Complex Numbers

AH Maths Exam Questions

Source: 2019 Specimen P1 Q5 AH Maths

(1) The complex number $z = 2 + i$ is a root of the polynomial equation $z^4 - 6z^3 + 16z^2 - 22z + 15 = 0$.

Find the remaining roots.

Answers:

•1 state second solution	•1 $z = 2 - i$
•2 create two linear factors	•2 $(z - 2 + i)$ and $(z - 2 - i)$
•3 create quadratic factor	•3 $z^2 - 4z + 5$
•4 set up algebraic division or equivalent	•4 $z^2 - 4z + 5 \overline{)\, z^4 - 6z^3 + 16z^2 - 22z + 15}$
•5 complete algebraic division	•5
•6 obtain remaining two solutions	•6 $z = 1 \pm \sqrt{2}\, i$

$$
\begin{array}{r}
z^2 - 2z + 3 \\
z^2 - 4z + 5 \overline{)\, z^4 - 6z^3 + 16z^2 - 22z + 15} \\
\underline{z^4 - 4z^3 + 5z^2 \qquad\qquad\quad} \\
-2z^3 + 11z^2 - 22z + 15 \\
\underline{-2z^3 + 8z^2 - 10z \qquad} \\
3z^2 - 12z + 15 \\
\underline{3z^2 - 12z + 15} \\
0
\end{array}
$$

Source: 2019 Specimen P2 Q7 AH Maths

(2)

Let $z=\sqrt{3}-i$.

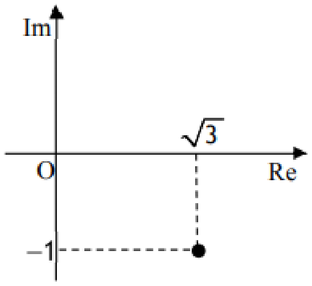
(a) Plot z on an Argand diagram.

(b) Let $w=az$ where $a>0$, $a\in\mathbb{R}$.

Express w in polar form.

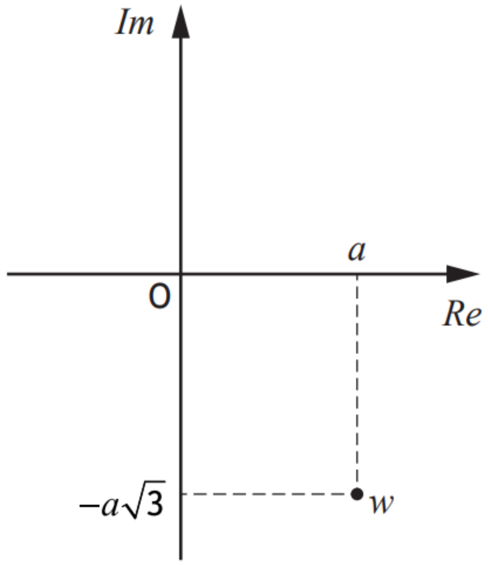
(c) Express w^8 in the form $ka^n\left(x+i\sqrt{y}\right)$ where $k,x,y\in\mathbb{Z}$.

Answers:

(a)		•[1] correctly plot z on Argand diagram	•[1] 
(b)		•[2] find modulus or argument	•[2] $\lvert w\rvert=2a$ or $\arg(w)=-\frac{\pi}{6}$
		•[3] express in polar form	•[3] $w=2a\left(\cos\left(-\frac{\pi}{6}\right)+i\sin\left(-\frac{\pi}{6}\right)\right)$
(c)		**Method 1**	
		•[4] process modulus	•[4] $256a^8$
		•[5] process argument	•[5] $\ldots\left(\cos\left(-\frac{8\pi}{6}\right)+i\sin\left(-\frac{8\pi}{6}\right)\right)$
		•[6] evaluate and express in form $ka^n\left(x+i\sqrt{y}\right)$	•[6] $w^8=128a^8\left(-1+i\sqrt{3}\right)$
(c)		**Method 2**	
		•[4] find w^2 and attempt to find a higher power of w	•[4] eg $w^2=a^2\left(2-2i\sqrt{3}\right)$ and $w^3=a^2\left(2-2i\sqrt{3}\right)\times a\left(\sqrt{3}-i\right)$.
		•[5] obtain w^4	•[5] $w^4=a^4\left(-8-8i\sqrt{3}\right)$
		•[6] complete expansion and express in form $ka^n\left(x+i\sqrt{y}\right)$	•[6] $w^8=128a^8\left(-1+i\sqrt{3}\right)$

Source: 2019 Q18 AH Maths

(3) The complex number w has been plotted on an Argand diagram, as shown below.



