$

Equation is

$$-4x - 2y = k$$

$$= -4(1) - 2(1) = -6 \qquad \mathbf{1}$$

$$\text{i.e. } -2x - y = -3$$

$$2x + y = 3$$

PTO for (b) & (c)

(b) In π_1: $2 \times 0 + a = 3 \Rightarrow a = 3$. **1**

In π_2: $0 + 3a - b = 2 \Rightarrow b = 3a - 2 = 7$. **1**

Hence the point of intersection is $(0, 3, 7)$.

Line of intersection: direction from

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -2 & 0 \\ 1 & 3 & -1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} - 10\mathbf{k}$$ **1**

$$x = 0 + 2t;\ y = 3 - 4t;\ z = 7 - 10t$$ **1**

There are many valid variations on this (including symmetric form) and these were marked on their merits.

(c) Let the angle be θ, then

$$\cos\theta = \left|\frac{(-4\mathbf{i} - 2\mathbf{j}).(\mathbf{i} + 3\mathbf{j} - \mathbf{k})}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}}\right| = \left|\frac{-4 - 6}{\sqrt{20 \times 11}}\right| = \frac{5}{\sqrt{55}}$$ **1M, 1**

or

$$\sin\theta = \left|\frac{(-4\mathbf{i} - 2\mathbf{j}) \times (\mathbf{i} + 3\mathbf{j} - \mathbf{k})}{\sqrt{4^2 + 2^2}\sqrt{1^2 + 3^2 + 1^2}}\right|$$ **1M**

$$= \frac{\sqrt{2^2 + 4^2 + 10^2}}{\sqrt{20}\sqrt{11}} = \sqrt{\frac{120}{20 \times 11}} = \sqrt{\frac{6}{11}}$$ **1**

Hence $\theta \approx 47.6°$. **1**

Source: 2007 Q15 AH Maths

(14) Lines L_1 and L_2 are given by the parametric equations

$$L_1 : x = 2 + s,\ y = -s,\ z = 2 - s \qquad L_2 : x = -1 - 2t,\ y = t,\ z = 2 + 3t.$$

(*a*) Show that L_1 and L_2 do not intersect.

(*b*) The line L_3 passes through the point $P(1, 1, 3)$ and its direction is perpendicular to the directions of both L_1 and L_2. Obtain parametric equations for L_3.

(*c*) Find the coordinates of the point Q where L_3 and L_2 intersect and verify that P lies on L_1.

(*d*) PQ is the shortest distance between the lines L_1 and L_2. Calculate PQ.

Answers:

(a) Equating the x-coordinates: $2 + s = -1 - 2t \Rightarrow s + 2t = -3$ (1)
Equating the y-coordinates: $-s = t \Rightarrow s = -t$ **1**
Substituting in (1): $-t + 2t = -3 \Rightarrow t = -3 \Rightarrow s = 3$. **1**
Putting $s = 3$ in L_1 gives $(5, -3, -1)$ and $t = -3$ in L_2, $(5, -3, -7)$.
As the z coordinates differ, L_1 and L_2 do not intersect. **1**

(b) Directions of L_1 and L_2 are: $\mathbf{i} - \mathbf{j} - \mathbf{k}$ and $-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$. The vector product of these gives the direction of L_3.

$$(\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (-2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = -2\mathbf{i} - \mathbf{j} - \mathbf{k}$$ **1M,1**

Equation of L_3:

$$\mathbf{r} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} + (-2\mathbf{i} - \mathbf{j} - \mathbf{k})u$$
$$= (1 - 2u)\mathbf{i} + (1 - u)\mathbf{j} + (3 - u)\mathbf{k}$$

Hence L_3 is given by $x = 1 - 2u,\ y = 1 - u,\ z = 3 - u$. **1**

(c) Solving the x and y coordinates of L_3 and L_2:

$$-1 - 2t = 1 - 2u \text{ and } t = 1 - u$$
$$\Rightarrow -1 = 3 - 4u \Rightarrow u = 1 \text{ and } t = 0$$ **1**

The point of intersection, Q, is $(-1, 0, 2)$ since $2 + 3t = 2$ and $3 - u = 2$. **1**
L_1 is $x = 2 + s, y = -s, z = 2 - s$. When $x = 1$, $s = -1$ and hence $y = 1$ and $z = 3$, i.e. P lies on L_1. **1**

(d) $PQ = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$. **1**