# Methods of Proof

## AH Maths Exam Questions

| Source: 2019 Specimen P1 Q7 AH Maths – Same as 2018 Q12 | |
|---|---|
| (1) | Prove by induction that, for all positive integers $n$, $$\sum_{r=1}^{n} 3^{r-1} = \frac{1}{2}\left(3^n - 1\right).$$ |

Answer:

| | | |
|---|---|---|
| •¹ show true for $n=1$ | •¹ LHS: $3^0 = 1$ RHS: $\frac{1}{2}(3-1)=1$<br>So true for $n=1$ | **5** |
| •² assume (statement) true for $n=k$ **AND** consider whether (statement) true for $n=k+1$ | •² Suitable statement **and** $\sum_{r=1}^{k} 3^{r-1} = \frac{1}{2}\left(3^k - 1\right)$<br>**AND** $\sum_{r=1}^{k+1} 3^{r-1} = \ldots$ | |
| •³ correct statement for sum to $(k+1)$ terms using inductive hypothesis | •³ $\ldots = \frac{1}{2}\left(3^k - 1\right) + 3^{(k+1)-1}$ | |
| •⁴ combine terms in $3^k$ | •⁴ $\frac{3}{2} \times 3^k - \frac{1}{2}$ | |
| •⁵ express sum explicitly in terms of $(k+1)$ or achieve stated aim/goal **AND** communicate | •⁵ $\frac{1}{2}\left(3^{(k+1)} - 1\right)$<br>If true for $n=k$ then true for $n=k+1$. Also shown true for $n=1$ therefore, by induction, true for all positive integers $n$. | |

| Source: 2019 Specimen P2 Q9 AH Maths – Same as 2016 Q10 | |
|---|---|
| (2) | For each of the following statements, decide whether it is true or false.<br>If true, give a proof; if false, give a counterexample.<br>A. If a positive integer $p$ is prime, then so is $2p+1$.<br>B. If a positive integer $n$ has remainder 1 when divided by 3, then $n^3$ also has remainder 1 when divided by 3. |

Answers:

| | | |
|---|---|---|
| •[1] give counterexample | •[1] for example choose $p=7$<br>$2(7)+1=15$, which is not prime.<br>$\therefore$ statement is false. | **4** |
| •[2] set up $n$ | •[2] $n=3a+1$ , $a \in \mathbb{W}$ | |
| •[3] consider expansion of $n^3$ | •[3] $n^3=27a^3+27a^2+9a+1$ | |
| •[4] complete proof with conclusion | •[4] $=3\left(9a^3+9a^2+3a\right)+1$ and statement such as “so $n^3$ has remainder 1 when divided by 3 $\therefore$ statement is true”. | |

Source: 2019 Q11 AH Maths

(3)

Let $n$ be a positive integer.

(a) Find a counterexample to show that the following statement is false.

$n^2+n+1$ is always a prime number.

(b) (i) Write down the contrapositive of:

If $n^2-2n+7$ is even then $n$ is odd.

(ii) Use the contrapositive to prove that if $n^2-2n+7$ is even then $n$ is odd.

Answers:

| | | | | |
|---|---|---|---|---|
| (a) | | $\bullet^1$ state counterexample [1,2] | $\bullet^1$ eg when $n=4$, $n^2+n+1=21$ which is not prime | **1** |

| | | | | |
|---|---|---|---|---|
| (b) | (i) | $\bullet^2$ write down contrapositive statement [1,2,8] | $\bullet^2$ If $n$ is even then $n^2-2n+7$ is odd | **1** |
| | (ii) | $\bullet^3$ write down appropriate form for $n$ **AND** substitute [1,3,4,5,9]<br><br>$\bullet^4$ show $n^2-2n+7$ is odd [1,6,7,9]<br><br>$\bullet^5$ communicate [1,8,9] | $\bullet^3$ $n=2k$, $k\in\mathbb{N}$ and $(2k)^2-2(2k)+7$<br><br>$\bullet^4$ eg $2(2k^2-2k+3)+1$ which is odd since $2k^2-2k+3\in\mathbb{N}$<br><br>$\bullet^5$ contrapositive statement is true **AND** therefore original statement is true | **3** |

| Source: 2019 Q14 AH Maths | |
|---|---|
| (4) | Prove by induction that $$\sum_{r=1}^{n} r!r = (n+1)! - 1$$ for all positive integers $n$. |

