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Matrices

AH Maths Exam Questions

Source: 2019 Specimen P1 Q1 AH Maths	
(1)	Matrix A is defined by $A = \begin{pmatrix} 5 & 6 \\ -1 & -2 \end{pmatrix}$ (a) Find A^{-1} (b) State A'.
Answers:	

$\bullet^1$ find determinant	$\bullet^1$ -4
$\bullet^2$ find inverse	$\bullet^2$ $-\frac{1}{4}\begin{pmatrix} -2 & -6 \\ 1 & 5 \end{pmatrix}$
$\bullet^3$ find transpose	$\bullet^3$ $\begin{pmatrix} 5 & -1 \\ 6 & -2 \end{pmatrix}$

Source: 2019 Specimen P2 Q5 AH Maths – Same as 2018 Q11	
(2)	(a) Obtain the matrix, A, associated with an anticlockwise rotation of $\frac{\pi}{3}$ radians about the origin. (b) Find the matrix, B, associated with a reflection in the x-axis. (c) Hence obtain the matrix, P, associated with an anticlockwise rotation of $\frac{\pi}{3}$ radians about the origin followed by reflection in the x-axis, expressing your answer using exact values. (d) Explain why matrix P is not associated with rotation about the origin.

Answer:

(a)		•1 obtain A	•1 $\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix}$
(b)		•2 obtain B	•2 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
(c)		•3 correct order for multiplication ($P = BA$)	•3 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$
		•4 multiplication completed and appearance of exact values	•4 $\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix}$
(d)		•5 valid explanation	•5 for example compare the elements of P with the general form of a rotation matrix

Source: 2019 Q2 AH Maths

(3)

Matrix A is defined by

$$A = \begin{pmatrix} 2 & 1 & 4 \\ -3 & p & 2 \\ -1 & -2 & 5 \end{pmatrix}$$

where $p \in \mathbb{R}$.

(a) Given that the determinant of A is 3, find the value of p.

Matrix B is defined by

$$B = \begin{pmatrix} 0 & 1 \\ q & 3 \\ 4 & 0 \end{pmatrix}$$

where $q \in \mathbb{R}$.

(b) Find AB.

(c) Explain why AB does not have an inverse.

Answers:

2.	(a)		•[1] begin process [1] •[2] find determinant [1,2] •[3] equate to 3 and find p [1]	•[1] eg $2\begin{vmatrix} p & 2 \\ -2 & 5 \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ -1 & 5 \end{vmatrix} + 4\begin{vmatrix} -3 & p \\ -1 & -2 \end{vmatrix}$ •[2] $14p+45$ •[3] -3
	(b)		•[4] any two simplified entries [1,2] •[5] complete multiplication [2]	•[4, 5] $\begin{pmatrix} q+16 & 5 \\ -3q+8 & -12 \\ -2q+20 & -7 \end{pmatrix}$
	(c)		•[6] explain [1,2]	•[6] AB is not a square matrix AND A general statement about square matrices

Source: 2018 Q7 AH Maths

(4)

Matrices C and D are given by:

$$C = \begin{pmatrix} -2 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 1 & 2 \\ k+3 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}, \text{ where } k \in \mathbb{R}.$$

(a) Obtain $2C' - D$ where C' is the transpose of C.

(b) (i) Find and simplify an expression for the determinant of D.

(ii) State the value of k such that D^{-1} does not exist.

Answers:

(a)		$\bullet^1$ state transpose of C	$\bullet^1$ $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{pmatrix}$ stated or implied by $\bullet^2$	2
		$\bullet^2$ obtain matrix	$\bullet^2$ $2C' - D = \begin{pmatrix} -5 & 1 & 0 \\ -k-1 & -2 & -2 \\ 3 & -1 & -3 \end{pmatrix}$	
(b)	(i)	$\bullet^3$ begin to find determinant	$\bullet^3$ $-(k+3)\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$	2
		$\bullet^4$ simplify expression	$\bullet^4$ $k+3$	
	(ii)	$\bullet^5$ state value of k	$\bullet^5$ -3	1

Source: 2017 Q7 AH Maths

(5)

Matrices P and Q are defined by $P = \begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix}$ and $Q = \begin{pmatrix} 2 & -3 \\ 4 & y \end{pmatrix}$, where $x, y \in \mathbb{R}$.

(a) Given the determinant of P is 2, obtain:

(i) The value of x.

(ii) P^{-1}.

(iii) $P^{-1}Q'$, where Q' is the transpose of Q.

(b) The matrix R is defined by $R = \begin{pmatrix} 5 & -2 \\ z & -6 \end{pmatrix}$, where $z \in \mathbb{R}$.