(a) Express w in

(i) Cartesian form

(ii) polar form.

(b) The complex number z_1 is a root of $z^3 = w$, where

$$z_1 = k\left(\cos\frac{\pi}{m} + i\sin\frac{\pi}{m}\right)$$

for integers k and m.

Given that $a = 4$,

(i) use de Moivre's theorem to obtain the values of k and m, and

(ii) find the remaining roots.

Answers:

(a)	(i)	•[1] write in Cartesian form	•[1] $a - a\sqrt{3}i$

(ii)	•² calculate modulus [1,6]	•² $2a$
	•³ calculate argument [2,3,4]	•³ $-\frac{\pi}{3}$
	•⁴ write in polar form [1,4,5,6]	•⁴ $2a\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)$

(b)	(i)	•⁵ begin process [1]	•⁵ $z_1=8^{\frac{1}{3}}\left(\cos\left(-\frac{\pi}{3}\right)+i\sin\left(-\frac{\pi}{3}\right)\right)^{\frac{1}{3}}$ stated or implied by •⁶
		•⁶ complete process [1]	•⁶ $z_1=8^{\frac{1}{3}}\left(\cos\left(-\frac{\pi}{9}\right)+i\sin\left(-\frac{\pi}{9}\right)\right)$
		•⁷ state value of k [1,2]	•⁷ $k=2$
		•⁸ state value of m [1,2]	•⁸ $m=-9$

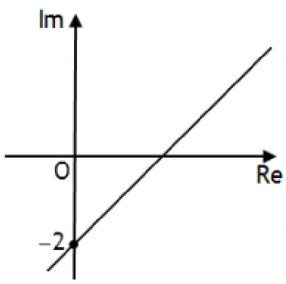
(ii)	•⁹ begin to add or subtract $\frac{2\pi}{3}$ to or from argument of z_1	•⁹ $\ldots\pm\frac{2\pi}{3}$ stated or implied by •¹⁰
	•¹⁰ state roots	•¹⁰ $z_2=2\left(\cos\frac{5\pi}{9}+i\sin\frac{5\pi}{9}\right)$ $z_3=2\left(\cos\left(-\frac{7\pi}{9}\right)+i\sin\left(-\frac{7\pi}{9}\right)\right)$

Source: 2018 Q4 AH Maths	
(4)	Given that $z_1 = 2 + 3i$ and $z_2 = p - 6i$, $p \in \mathbb{R}$, find: (a) $z_1\bar{z}_2$; (b) the value of p such that $z_1\bar{z}_2$ is a real number.
Answers: (a) $(2p - 18) + (3p + 12)i$ (b) $p = -4$	

Source: 2018 Q10 AH Maths	
(5)	Given $z = x + iy$, sketch the locus in the complex plane given by $\lvert z\rvert = \lvert z - 2 + 2i\rvert$.

Answer:

•¹ substitute, collect real and imaginary parts and equate moduli	•¹ $\lvert x + iy\rvert = \lvert (x-2) + (y+2)i\rvert$
•² process to obtain a linear equation in x and y	•² eg $y = x - 2$
•³ sketch consistent with equation [1,2]	•³ complete sketch



Other methods available – see Marking Scheme

Source: 2017 Q17 AH Maths

(6)

The complex number $z = 2 + i$ is a root of the polynomial equation $z^4 - 6z^3 + 16z^2 - 22z + q = 0$, where $q \in \mathbb{Z}$.

(a) State a second root of the equation.

(b) Find the value of q and the remaining roots.

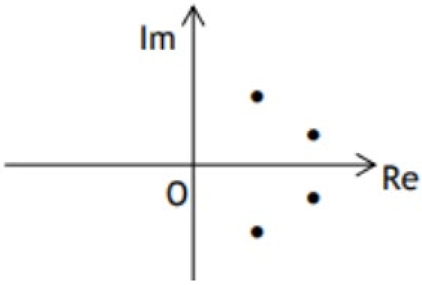
(c) Show the solutions to $z^4 - 6z^3 + 16z^2 - 22z + q = 0$ on an Argand diagram.

