Sequences & Series

AH Maths Exam Questions

Source: 2019 Specimen P2 Q11 AH Maths – Same as 2018 Q14	
(1)	A geometric sequence has first term 80 and common ratio $\frac{1}{3}$. (a) For this sequence calculate (i) the 7th term (ii) the sum to infinity of the associated geometric series. The first term of this geometric sequence is equal to the first term of an arithmetic sequence. The sum of the first five terms of this arithmetic sequence is 240. (b) Find the common difference of this sequence. Let S_n represent the sum to n terms of this arithmetic sequence. (c) Find the values of n for which $S_n = 144$.
Answers:	$(a)\ (i)\ 7th\ term = \frac{80}{729}$ $\quad (ii)\ Sum\ to\ Infinity = 120$ $(b)\ Common\ Difference = -16$ $(c)\ n = 2, \quad n = 9$

Source: 2019 Q7 AH Maths	
(2)	(a) Find an expression for $\sum_{r=1}^{n}(6r+13)$ in terms of n. (b) Hence, or otherwise, find $\sum_{r=p+1}^{20}(6r+13)$.
Answers:	$(a)\ \ 3n^2+16n$ $(b)\ \ 1520-3p^2-16p$

Source: 2017 Q4 AH Maths	
(3)	The fifth term of an arithmetic sequence is -6 and the twelfth term is -34. (a) Determine the values of the first term and the common difference. (b) Obtain algebraically the value of n for which $S_n=-144$.
Answers:	$(a)\ \ a=10\ \ d=-4$ $\qquad (b)\ \ n=12$

Source: 2017 Q10 AH Maths	
(4)	S_n is defined by $\sum_{r=1}^{n}\left(r^2+\frac{1}{3}r\right)$. (a) Find an expression for S_n, fully factorising your answer. (b) Hence find an expression for $\sum_{r=10}^{2p}\left(r^2+\frac{1}{3}r\right)$ where $p>5$.
Answers:	$(a)\ \frac{n(n+1)^2}{3}$ $\qquad (b)\ \frac{2p(2p+1)^2}{3}-300$

Source: 2019 Q17 AH Maths

(5)

The first three terms of a sequence are given by

$$5x+8,\ -2x+1,\ x-4$$

(a) When $x=11$, show that the first three terms form the start of a geometric sequence, and state the value of the common ratio.

(b) Given that the entire sequence is geometric for $x=11$

(i) state why the associated series has a sum to infinity

(ii) calculate this sum to infinity.

(c) There is a second value for x that also gives a geometric sequence.

For this second sequence

(i) show that $x^2-8x-33=0$

(ii) find the first three terms

(iii) state the value of S_{2n} and justify your answer.

Answers:

(a)

•[1] substitute and calculate one ratio [1,2,3,4]	•[1] $\frac{-21}{63}=-\frac{1}{3}$ or $\frac{7}{-21}=-\frac{1}{3}$
•[2] calculate second ratio and state common ratio [1,5]	•[2] $\frac{7}{-21}=-\frac{1}{3}$ or $\frac{-21}{63}=-\frac{1}{3}$ So $r=-\frac{1}{3}$

$(b)(i)$ $\left|-\frac{1}{3}\right|<1$

(ii)	•[4] begin to substitute [1,2,3]	•[4] $\frac{\dots}{1-\left(-\frac{1}{3}\right)}$
	•[5] calculate sum [1,2,3]	•[5] $\frac{189}{4}$ or $47\cdot25$

(c)

(i)	•[6] equate ratios	•[6] $\frac{-2x+1}{5x+8}=\frac{x-4}{-2x+1}$
	•[7] perform algebraic manipulation leading to formation of quadratic equation [1]	•[7] $x^2-8x-33=0$
(ii)	•[8] calculate second value of x	•[8] $x=-3$
	•[9] find first three terms	•[9] $-7,\ 7,\ -7$
(iii)	•[10] state S_{2n} and justify [1,2]	•[10] 0 since eg $2n$ is even and so pairs of terms cancel each other out

Source: 2016 Q2 AH Maths

(6) A geometric sequence has second and fifth terms 108 and 4 respectively.

(a) Calculate the value of the common ratio.

(b) State why the associated geometric series has a sum to infinity.

(c) Find the value of this sum to infinity.

Answers:

(a) $r = \frac{1}{3}$

(b) $-1 < \frac{1}{3} < 1$

(c) Value = 486

Source: 2015 Q3 AH Maths

(7) The sum of the first twenty terms of an arithmetic sequence is 320.

The twenty-first term is 37.

What is the sum of the first ten terms?

Answer: $S_n = 60$

Source: 2012 Q2 AH Maths

(8) The first and fourth terms of a geometric series are 2048 and 256 respectively. Calculate the value of the common ratio.

Given that the sum of the first n terms is 4088, find the value of n.

