Differential Equations

AH Maths Exam Questions

Source: 2019 Specimen P2 Q6 AH Maths – Same as 2017 Q9	
(1)	Solve $\frac{dy}{dx} = e^{2x}\left(1+y^2\right)$ given that when $x=0$, $y=1$. Express y in terms of x.

Answers:

•1 separate variables and write down integral equation	•1 $\int \frac{dy}{1+y^2} = \int e^{2x}\, dx$
•2 integrate LHS	•2 $\tan^{-1} y$
•3 integrate RHS	•3 $\frac{1}{2}e^{2x} + c$
•4 evaluate constant of integration	•4 $c = \frac{\pi}{4} - \frac{1}{2}$
•5 express y in terms of x	•5 $y = \tan\left(\frac{1}{2}e^{2x} + \frac{\pi}{4} - \frac{1}{2}\right)$

Source: 2019 Q13 AH Maths	
(2)	An electronic device contains a timer circuit that switches off when the voltage, V, reaches a set value. The rate of change of the voltage is given by $$\frac{dV}{dt} = k(12 - V),$$ where k is a constant, t is the time in seconds, and $0 \leq V < 12$. Given that $V = 2$ when $t = 0$, express V in terms of k and t.

Answers:

•1 separate variables and write integral equation 1	•1 $\int \frac{1}{12 - V} dV = \int k dt$
•2 integrate LHS	•2 $-\ln(12 - V)$
•3 integrate RHS 2	•3 $kt + c$
•4 evaluate constant of integration 2	•4 $-\ln 10$
•5 express V in terms of k and t 2,3,4	•5 $V = 12 - 10e^{-kt}$

Source: 2016 Q16 AH Maths

(3)

A beaker of liquid was placed in a fridge.

The rate of cooling is given by

$$\frac{dT}{dt} = -k\left(T - T_F\right), \quad k > 0,$$

where T_F is the constant temperature in the fridge and T is the temperature of the liquid at time t.

- The constant temperature in the fridge is 4 °C.
- When first placed in the fridge, the temperature of the liquid was 25 °C.
- At 12 noon, the temperature of the liquid was 9·8 °C.
- At 12:15 pm, the temperature of the liquid had dropped to 6·5 °C.

At what time, to the nearest minute, was the liquid placed in the fridge?

Answers:

Question			Generic Scheme	Illustrative Scheme	Max Mark
16.			**Method 1** - working in minutes ($t = 0$ at noon)		9
			•[1] construct integral equation Note 1	•[1] $\int \frac{1}{(T - T_F)} dT = \int -k\, dt$	
			•[2] integrate 2	•[2] $\ln\left(T - T_F\right) = -kt + c$	
			•[3] find constant, c	•[3] $\ln\left(9 \cdot 8 - 4\right) = -k(0) + c$ $c = \ln 5 \cdot 8$	
			•[4] substitute using given information 4	•[4] $\ln\left(6 \cdot 5 - 4\right) = -15k + \ln 5 \cdot 8$	
			•[5] find constant, k	•[5] $k = \frac{\ln 2 \cdot 5 - \ln 5 \cdot 8}{-15} = 0 \cdot 05610...$	
			•[6] substitute given condition	•[6] $\ln\left(25 - 4\right) = -0 \cdot 05610...t + \ln 5 \cdot 8$	
			•[7] know how to find time	•[7] $t = \frac{\ln 21 - \ln 5 \cdot 8}{-0 \cdot 05610...}$	
			•[8] calculate time	•[8] $t = -22 \cdot 93...$	
			•[9] state the time to the nearest minute 3	•[9] The liquid was placed in the fridge at 11:37 (am)	

Question			Generic Scheme	Illustrative Scheme	Max Mark
			Method 2 - working in minutes ($t=0$ when $T=25$)		
			•[1] construct integral equation Note 1	•[1] $\int \frac{1}{(T-T_F)}dT = \int -k\,dt$	
			•[2] integrate 2	•[2] $\ln(T-T_F) = -kt+c$	
			•[3] find constant, c.	•[3] $\ln(25-4) = -k(0)+c$, $c = \ln 21$	
			•[4] substitute using given information	•[4] $\ln(9\cdot 8-4) = -k(t)+\ln 21$	
			•[5] know to use $t+15$ Note 5	•[5] appearance of $(t+15)$	
			•[6] use given condition	•[6] $\ln(6\cdot 5-4) = -k(t+15)+\ln 21$	
			•[7] find constant, k Note 6	•[7] $k = -\frac{1}{15}\ln\left(\frac{2\cdot 5}{5\cdot 8}\right) = 0\cdot 05610...$	
			•[8] calculate time	•[8] $t = \ln\left(\frac{21}{5\cdot 8}\right) \div 0\cdot 05610... = 22\cdot 93$	
			•[9] state the time to the nearest minute 3	•[9] The liquid was placed in the fridge at 11:37 (am).	