Answers:

(a) *second root is* $2 - i$

(b) $q = 15$, *remaining roots* $1 \pm \sqrt{2}i$

(c)



Source: 2016 Q8 AH Maths

(7)

Let $z = \sqrt{3} - i$.

(a) Plot z on an Argand diagram.

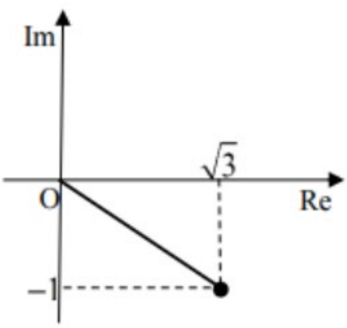
(b) Let $w = az$ where $a > 0$, $a \in \mathbb{R}$.

Express w in polar form.

(c) Express w^8 in the form $ka^n\left(x + i\sqrt{y}\right)$ where $k, x, y \in \mathbb{Z}$.

Answers:

(a)



(b) $w = 2a\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$

(c) $w^8 = 128a^8(-1 + i\sqrt{3})$

Source: 2015 Q13 AH Maths

(8) By writing z in the form $x + iy$:

(a) solve the equation $z^2 = |z|^2 - 4$;

(b) find the solutions to the equation $z^2 = i(|z|^2 - 4)$.

Answers:

(a) $z = \pm\sqrt{2}i$

(b) $z = 1 - i$ & $z = -1 + i$

Source: 2013 Q7 AH Maths

(9) Given that $z = 1 - \sqrt{3}\,i$, write down $\bar{z}$ and express $\bar{z}^2$ in polar form.

Answers:

$1 + \sqrt{3}\,i$

$4(cos\frac{2\pi}{3} + isin\frac{2\pi}{3})$

Source: 2013 Q10 AH Maths

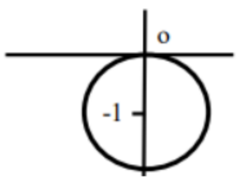
(10) Describe the loci in the complex plane given by:

(a) $|z + i| = 1$;

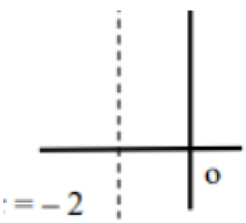
(b) $|z - 1| = |z + 5|$.

Answers:

(a) *Circle centre* $(0, -1)$ *radius* 1



(b) *Straight line with equation* $x = -2$



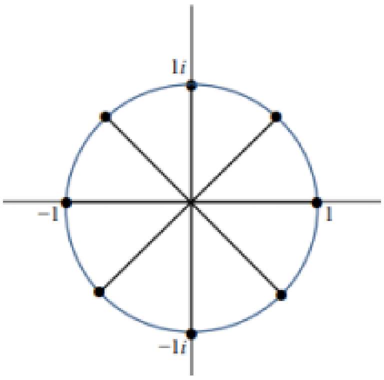
Source: 2014 Q16 AH Maths	
(11)	(*a*) Express −1 as a complex number in polar form and hence determine the solutions to the equation $z^4 + 1 = 0$. (*b*) Write down the four solutions to the equation $z^4 - 1 = 0$. (*c*) Plot the solutions of both equations on an Argand diagram. (*d*) Show that the solutions of $z^4 + 1 = 0$ and the solutions of $z^4 - 1 = 0$ are also solutions of the equation $z^8 - 1 = 0$. (*e*) Hence identify all the solutions to the equation $z^6 + z^4 + z^2 + 1 = 0.$

Answers:

(a) $z = cos\left(\frac{\pi}{4}\right) \pm isin\left(\frac{\pi}{4}\right), \quad cos\left(\frac{3\pi}{4}\right) \pm isin\left(\frac{3\pi}{4}\right)$

(b) $z = \pm i, \quad \pm 1$

(c)



(d) $z^8 - 1 = (z^4 + 1)(z^4 - 1)$

Then the solutions to $z^4 + 1 = 0$ and $z^4 - 1 = 0$ are also the solutions to $z^8 - 1 = 0$.