Answer:

| Generic scheme | Illustrative scheme | Max mark |
|---|---|---|
| •[1] show true when $n=1$ [1] | •[1] when $n=1$ LHS $= 1!\times 1 = 1$ RHS $= (1+1)! - 1 = 1$ so result is true when $n=1$. | 5 |
| •[2] assume (statement) true for $n=k$ **AND** consider whether (statement) true for $n=k+1$ [2] | •[2] suitable statement AND $\sum_{r=1}^{k} r!r = (k+1)! - 1$ AND $\sum_{r=1}^{k+1} r!r = \ldots$ | |
| •[3] state sum to $(k+1)$ terms using inductive hypothesis [5] | •[3] $(k+1)! - 1 + (k+1)!(k+1)$ | |
| •[4] extract $(k+1)!$ as common factor [3,5] | •[4] $(k+1)!(k+2) - 1$ | |
| •[5] express sum explicitly in terms of $(k+1)$ or achieve stated aim/goal **AND** communicate [4,5,6] | •[5] $((k+1)+1)! - 1$ AND If true for $n=k$ then true for $n=k+1$. Also shown true for $n=1$ therefore, by induction, true for all positive integers $n$. | |

Source: 2018 Q9 AH Maths

(5) Prove directly that:

(a) the sum of any three consecutive integers is divisible by 3;

(b) any odd integer can be expressed as the sum of two consecutive integers.

Answers:

| | | | |
|---|---|---|---|
| (a) | | •¹ form the sum of three consecutive integers [1,2,3,4,5] | •¹ $(n-1)+n+(n+1)$ |
| | | •² communication [1,5] | •² $3n$ which is divisible by 3 |
| (b) | | •³ appropriate form for odd number, decomposed into two consecutive integers [1,2,3] | •³ $2k+1=k+(k+1)$ , $k \in \mathbb{Z}$ |

Source: 2017 Q13 AH Maths

(6)

Let $n$ be an integer.

Using proof by contrapositive, show that if $n^2$ is even, then $n$ is even.

Answer:

| Generic scheme | Illustrative scheme | Max mark |
|---|---|---|
| •[1] write down contrapositive statement [1,2,7,8] | •[1] The contrapositive of the original statement is :<br>If $n$ is odd then $n^2$ is odd | 4 |
| •[2] write down appropriate form for $n$ [3,4,7] | •[2] $n = 2k + 1\ ,\ k \in \mathbb{Z}$ | |
| •[3] show $n^2$ is odd [5,6,7] | •[3] $n^2 = 2(2k^2 + 2k) + 1$ which is odd | |
| •[4] communicate | •[4] contrapositive statement is true therefore original statement is true | |

Source: 2016 Q5 AH Maths

(7) Prove **by induction** that

$$\sum_{r=1}^{n} r(3r-1) = n^2(n+1), \qquad \forall n \in \mathbb{N}.$$

Answer:

| Generic Scheme | Illustrative Scheme | Max Mark |
|---|---|---|
| •[1] show true for $n=1$ [1] | •[1] LHS: $1(3-1)=2$ RHS: $1^2(1+1)=2$ So true for $n=1$ | 4 |
| •[2] assume true for $n=k$ [2] AND consider $n=k+1$ | •[2] $\sum_{r=1}^{k} r(3r-1) = k^2(k+1)$ and $\sum_{r=1}^{k+1} r(3r-1) =$ <br> $\ldots = \sum_{r=1}^{k} r(3r-1) + (k+1)(3(k+1)-1)$ | |
| •[3] correct statement of sum to $(k+1)$ terms using inductive hypothesis | •[3] $= k^2(k+1)+(k+1)(3k+2)$ <br> $=(k+1)\left[k^2+3k+2\right]$ <br> $=(k+1)(k+1)(k+2)$ | |
| •[4] express explicitly in terms of $(k+1)$ or achieve stated aim/goal [3,4] AND communicate | •[4] $=(k+1)^2((k+1)+1)$, thus if true for $n=k$ then true for $n=k+1$ but since true for $n=1$, then by induction true for all $n \in \mathbb{N}$ | |

| Source: 2015 Q12 AH Maths | |
|---|---|
| (8) | Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8. |