Determine the value of z such that R is singular.

Answers:

(a)	(i)	•1 determine value of x	•1 $x = 8$	1
	(ii)	•2 find inverse [1]	•2 $P^{-1} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ 5 & 8 \end{pmatrix}$	1
	(iii)	•3 state transpose •4 obtain product [2,3]	•3 $Q' = \begin{pmatrix} 2 & 4 \\ -3 & y \end{pmatrix}$ •4 $P^{-1}Q' = \begin{pmatrix} 2 & -2-y \\ -7 & 10+4y \end{pmatrix}$	2

(b)		•5 state condition for singularity [1,2] •6 obtain value for z [2]	•5 $\det R = 0$ or one row is a multiple of the other •6 $z = 15$	2

Source: 2016 Q7 AH Maths

(6) A is the matrix $\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix}$.

(a) Find the determinant of matrix A.

(b) Show that A^2 can be expressed in the form $pA + qI$, stating the values of p and q.

(c) Obtain a similar expression for A^4.

Answers:

$(a)\ Determinant\ of\ matrix\ A = -2$

(b)		**Method 1**		3
		•[2] find A^2	•[2] $A^2 = \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix}$	
		•[3] use an appropriate method	•[3] $A^2 = \begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ $A^2 = A + 2I$	
		•[4] write in required form and explicitly state values of p and q Note 1	•[4] $p = 1$ and $q = 2$	
		Method 2		
		•[2] find A^2	•[2] $A^2 = \begin{pmatrix} 4 & 0 \\ \lambda & 1 \end{pmatrix}$	
		•[3] use an appropriate method	•[3] $A^2 = p\begin{pmatrix} 2 & 0 \\ \lambda & -1 \end{pmatrix} + q\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	
		•[4] write in required form and explicitly state values of p and q Note 1	•[4] $A^2 = A + 2I$ $p = 1$ and $q = 2$	
(c)		•[5] square expression found in (b) 1,2,3	•[5] $A^4 = (A + 2I)^2$ $= A^2 + 4AI + 4I^2$ $= A + 2I + 4A + 4I$	2
		•[6] substitute for A^2 and complete process	•[6] $= 5A + 6I$	

Source: 2015 Q5 AH Maths	
(7)	Obtain the value(s) of p for which the matrix $A = \begin{pmatrix} p & 2 & 0 \\ 3 & p & 1 \\ 0 & -1 & -1 \end{pmatrix}$ is singular.

Answer:

Singular when det A = 0 $p\begin{vmatrix} p & 1 \\ -1 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 0 & -1 \end{vmatrix} + 0\begin{vmatrix} 3 & p \\ 0 & -1 \end{vmatrix} = 0$ $p(-p+1)-2(-3)=0$ $p^2-p-6=0$ $(p-3)(p+2)=0$ $p=3$ **or** $p=-2$	**4**	•[1] evidence of any valid method for obtaining det$A$ *and* setting = 0^1. •[2] expansion or equivalent method by first row •[3] for polynomial •[4] solutions (both)

Source: 2015 Q11 AH Maths

(8)

Write down the 2×2 matrix, M_1, associated with a reflection in the y-axis.

Write down a second 2×2 matrix, M_2, associated with an anticlockwise rotation through an angle of $\frac{\pi}{2}$ radians about the origin.

Find the 2×2 matrix, M_3, associated with an anticlockwise rotation through $\frac{\pi}{2}$ radians about the origin followed by a reflection in the y-axis.

What single transformation is associated with M_3?

Answers:

$M_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	4	•[1] for M_1
$M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$		•[2] for M_2.
$M_3 = M_1 M_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		•[3] for M_3.[1]
Reflection in the line $y = x$		•[4] correct interpretation[3].

Source: 2013 Q3 AH Maths

(9) Matrices A and B are defined by $A=\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}$.

(a) Find A^2.

(b) Find the value of p for which A^2 is singular.

(c) Find the values of p and x if $B = 3A'$.