(e) Observe that $z^6 + z^4 + z^2 + 1 = (z^2 + 1)(z^4 + 1)$

OR

$$z^8 - 1 = (z^4 + 1)(z^2 + 1)(z^2 - 1)$$

$\therefore$ Six solutions are those above except $z = \pm 1$

Source: 2012 Q13 AH Maths

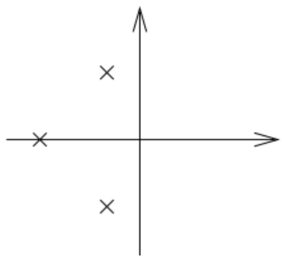
(12) Given that $(-1 + 2i)$ is a root of the equation

$$z^3 + 5z^2 + 11z + 15 = 0,$$

obtain all the roots.

Plot all the roots on an Argand diagram.

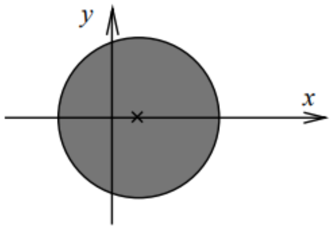
Answers:

Since w is a root, $\overline{w} = -1 - 2i$ is also a root.	**1**	for conjugate
The corresponding factors are $(z + 1 - 2i)$ and $(z + 1 + 2i)$		
from which $((z+1)-2i)((z+1)+2i) = (z+1)^2 + 4$		
$= z^2 + 2z + 5$	**1**	
$z^3 + 5z^2 + 11z + 15 = (z^2 + 2z + 5)(z + 3)$	**1**	evidence needed
The roots are $(-1 + 2i)$, $(-1 - 2i)$ and -3.	**1**	for stating roots together
	**1**	for two correct points
	1	for third correct point

Source: 2011 Q10 AH Maths	
(13)	Identify the locus in the complex plane given by $$\lvert z - 1 \rvert = 3.$$ Show in a diagram the region given by $\lvert z - 1 \rvert \le 3$.

Answers:

Let $z = x + iy$, so		
$z - 1 = (x - 1) + iy$.	1	
$\lvert z - 1 \rvert^2 = (x - 1)^2 + y^2 = 9$.	1	
The locus is the circle with centre $(1, 0)$ and radius 3.	1	Can subsume the first two marks.
	1	for circle
	1	for shading or other indication



Source: 2010 Q16 AH Maths	
(14)	Given $z = r(\cos\theta + i\sin\theta)$, use de Moivre's theorem to express z^3 in polar form. Hence obtain $\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)^3$ in the form $a + ib$. Hence, or otherwise, obtain the roots of the equation $z^3 = 8$ in Cartesian form. Denoting the roots of $z^3 = 8$ by z_1, z_2, z_3: (a) state the value $z_1 + z_2 + z_3$; (b) obtain the value of $z_1^6 + z_2^6 + z_3^6$.

Answers:

$z^3 = r^3(\cos 3\theta + i \sin 3\theta)$	1	
$(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})^3 = \cos 2\pi + i \sin 2\pi$	1	
$a = 1;\ b = 0$	1	
Method 1		
$r^3(\cos 3\theta + i \sin 3\theta) = 8$		
$r^3 \cos 3\theta = 8$ and $r^3 \sin 3\theta = 0$	1	
$\Rightarrow r = 2;\ 3\theta = 0,\ 2\pi,\ 4\pi$	1	See Marking Scheme for alternative method
Roots are $2,\ 2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}),\ 2(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3})$.	1	
In cartesian form: $2,\ (-1 + i\sqrt{3}),\ (-1 - i\sqrt{3})$	1	

Source: 2009 Q6 AH Maths	
(15)	Express $z = \frac{(1+2i)^2}{7-i}$ in the form $a + ib$ where a and b are real numbers. Show z on an Argand diagram and evaluate $\lvert z \rvert$ and $\arg(z)$.

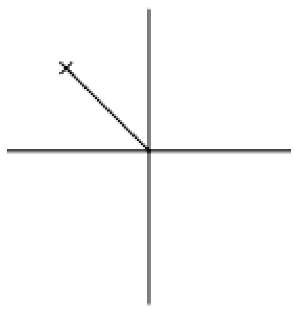
Answers:

$$\frac{(1+2i)^2}{7-i} = \frac{1+4i-4}{7-i}$$

$$= \frac{-3+4i}{7-i} \times \frac{7+i}{7+i}$$

$$= \frac{(-3+4i)(7+i)}{50}$$

$$= -\frac{1}{2} + \frac{1}{2}i$$



$$|z| = \sqrt{\frac{1}{4} + \frac{1}{4}} = \frac{1}{2}\sqrt{2}$$

$$\arg z = \tan^{-1}\frac{\frac{1}{2}}{-\frac{1}{2}} = \tan^{-1}(-1) = \frac{3\pi}{4} \text{ (or } 135^\circ\text{)}.$$

Source: 2008 Q16 AH Maths

(16) Given $z = \cos\theta + i\sin\theta$, use de Moivre's theorem to write down an expression for z^k in terms of θ, where k is a positive integer.