Answers: $r = \frac{1}{2}$, $n = 9$

Source: 2014 Q14 AH Maths

(9)

(*a*) Given the series $1 + r + r^2 + r^3 + \ldots$, write down the sum to infinity when $|r| < 1$.

Hence obtain an infinite geometric series for $\dfrac{1}{2-3r}$.

For what values of r is this series valid?

(*b*) Express $\dfrac{1}{3r^2-5r+2}$ in partial fractions.

Hence, or otherwise, determine the first three terms of an infinite series for $\dfrac{1}{3r^2-5r+2}$.

For what values of r does the series converge?

Answers:

(*a*)

$$1+r+r^2+r^3+\ldots=\frac{1}{1-r}$$

$$\frac{1}{2-3r}=\frac{1}{2\left(1-\frac{3r}{2}\right)} \textbf{ OR } \frac{1}{1-(3r-1)} \textbf{ OR } \frac{\frac{1}{2}}{1-\frac{3}{2}r}$$

$$=\frac{1}{2}\left(\frac{1}{1-\frac{3r}{2}}\right)=\frac{1}{2}\left(1+\frac{3r}{2}+\left(\frac{3r}{2}\right)^2+\ldots\right)$$

$$=\frac{1}{2}\left(1+\frac{3r}{2}+\frac{9r^2}{4}+..\right)$$

$$\left|\frac{3r}{2}\right|<1, \quad \therefore |r|<\frac{2}{3}$$

4

- •1 Correct statement of sum.
- •2 Valid rearrangement of expression.[2, 5]
- •3 Makes correct substitution for r in series at •1.[1,3,5]
- •4 Correct statement of range.[5]

(*b*)

$$\frac{1}{3r^2-5r+2}=\frac{A}{(3r-2)}+\frac{B}{(r-1)}$$

$$\therefore A(r-1)+B(3r-2)\equiv 1; \quad B=1$$
$$A=-3$$

$$\frac{1}{3r^2-5r+2}=\frac{-3}{(3r-2)}+\frac{1}{(r-1)}$$
$$=\frac{3}{(2-3r)}-\frac{1}{(1-r)}$$

$$=3\left(\frac{1}{2}\left(1+\frac{3r}{2}+\frac{9r^2}{4}+\ldots\right)\right)-\left(1+r+r^2+\ldots\right)$$
$$=\frac{1}{2}+\frac{5r}{4}+\frac{19r^2}{8}\ldots$$

$$\left|\frac{3r}{2}\right|<1 \text{ and } |r|<1, \text{ so } |r|<\frac{2}{3}$$

6

- •5 Correct form of partial fractions.[7]
- •6 Either coefficient correct.[7]
- •7 Second coefficient correct.[7]
- •8 Recognising form *and* manipulating correctly.
- •9 Simplifying to obtain first three terms.
- •10 Correct statement of range of convergence.[6]

Source: 2013 Q17 AH Maths	
(10)	Write down the sums to infinity of the geometric series $1 + x + x^2 + x^3 + \ldots\ldots$ and $1 - x + x^2 - x^3 + \ldots\ldots$ valid for $\lvert x \rvert < 1$. Assuming that it is permitted to integrate an infinite series term by term, show that, for $\lvert x \rvert < 1$, $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right).$ Show how this series can be used to evaluate ln 2. Hence determine the value of ln 2 correct to 3 decimal places.

Answers:

$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$	•[1] Correct statement of sum.
$1 - x + x^2 - x^3 + \ldots = \frac{1}{1+x}$	•[2] Correct statement of sum.
Integrating the first of these gives: $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \ldots = -\ln(1-x) + c$	•[3] Correct integration of both sides.[1]
Putting $x = 0$ gives $c = 0$.	•[4] Correct evaluation of c.[3]
Similarly, $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \ldots = \ln(1+x)$	•[5] Correct integration of both sides.[1]
Adding together gives: $2\left(x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \ldots\right) = \ln(1+x) - \ln(1-x)$	•[6] Evidence of appropriate method.
$\left[= \ln\frac{1+x}{1-x}\right]$ as required.	•[7] Appropriate intermediate step. •[3] Adds series.

See Marking Scheme for alternative method

Source: 2011 Q8 AH Maths	
(11)	Write down an expression for $\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2$ and an expression for $$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2.$$

Answers:

$$\sum_{r=1}^{n} r^3 - \left(\sum_{r=1}^{n} r\right)^2 = \frac{n^2(n+1)^2}{4} - \left(\frac{n(n+1)}{2}\right)^2 = 0 \qquad \mathbf{1}$$

$$\sum_{r=1}^{n} r^3 + \left(\sum_{r=1}^{n} r\right)^2 = \frac{n^2(n+1)^2}{4} + \left(\frac{n(n+1)}{2}\right)^2 \qquad \mathbf{1}$$

$$= \frac{n^2(n+1)^2}{4} + \frac{n^2(n+1)^2}{4} \qquad \mathbf{1}$$

$$= \frac{n^2(n+1)^2}{2} \qquad \mathbf{1}$$

Source: 2011 Q13 AH Maths	
(12)	The first three terms of an arithmetic sequence are $a, \frac{1}{a}, 1$ where $a < 0$. Obtain the value of a and the common difference. Obtain the smallest value of n for which the sum of the first n terms is greater than 1000.