Answers:

a	$A^2=\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 & p \\ -2 & 1 \end{pmatrix}=\begin{pmatrix} 16-2p & 4p+p \\ -8-2 & -2p+1 \end{pmatrix}$	•[1] Correct answer.[3,1]
	$=\begin{pmatrix} 16-2p & 5p \\ -10 & 1-2p \end{pmatrix}$	Improved alternative.
b	A^2 is singular when $\det A^2 = 0$ $(16-2p)(1-2p)+50p = 0$ $16-34p+4p^2+50p = 0$ $4p^2+16p+16 = 0$ $4(p+2)^2 = 0$	•[2] Property stated or implied.[4]
	$\mathbf{p = -2}$	•[3] Correct value of p.[5,1]
	OR A^2 is singular when A is singular, [i.e. when $\det A = 0$]	•[2] Explicitly states property. [not essential, but preferred]
	$4+2p = 0$ $\mathbf{p = -2}$	•[3] Correct value of p[1]
c	$A'=\begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix}$	•[4] A transpose (A^T) correct. Does not have to be explicitly stated.
	$\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}=3\begin{pmatrix} 4 & -2 \\ p & 1 \end{pmatrix}$	
	$\begin{pmatrix} x & -6 \\ 1 & 3 \end{pmatrix}=\begin{pmatrix} 12 & -6 \\ 3p & 3 \end{pmatrix}$ $x=12,\ p=\frac{1}{3}$	•[5] Values of p <u>and</u> x correct.[1,2]

Source: 2012 Q9 AH Maths	
(10)	A non-singular $n \times n$ matrix A satisfies the equation $A + A^{-1} = I$, where I is the $n \times n$ identity matrix. Show that $A^3 = kI$ and state the value of k.

Answers:

Method 1		
$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$		
$\int \frac{dy}{1+y} = 3\int (1+x)^{\frac{1}{2}}\,dx$	**M1**	separating variables
$\ln(1+y) = 2(1+x)^{\frac{3}{2}} + c$	1 1	for LHS for term in x
$1 + y = \exp\left(2(1+x)^{\frac{3}{2}} + c\right)$	1	for the constant
$y = \exp\left(2(1+x)^{\frac{3}{2}} + c\right) - 1.$	1	
$= A\exp\left(2(1+x)^{\frac{3}{2}}\right) - 1.$		
Method 2		
$\frac{dy}{dx} - 3\sqrt{1+x}\,y = 3\sqrt{1+x}$	1	
Integrating Factor		
$\exp\left(-3\int\sqrt{1+x}\,dx\right) = \exp\left(-2(1+x)^{3/2}\right)$	1	
$\frac{d}{dx}\left(y\exp\left(-2(1+x)^{3/2}\right)\right) =$ $3\sqrt{1+x}\left(\exp\left(-2(1+x)^{3/2}\right)\right)$	1	
$y\left(\exp\left(-2(1+x)^{3/2}\right)\right) =$ $-\int\left(-3\sqrt{1+x}\right)\exp\left(-2(1+x)^{3/2}\right)dx$		
$= -\exp\left(-2(1+x)^{3/2}\right) + c$	1	
$y = -1 + c\exp\left(2(1+x)^{3/2}\right)$	1	

Source: 2011 Q4 AH Maths

(11)

(*a*) For what value of λ is $\begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \\ -1 & \lambda & 6 \end{pmatrix}$ singular?

(*b*) For $A = \begin{pmatrix} 2 & 2\alpha - \beta & -1 \\ 3\alpha + 2\beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$, obtain values of α and β such that

$$A' = \begin{pmatrix} 2 & -5 & -1 \\ -1 & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}.$$

Answers:

(a) Singular when the determinant is 0.

$$1 \det\begin{pmatrix} 0 & 2 \\ \lambda & 6 \end{pmatrix} - 2 \det\begin{pmatrix} 3 & 2 \\ -1 & 6 \end{pmatrix} + (-1) \det\begin{pmatrix} 3 & 0 \\ -1 & \lambda \end{pmatrix} = 0 \quad \textbf{M1}$$

$$-2\lambda - 2(18 + 2) - 1(3\lambda - 0) = 0 \quad \textbf{1}$$

$$-5\lambda - 40 = 0 \text{ when } \lambda = -8 \quad \textbf{1}$$

(b) From A, $A' = \begin{pmatrix} 2 & 3\alpha + 2\beta & -1 \\ 2\alpha - \beta & 4 & 3 \\ -1 & 3 & 2 \end{pmatrix}$. **1** for transpose

Hence $2\alpha - \beta = -1$ and $3\alpha + 2\beta = -5$. **1**

$$4\alpha - 2\beta = -2$$

$$3\alpha + 2\beta = -5$$

$7\alpha = -7 \Rightarrow \alpha = -1$ and $\beta = -1$. **1** for both values

Source: 2010 Q4 AH Maths

(12) Obtain the 2×2 matrix M associated with an enlargement, scale factor 2, followed by a clockwise rotation of $60°$ about the origin.