Hence show that $\dfrac{1}{z^k} = \cos k\theta - i\sin k\theta$.

Deduce expressions for $\cos k\theta$ and $\sin k\theta$ in terms of z.

Show that $\cos^2\theta\ \sin^2\theta = -\dfrac{1}{16}\left(z^2 - \dfrac{1}{z^2}\right)^2$.

Hence show that $\cos^2\theta\ \sin^2\theta = a + b\cos 4\theta$, for suitable constants a and b.

Answers:

$$z^k = \cos k\theta + i\sin k\theta,$$

$$\text{so } \frac{1}{z^k} = \frac{1}{\cos k\theta + i\sin k\theta} = \frac{\cos k\theta - i\sin k\theta}{\cos^2 k\theta + \sin^2 k\theta} = \cos k\theta - i\sin k\theta.$$

Adding the expressions for z^k and $\frac{1}{z^k}$ gives $z^k + \frac{1}{z^k} = 2\cos k\theta$ so $\cos k\theta = \frac{1}{2}(z^k + z^{-k})$.

Subtracting the expressions for z^k and $\frac{1}{z^k}$ gives $z^k - \frac{1}{z^k} = 2i\sin k\theta$ so $\sin k\theta = \frac{1}{2i}(z^k - z^{-k})$.

For $k = 1$

$$\cos^2\theta\ \sin^2\theta = (\cos\theta\ \sin\theta)^2$$
$$= \left(\frac{(z + \frac{1}{z})(z - \frac{1}{z})}{4i}\right)^2$$
$$= -\frac{1}{16}\left(z^2 - \frac{1}{z^2}\right)^2.$$

$$\left(z^2 - \frac{1}{z^2}\right)^2 = z^4 + \frac{1}{z^4} - 2 = 2\cos 4\theta - 2$$
$$\Rightarrow \cos^2\theta\ \sin^2\theta = \tfrac{1}{8} - \tfrac{1}{8}\cos 4\theta,$$

i.e. $a = \frac{1}{8}$ and $b = \frac{1}{8}$.

OR

A correct trigonometric proof that $\cos^2\theta\ \sin^2\theta = \frac{1}{8} - \frac{1}{8}\cos 4\theta$.

Source: 2007 Q3 AH Maths	
(17)	Show that $z = 3 + 3i$ is a root of the equation $z^3 - 18z + 108 = 0$ and obtain the remaining roots of the equation.

Answers:

$(3 + 3i)^3 = 27 + 81i + 81i^2 + 27i^3 = -54 + 54i$. Thus

$$(3 + 3i)^3 - 18(3 + 3i) + 108 =$$

$$-54 + 54i - 54 - 54i + 108 = 0$$

Since $3 + 3i$ is a root, $3 - 3i$ is a root.

These give a factor $(z - (3 + 3i))(z - (3 - 3i)) = (z - 3)^2 + 9 = z^2 - 6z + 18$.

$$z^3 - 18z + 108 = (z^2 - 6z + 18)(z + 6)$$

The remaining roots are $3 - 3i$ and -6.

Source: 2007 Q11 AH Maths	
(18)	Given that $\lvert z - 2 \rvert = \lvert z + i \rvert$, where $z = x + iy$, show that $ax + by + c = 0$ for suitable values of a, b and c. Indicate on an Argand diagram the locus of complex numbers z which satisfy $\lvert z - 2 \rvert = \lvert z + i \rvert$.

Answers:

$$\lvert z - 2 \rvert = \lvert z + i \rvert$$

$$\lvert (x - 2) + iy \rvert = \lvert x + (y + 1)i \rvert$$

$$(x - 2)^2 + y^2 = x^2 + (y + 1)^2$$

$$-4x + 4 = 2y + 1$$

$$4x + 2y - 3 = 0$$