Answer:

| | | |
|---|---|---|
| Let numbers be $2n-1,\ 2n+1,\quad n \in ℵ$ | 3 | •[1] correct form for any two consecutive odd numbers[1,2]. |
| $(2n+1)^2-(2n-1)^2$ <br> $=(4n^2+4n+1)-(4n^2-4n+1)$ | | •[2] correct expressions squared out. |
| $=8n$ which is divisible by 8 | | •[3] multiple of 8 and communication. |

Source: 2014 Q7 AH Maths

(9)

Given $A$ is the matrix $\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$,

prove by induction that

$$A^n = \begin{pmatrix} 2^n & a\left(2^n - 1\right) \\ 0 & 1 \end{pmatrix}, \ n \geq 1.$$

Answer:

| Expected Answer/s | Max Mark | Additional Guidance |
|---|---|---|
| For $n = 1$ RHS $= \begin{pmatrix} 2^1 & a\left(2^1 - 1\right) \\ 0 & 1 \end{pmatrix}$ $= \begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ $= A$ <br> LHS $= A^1 = A =$ RHS. | 4 | •¹ Substituting $n = 1$.[1] |
| Assume true for $n = k$, <br> $A^k = \begin{pmatrix} 2^k & a\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> Consider $n = k + 1$, <br> $A^{k+1} = A^k A^1$ [OR $A^1 A^k$] | | See note 5. <br> •² Inductive hypothesis (must include "Assume true for $n = k$…" or equivalent phrase) *and* expansion of. $A^{k+1}$.[2,5] |
| $= \begin{pmatrix} 2^k & a\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & a \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2^k.2 & 2^k.a + a\left(2^k - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> $= \begin{pmatrix} 2^{k+1} & 2^k.a + 2^k.a - a \\ 0 & 1 \end{pmatrix}$ | | •³ Correct multiplication of two matrices *and* accurate manipulation of indices and brackets.[3] |
| $= \begin{pmatrix} 2^{k+1} & a\left(2^k + 2^k - 1\right) \\ 0 & 1 \end{pmatrix}$ | | * |
| $= \begin{pmatrix} 2^{k+1} & a\left(2^{k+1} - 1\right) \\ 0 & 1 \end{pmatrix}$ <br> Hence, if true for $n = k$, then true for $n = k + 1$, but since true for $n = 1$, then by induction true for all positive integers $n$. | | •⁴ Line * *and* statement of result in terms of $(k + 1)$ *and* valid statement of conclusion.[4,6] |

| Source: 2013 Q9 AH Maths | |
|---|---|
| (10) | Prove by induction that, for all positive integers $n$, $$\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)=n(n+1)^3$$ |

Answer:

For $n = 1$

L.H.S

$\sum_{r=1}^{n}\left(4r^3+3r^2+r\right)$

$= 4 + 3 + 1 = 8$

R.H.S

$n(n+1)^3$

$= 1 \times 2^3 = 8$

$\Rightarrow$ true for $n=1$

•[1] Evaluation of both sides independently to 8.[8]

Assume true for $n = k$,

$\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)=k(k+1)^3$

•[2] Inductive hypothesis (must include "Assume true…" or equivalent phrase).[3,4]

Consider $n = k + 1$,

$$\sum_{r=1}^{k+1}\left(4r^3+3r^2+r\right)$$

$$=\sum_{r=1}^{k}\left(4r^3+3r^2+r\right)+4(k+1)^3+3(k+1)^2+(k+1)$$

•[3] Addition of $(k + 1)$th term.[5]

$$=k(k+1)^3+4(k+1)^3+3(k+1)^2+(k+1)$$

$$=(k+1)\left[k(k+1)^2+4(k+1)^2+3(k+1)+1\right]$$

•[4] Use of inductive hypothesis *and* first step in factorisation process.[1,6]

$$=(k+1)\left[k\left(k^2+2k+1\right)+4\left(k^2+2k+1\right)+3(k+1)+1\right]$$

$$=(k+1)\left[k^3+2k^2+k+4k^2+8k+4+3k+3+1\right]$$

$$=(k+1)\left(k^3+6k^2+12k+8\right)$$

•[5] Processing and simplifying to arrive at second factor.[1]

$$=(k+1)(k+2)^3$$

$$=(k+1)((k+1)+1)^3$$

Hence, if true for $n = k$, then true for $n = k + 1$, but since true for $n = 1$, then by induction true for all positive integers $n$.