Source: 2015 Q18 AH Maths	
(4)	Vegetation can be irrigated by putting a small hole in the bottom of a cylindrical tank, so that the water leaks out slowly. Torricelli's Law states that the rate of change of volume, V, of water in the tank is proportional to the square root of the height, h, of the water above the hole. This is given by the differential equation: $$\frac{dV}{dt} = -k\sqrt{h},\ k > 0.$$ (*a*) For a cylindrical tank with constant cross-sectional area, A, show that the rate of change of the height of the water in the tank is given by $$\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}.$$ (*b*) Initially, when the height of the water is 144 cm, the rate at which the height is changing is $-0{\cdot}3$ cm/hr. By solving the differential equation in part (*a*), show that $h = \left(12 - \frac{1}{80}t\right)^2$. (*c*) How many days will it take for the tank to empty? (*d*) Given that the tank has radius 20 cm, find the rate at which the water was being delivered to the vegetation (in cm^3/hr) at the end of the fourth day.

Answers over the page

Question			Expected Answer/s	Max Mark	Additional Guidance
18	a		**Method 1** $V = Ah$ (here of below) $\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$ •[1] $\frac{dV}{dh} = A$* $\therefore \frac{dh}{dV} = \frac{1}{A}$* $= \frac{1}{A} . -k\sqrt{h}$ •[2] $= \frac{-k}{A}\sqrt{h}$ or **Method 2** $V = Ah$ $\frac{dV}{dt} = \frac{d}{dt}(Ah)$ •[1] $-k\sqrt{h} = A\frac{dh}{dt}$ $\frac{Adh}{dt} = -k\sqrt{h}$ •[2] $\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}$ **Method 3** $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$ •[1] $-k\sqrt{h} = A\frac{dh}{dt}$ •[2] or **Method 4** $h = \frac{V}{A}$ •[1] $\frac{dh}{dt} = \frac{dV}{dt} \Big/ A$ •[2]	2	•[1] (Ah) in brackets and/or *A* following line needed for Method 2, since taking *A* out as a constant necessary to illustrate understanding of validity of step. •[2] One or both of *lines needed for Method 1.

Question			Expected Answer/s	Max Mark	Additional Guidance
	b		$\frac{dh}{dt} = -0 \cdot 3\text{cm / hr}$ when $h = 144$ $-0 \cdot 3 = -\frac{k}{A}\sqrt{144}$ $\frac{k}{A} = \frac{1}{40} \therefore A = 40k$ $\frac{dh}{dt} = \frac{-k}{A}\sqrt{h}$ $\int \frac{1}{\sqrt{h}}dh = \int \frac{-k}{A}dt$ OR $\int \frac{1}{\sqrt{h}}dh = \int -\frac{1}{40}dt$ $2\sqrt{h} = \frac{-k}{A}t + c$ $2\sqrt{144} = c \quad c = 24$ $2\sqrt{h} = \frac{-k}{A}t + 24$ $\sqrt{h} = \frac{-k}{2A}t + 12$ $h = \left(\frac{-k}{2A}t + 12\right)^2$ $h = \left(\frac{-1}{80}t + 12\right)^2$	4	•[3] Subs in $\frac{dh}{dt} = -0 \cdot 3$ *and* $h = 144$. Award this mark if substitution appears in part (d). •[4] separating variables[3]. •[5] integrating correctly. •[6] evaluating constant of integration *and* completion

Question			Expected Answer/s	Max Mark	Additional Guidance
18	c		$0=\left(-\frac{1}{80}t+12\right)^2$ $-\frac{1}{80}t+12=0$ $t=960$ hours number days $=\frac{960}{24}=40$ days	2	•[7] knowing to set correct expression to zero •[8] Processing to obtain number of days [4]
18	d		$A=400\pi$ $\frac{k}{A}=\frac{1}{40}$ $k=10\pi$ $h=\left(\frac{-1}{80}.96+12\right)^2$ $\frac{dV}{dt}=-108\pi$ $\therefore$ Rate to vegetation is $\mathbf{108\pi}\ \text{cm}^3/\text{hr}$	3	•[9] for finding *k*. •[10] obtaining h or $\sqrt{h}$ •[11] processing to answer *with* interpretation.