Answer:

The matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ gives an enlargement, scale factor 2.	**1**	correct matrix
The matrix $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$ gives a clockwise rotation of $60°$ about the origin.	**1**	correct matrix
$M = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$	**1**	correct order
$= \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$	**1**	

Source: 2010 Q14 AH Maths	
(13)	Use Gaussian elimination to show that the set of equations $$\begin{aligned} x - y + z &= 1 \\ x + y + 2z &= 0 \\ 2x - y + az &= 2 \end{aligned}$$ has a unique solution when $a \neq 2{\cdot}5$. Explain what happens when $a = 2{\cdot}5$. Obtain the solution when $a = 3$. Given $A = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, calculate AB. Hence, or otherwise, state the relationship between A and the matrix $$C = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 2 & -1 & 3 \end{pmatrix}.$$

Answers:

$\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & -1 & a & 2 \end{array}\right)$	1	for a structured approach
$\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 1 & a-2 & 0 \end{array}\right)$	1	
$\left(\begin{array}{ccc\|c} 1 & -1 & 1 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 2a-5 & 1 \end{array}\right)$	1	for triangular form
$z = \dfrac{1}{2a-5}$;	1	one correct variable
$2y + \dfrac{1}{2a-5} = -1 \Rightarrow 2y = \dfrac{-2a+5-1}{2a-5}$ $\Rightarrow y = \dfrac{2-a}{2a-5}$;		

$x - \frac{2-a}{2a-5} + \frac{1}{2a-5} = 1$ $\Rightarrow x = \frac{2a-5}{2a-5} + \frac{1-a}{2a-5} = \frac{\mathbf{a-4}}{\mathbf{2a-5}}$.	**1**	for the two other variables {*other justifications for uniqueness are possible*}
which exist when $2a - 5 \neq 0$.		
From the third row of the final tableau, when $a = 2{\cdot}5$, there are no solutions	**1**	
When $a = 3, x = -1, y = -1, z = 1$.	**1**	
$AB = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 1 & -1 \\ -3 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$	**1**	
From above, we have $C\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ and also $A\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ which suggests $AC = I$ and this can be verified directly. Hence A is the inverse of C (or vice versa).	**2**	A candidate who obtains $AC = I$ directly may be awarded full marks.

Source: 2009 Q2 AH Maths

(14)

Given the matrix $A = \begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix}$.

(*a*) Find A^{-1} in terms of t when A is non-singular.

(*b*) Write down the value of t such that A is singular.

(*c*) Given that the transpose of A is $\begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix}$, find t.

Answers:

(a)

$$\det\begin{pmatrix} t+4 & 3t \\ 3 & 5 \end{pmatrix} = 5(t+4) - 9t$$ **1**

$$= 20 - 4t$$

$$A^{-1} = \frac{1}{20-4t}\begin{pmatrix} 5 & -3t \\ -3 & t+4 \end{pmatrix}$$ **1,1**

(b)

$$20 - 4t = 0 \Rightarrow t = 5$$ **1**

(c)

$$\begin{pmatrix} t+4 & 3 \\ 3t & 5 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 6 & 5 \end{pmatrix} \Rightarrow t = 2$$ **1**

Source: 2008 Q6 AH Maths	
(15)	Let the matrix $A = \begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix}$. (*a*) Obtain the value(s) of x for which A is singular. (*b*) When $x = 2$, show that $A^2 = pA$ for some constant p. Determine the value of q such that $A^4 = qA$.

Answers:

(a)

$$\det\begin{pmatrix} 1 & x \\ x & 4 \end{pmatrix} = 4 - x^2$$ **1**

A matrix is singular when its determinant is 0, hence $x = \pm 2$. **1**

(b) When $x = 2$, $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & 20 \end{pmatrix} = 5A.$$ **1**

$$A^4 = (A^2)^2 = (5A)^2 = 25A^2 = 125A.$$ **2E1**

Evaluating $A^4 = \begin{pmatrix} 125 & 250 \\ 250 & 500 \end{pmatrix} = 125A$ was accepted.

Source: 2007 Q5 AH Maths

(16) Matrices A and B are defined by

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

(*a*) Find the product AB.

(*b*) Obtain the determinants of A and of AB.

Hence, or otherwise, obtain an expression for det B.

Answers:

(a)

$$AB = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x+2 & x-2 & x+3 \\ -4 & 4 & 2 \\ 2 & -2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} x & x & x \\ -6 & 6 & -1 \\ 0 & 0 & 8 \end{pmatrix}$$ **2E1**

(b)

$$\det A = 1 \times (2+1) - 0 - 1 \times 0 = 3$$ **1**

$$\det AB = x(48-0) - x(-48-0) + x(0-0) = 96x$$ **1**

Since $\det AB = \det A \det B$

$$\det B = \frac{\det AB}{\det A} = \frac{96x}{3} = 32x$$ **1**