•[6] Statement of result in terms of $(k + 1)$ and valid statement of conclusion.[1,7]

| Source: 2013 Q12 AH Maths | |
|---|---|
| (11) | Let $n$ be a natural number.<br>For each of the following statements, decide whether it is true or false.<br>If true, give a proof; if false, give a counterexample.<br><br>**A** If $n$ is a multiple of 9 then so is $n^2$.<br><br>**B** If $n^2$ is a multiple of 9 then so is $n$. |

Answer:

| | | | | |
|---|---|---|---|---|
| **A** | | Suppose $n = 9m$ for some natural number [positive integer], $m$. | $\bullet^1$ | Generalisation, using *different* letter.[3, 6] |
| | | Then $n^2 = 81m^2 = 9(9m^2)$ | $\bullet^2$ | Correct multiplication *and* 9 extracted as a factor. |
| | | Hence $n^2$ is a multiple of 9, so **A** is **true**. | $\bullet^3$ | Conclusion of proof *and* state A true.[1] |
| **B** | | **False.** Accept any valid counterexample: $n = 3, 6, 12, 15, 21$ etc | $\bullet^4$ | Valid counterexample *and* conclusion.[5] |

Source: 2012 Q16a AH Maths

(12)

(*a*) Prove by induction that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

for all integers $n \geq 1$.

(*b*) Show that the real part of $\dfrac{\left(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18}\right)^{11}}{\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)^{4}}$ is zero.

Answers:

| | | |
|---|---|---|
| (a) For $n = 1$, the LHS $= \cos\theta + i\sin\theta$ and the RHS $= \cos\theta + i\sin\theta$. Hence the result is true for $n = 1$. | **1** | |
| Assume the result is true for $n = k$, i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$. | **1** | working with $n$ is penalised. |
| Now consider the case when $n = k + 1$: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$ | **1** | |
| $= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$ | **1** | for applying the inductive hypothesis |
| $= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$ | **1** | multiplying and collecting |
| $= \cos(k+1)\theta + i\sin(k+1)\theta$ <br> Thus, if the result is true for $n = k$ the result is true for $n = k + 1$. <br> Since it is true for $n = 1$, the result is true for all $n \geqslant 1$. | **1** | |
| (b) $\dfrac{(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})^{11}}{(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36})^4} = \dfrac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}}$ | **1** | using result from above |
| $= \dfrac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}} \times \dfrac{\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}}{\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}}$ | **1** | |
| $= \dfrac{\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}}{\cos^2\frac{\pi}{9} + \sin^2\frac{\pi}{9}} + \text{imaginary term}$ | | |
| $= \cos\left(\dfrac{11\pi}{18} - \dfrac{\pi}{9}\right) + \text{imaginary term}$ | **1** | |
| $= \cos\dfrac{\pi}{2} + \text{imaginary term}$ <br> Thus the real part is zero as required. | **1** | or equivalent |

| Source: 2011 Q12 AH Maths | |
|---|---|
| (13) | Prove by induction that $8^n + 3^{n-2}$ is divisible by 5 **for all integers $n \geq 2$**. |

Answer:

| | | |
|---|---|---|
| For $n = 2$, $8^2 + 3^0 = 64 + 1 = 65$.<br>True when $n = 2$. | **1** | |
| Assume true for $k$, i.e. that $8^k + 3^{k-2}$ is divisible by 5, i.e. can be expressed as $5p$ for an integer $p$. | **1** | for the inductive hypothesis |
| Now consider $8^{k+1} + 3^{k-1}$ | | |
| $= 8 \times 8^k + 3^{k-1}$ | **1** | |
| $= 8 \times (5p - 3^{k-2}) + 3^{k-1}$ | **1** | for replacing $8^k$ |
| $= 40p - 3^{k-2}(8 - 3)$ | | |
| $= 5(8p - 3^{k-2})$ which is divisible by 5. | **1** | |
| So, since it is true for $n = 2$, it is true for all $n \geqslant 2$. | | |

Source: 2010 Q8 AH Maths

(14)

(*a*) Prove that the product of two odd integers is odd.

(*b*) Let $p$ be an odd integer. Use the result of (*a*) to prove by induction that $p^n$ is odd for all positive integers $n$.