Source: 2013 Q16 AH Maths

(5)

In an environment without enough resources to support a population greater than 1000, the population $P(t)$ at time t is governed by Verhurst's law

$$\frac{dP}{dt} = P(1000 - P).$$

Show that

$$\ln\frac{P}{1000 - P} = 1000t + C \quad \text{for some constant } C.$$

Hence show that

$$P(t) = \frac{1000K}{K + e^{-1000t}} \quad \text{for some constant } K.$$

Given that $P(0) = 200$, determine at what time t, $P(t) = 900$.

Answers:

$\frac{dP}{dt} = P(1000 - P)$		
So $\int \frac{dP}{P(1000-P)} = \int dt$		•[1] Separates variables.[5]
$\frac{1}{P(1000-P)} = \frac{A}{P} + \frac{B}{1000-P}$		•[2] Appropriate form of partial fractions.
$A = \frac{1}{1000}, B = \frac{1}{1000}$		•[3] Obtains correct values of both A and B.
$\frac{1}{1000}\int\left(\frac{1}{P} + \frac{1}{1000-P}\right)dP = \int dt$		
$\ln P - \ln(1000 - P) = 1000t + c$		•[4] Integrates correctly, including '$+c$'.[6]
$\ln\frac{P}{1000-P} = 1000t + c$		
$\frac{P}{1000-P} = Ke^{1000t}\left(where\ K = e^{c}\right)$		•[5] Accurately converts to exponential form.[1]

(continued) $P = 1000Ke^{1000t} - PKe^{1000t}$, $P + PKe^{1000t} = 1000Ke^{1000t}$,		•[6] Multiplies out fractions and collects P terms.
$P = \dfrac{1000Ke^{1000t}}{1+Ke^{1000t}}$ $= \dfrac{1000K}{e^{-1000t}+K} \quad \left(\text{or } \dfrac{1000e^{c}}{e^{-1000t}+e^{c}}\right)$		•[7] Factorises and divides to obtain required form.[2]
Since P(0) = 200, $200 = \dfrac{1000K}{1+K}$ $K = \dfrac{1}{4} \quad (\text{or } 0\cdot25)$ Require $900 = \dfrac{1000\times0\cdot25}{0\cdot25+e^{-1000t}}$ $225 + 900e^{-1000t} = 250$ $e^{1000t} = 36$ $1000t = \ln 36$ $t = \dfrac{1}{1000}\ln 36$ [or 0·003584 (4sf)]		•[8] Equates and process to obtain value of K.[3] •[9] Inserts value of K *and* equates. •[10] Solves to obtain value for t.[4]

Source: 2011 Q9 AH Maths	
(6)	Given that $y > -1$ and $x > -1$, obtain the general solution of the differential equation $$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$$ expressing your answer in the form $y = f(x)$.

Answer:

Method 1		
$\frac{dy}{dx} = 3(1+y)\sqrt{1+x}$		
$\int \frac{dy}{1+y} = 3\int (1+x)^{\frac{1}{2}}\,dx$	**M1**	separating variables
$\ln(1+y) = 2(1+x)^{\frac{3}{2}} + c$	**1** **1**	for LHS for term in x
$1 + y = \exp\left(2(1+x)^{\frac{3}{2}} + c\right)$	**1**	for the constant
$y = \exp\left(2(1+x)^{\frac{3}{2}} + c\right) - 1.$	**1**	
$= A\exp\left(2(1+x)^{\frac{3}{2}}\right) - 1.$		
Method 2		
$\frac{dy}{dx} - 3\sqrt{1+x}\,y = 3\sqrt{1+x}$	**1**	
Integrating Factor		
$\exp\left(-3\int\sqrt{1+x}\,dx\right) = \exp\left(-2(1+x)^{3/2}\right)$	**1**	
$\frac{d}{dx}\left(y\exp\left(-2(1+x)^{3/2}\right)\right) =$ $3\sqrt{1+x}\left(\exp\left(-2(1+x)^{3/2}\right)\right)$	**1**	
$y\left(\exp\left(-2(1+x)^{3/2}\right)\right) =$ $-\int\left(-3\sqrt{1+x}\right)\exp\left(-2(1+x)^{3/2}\right)dx$		
$= -\exp\left(-2(1+x)^{3/2}\right) + c$	**1**	
$y = -1 + c\exp\left(2(1+x)^{3/2}\right)$	**1**	

Source: 2009 Q3 AH Maths	
(7)	Given that $$x^2e^y\frac{dy}{dx}=1$$ and $y=0$ when $x=1$, find y in terms of x.