Answers:

$$a = -2, \quad d = \frac{3}{2}$$

smallest value of n is 39

Source: 2010 Q2 AH Maths	
(13)	The second and third terms of a geometric series are –6 and 3 respectively. Explain why the series has a sum to infinity, and obtain this sum.

Answers:

Let the first term be a and the common ratio be r. Then		
$ar = -6$ and $ar^2 = 3$	**1**	{both terms needed}
Hence		
$r = \frac{ar^2}{ar} = \frac{3}{-6} = -\frac{1}{2}$.	**1**	evaluating r
So, since $\lvert r\rvert < 1$, the sum to infinity exists.	**1**	justification
$S = \frac{a}{1-r}$	**1**	correct formula
$= \frac{12}{1-(-\frac{1}{2})} = \frac{12}{\frac{3}{2}}$		
$= 8.$	**1**	the sum to infinity

Source: 2008 Q1 AH Maths	
(14)	The first term of an arithmetic sequence is 2 and the 20th term is 97. Obtain the sum of the first 50 terms.

Answers:

Let the common difference be d. General term is $a + (n-1)d$.
So $2 + 19d = 97 \Rightarrow d = 5$.
Sum of an arithmetic series is $\frac{n}{2}[2a + (n-1)d]$.
Required sum is $\frac{50}{2}\{4 + 49 \times 5\} = 6225$.

Source: 2009 Q12 AH Maths	
(15)	The first two terms of a geometric sequence are $a_1 = p$ and $a_2 = p^2$. Obtain expressions for S_n and S_{2n} in terms of p, where $S_k = \sum_{j=1}^{k} a_j$. Given that $S_{2n} = 65S_n$ show that $p^n = 64$. Given also that $a_3 = 2p$ and that $p > 0$, obtain the exact value of p and hence the value of n.

Answer:

$$a_j = p^j \Rightarrow S_k = p + p^2 + \ \dots \ + p^k = \frac{p(p^k - 1)}{p - 1}$$

$$S_n = \frac{p(p^n - 1)}{p - 1} \qquad \mathbf{1}$$

$$S_{2n} = \frac{p(p^{2n} - 1)}{p - 1} \qquad \mathbf{1}$$

$$\frac{p(p^{2n} - 1)}{p - 1} = \frac{65p(p^n - 1)}{p - 1}$$

$$(p^n + 1)(p^n - 1) = 65(p^n - 1) \qquad \mathbf{1}$$

$$p^n + 1 = 65 \qquad \mathbf{1}$$

$$\Rightarrow p^n = 64$$

$$a_2 = p^2 \Rightarrow a_3 = p^3 \text{ but } a_3 = 2p \text{ so } p^3 = 2p$$

$$\Rightarrow p^2 = 2 \Rightarrow p = \sqrt{2} \text{ since } p > 0. \qquad \mathbf{1}$$

$$p^n = 64 = 2^6 = (\sqrt{2})^{12}$$

$$n = 12 \qquad \mathbf{1}$$

Source: 2007 Q9 AH Maths

(16)

Show that $\sum_{r=1}^{n}(4-6r) = n-3n^2$.

Hence write down a formula for $\sum_{r=1}^{2q}(4-6r)$.

Show that $\sum_{r=q+1}^{2q}(4-6r) = q-9q^2$.

Answers:

$$\sum_{r=1}^{n}(4-6r) = 4\sum_{r=1}^{n} - 6\sum_{r=1}^{n} r \quad \textbf{1M}$$

$$= 4n - 3n(n+1) \quad \textbf{1}$$

$$= n - 3n^2$$

$$\sum_{r=1}^{2q}(4-6r) = 2q - 12q^2 \quad \textbf{1}$$

$$\sum_{r=q+1}^{2q}(4-6r) = \sum_{r=1}^{2q}(4-6r) - \sum_{r=1}^{q}(4-6r) \quad \textbf{1M}$$

$$= (2q - 12q^2) - (q - 3q^2) \quad \textbf{1}$$

$$= q - 9q^2.$$

Arithmetic Series could be used, so, for the first two marks:

$$a = -2, d = -6 \Rightarrow S_n = \tfrac{n}{2}\{2(-2) + (n-1)(-6)\} \quad \textbf{1}$$

$$= -2n - 3n^2 + 3n = n - 3n^2 \quad \textbf{1}$$