Answers:

| | | |
|---|---|---|
| (a) Write the odd integers as: $2n + 1$ and $2m + 1$ where $n$ and $m$ are integers. | **1M** | for unconnected odd integers |
| Then $(2n+1)(2m+1) = 4nm + 2n + 2m + 1$ | | |
| $= 2(2nm + n + m) + 1$ | **1** | demonstrating clearly |
| which is odd. | | |
| (b) Let $n = 1$, $p^1 = p$ which is given as odd. | **1** | |
| Assume $p^k$ is odd and consider $p^{k+1}$. | **1M** | |
| $p^{k+1} = p^k \times p$ | **1** | |
| Since $p^k$ is assumed to be odd and $p$ is odd, $p^{k+1}$ is the product of two odd integers is therefore odd. | **1** | for a valid explanation from a previous correct argument |
| Thus $p^{n+1}$ is an odd integer for all $n$ if $p$ is an odd integer. | | |

Source: 2010 Q12 AH Maths

(15)

Prove by contradiction that if $x$ is an irrational number, then $2 + x$ is irrational.

Answer:

| | | |
|---|---|---|
| Assume $2 + x$ is rational | **1** | |
| and let $2 + x = \frac{p}{q}$ where $p$, $q$ are integers. | **1** | |
| So $x = \frac{p}{q} - 2$ | | |
| $= \frac{p - 2q}{q}$ | **1** | as a single fraction |
| Since $p - 2q$ and $q$ are integers, it follows that $x$ is rational. This is a contradiction. | **1** | |

| Source: 2009 Q4 AH Maths | |
|---|---|
| (16) | Prove by induction that, for all positive integers $n$, $$\sum_{r=1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1}.$$ |

Answer:

When $n = 1$, LHS $= \frac{1}{1 \times 2} = \frac{1}{2}$, RHS $= 1 - \frac{1}{2} = \frac{1}{2}$. So true when $n = 1$. **1**

Assume true for $n = k$, $\sum_{r=1}^{k} \frac{1}{r(r+1)} = 1 - \frac{1}{k+1}$. **1**

Consider $n = k + 1$

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$ **1**

$$= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= 1 - \frac{k+2-1}{(k+1)(k+2)} = 1 - \frac{k+1}{(k+1)((k+1)+1)}$$ **1**

$$= 1 - \frac{1}{((k+1)+1)}$$ **1**

Thus, if true for $n = k$, statement is true for $n = k + 1$, and, since true for $n = 1$, true for all $n \geqslant 1$.

| Source: 2008 Q11 AH Maths | |
|---|---|
| (17) | For each of the following statements, decide whether it is true or false and prove your conclusion.<br>A For all natural numbers $m$, if $m^2$ is divisible by 4 then $m$ is divisible by 4.<br>B The cube of any odd integer $p$ plus the square of any even integer $q$ is always odd. |

Answers:

(a) Counter example $m = 2$. **1,1**

So statement is false.

(b) Let the numbers be $2n + 1$ and $2m$ then **1M**

$$(2n+1)^3 + (2m)^2 = 8n^3 + 12n^2 + 6n + 1 + 4m^2$$ **1**

$$= 2(4n^3 + 6n^2 + 3n + 2m^2) + 1$$ **1**

which is odd.

*OR*

Proving algebraically that either the cube of an odd number is odd or the square of an even number is even. **1**

Odd cubed is odd and even squared is even. **1**

So the sum of them is odd. **1**

| Source: 2007 Q12 AH Maths | |
|---|---|
| (18) | Prove by induction that for $a > 0$, $$(1 + a)^n \geq 1 + na$$ for all positive integers $n$. |

Answers:

| | |
|---|---|
| Consider $n = 1$, LHS $= (1 + a)$, RHS $= 1 + a$ so true for $n = 1$. | **1** |
| Assume that $(1 + a)^k \geqslant 1 + ka$ and consider $(1 + a)^{k+1}$. | **1** |
| $(1 + a)^{k+1} = (1 + a)(1 + a)^k$ | **1** |
| $\geqslant (1 + a)(1 + ka)$ | **1** |
| $= 1 + a + ka + ka^2$ | |
| $= 1 + (k + 1)a + ka^2$ | |
| $> 1 + (k + 1)a$ since $ka^2 > 0$ | **1** |

as required. So since true for $n = 1$, by mathematical induction statement is true for all $n \geqslant 1$.