Answer:

$$e^y x^2\frac{dy}{dx} = 1$$

$$e^y\frac{dy}{dx} = x^{-2}$$

$$\int e^y dy = \int x^{-2}dx$$

$$e^y = -x^{-1} + c$$

$y = 0$ when $x = 1$ so

$$1 = -1 + c \Rightarrow c = 2$$

$$e^y = 2 - \frac{1}{x} \Rightarrow y = \ln\left(2 - \frac{1}{x}\right)$$

Source: 2007 Q14 AH Maths

(8)

A garden centre advertises young plants to be used as hedging.

After planting, the growth G metres (ie the increase in height) after t years is modelled by the differential equation

$$\frac{dG}{dt} = \frac{25k - G}{25}$$

where k is a constant and $G = 0$ when $t = 0$.

(*a*) Express G in terms of t and k.

(*b*) Given that a plant grows 0·6 metres by the end of 5 years, find the value of k correct to 3 decimal places.

(*c*) On the plant labels it states that the expected growth after 10 years is approximately 1 metre. Is this claim justified?

(*d*) Given that the initial height of the plants was 0·3 m, what is the likely long-term height of the plants?

Answers:

(a)
$$\frac{dG}{dt} = \frac{25k - G}{25}$$

$$\int \frac{dG}{25k - G} = \int \frac{1}{25}dt \qquad \mathbf{1}$$

$$-\ln(25k - G) = \frac{t}{25} + C \qquad \mathbf{1}$$

When $t = 0, G = 0$, so $C = -\ln 25k$ **1**

$$25k - G = 25ke^{-t/25}$$

$$G = 25k\left(1 - e^{-t/25}\right) \qquad \mathbf{1}$$

(b) When $t = 5, G = 0.6$. Therefore

$$0.6 = 25k\left(1 - e^{-0.2}\right) \qquad \mathbf{1}$$

$$k = 0.6/\left(25\left(1 - e^{-0.2}\right)\right) \approx 0{\cdot}132 \qquad \mathbf{1}$$

(c) When $t = 10$

$$G \approx 3.3\left(1 - e^{-0.4}\right) \qquad \mathbf{1}$$

$$\approx 1.09$$

The claim seems to be justified, **1**

(d) As $t \to \infty, G \to 25k \approx 3.3$ metres **1**

so the limit is 3.6 metres. **1**

Source: 2003 Q1 AH Maths	
(9)	The volume $V(t)$ of a cell at time t changes according to the law $\frac{dV}{dt}=V(10-V)$ for $0<V<10$ Show that $\frac{1}{10}\ln V-\frac{1}{10}\ln(10-V)=t+C$ for some constant C. Given that $V(0)=5$, show that $V(t)=\frac{10e^{10t}}{1+e^{10t}}$. Obtain the limiting value of $V(t)$ as $t\to\infty$.

Answers:

$$\frac{dV}{dt} = V(10-V)$$

$$\int \frac{dV}{V(10-V)} = \int 1\,dt \qquad \mathbf{1}$$

$$\frac{1}{10}\int \frac{1}{V} + \frac{1}{10-V}\,dV = \int 1\,dt \qquad \mathbf{2}$$

$$\frac{1}{10}(\ln V - \ln(10-V)) = t + C \qquad \mathbf{1}$$

$$\frac{1}{10}\ln V - \frac{1}{10}\ln(10-V) = t + C$$

$V(0) = 5$, so

$$\tfrac{1}{10}\ln 5 - \tfrac{1}{10}\ln 5 = 0 + C$$

$$C = 0 \qquad \mathbf{1}$$

$$\ln V - \ln(10-V) = 10t$$

$$\ln\left(\frac{V}{10-V}\right) = 10t$$

$$\frac{V}{10-V} = e^{10t}$$

$$V = 10e^{10t} - Ve^{10t}$$

$$V(1+e^{10t}) = 10e^{10t} \qquad \mathbf{2E1}$$

$$V = \frac{10e^{10t}}{1+e^{10t}}$$

$$V = \frac{10e^{10t}}{1+e^{10t}} = \frac{10}{e^{-10t}+1} \qquad \mathbf{1}$$

$$\to 10 \text{ as } t \to \infty. \qquad \mathbf{1}$$