Functions & Graphs

AH Maths Exam Questions

Source: 2019 Specimen P1 Q8 AH Maths

(1) A function is defined on a suitable domain by $f(x)=\dfrac{3x^2+2}{x^2-2}$.

(a) Obtain equations for the asymptotes of the graph of $y=f(x)$.

(b) Determine whether the graph of $y=f(x)$ has any points of inflection. Justify your answer.

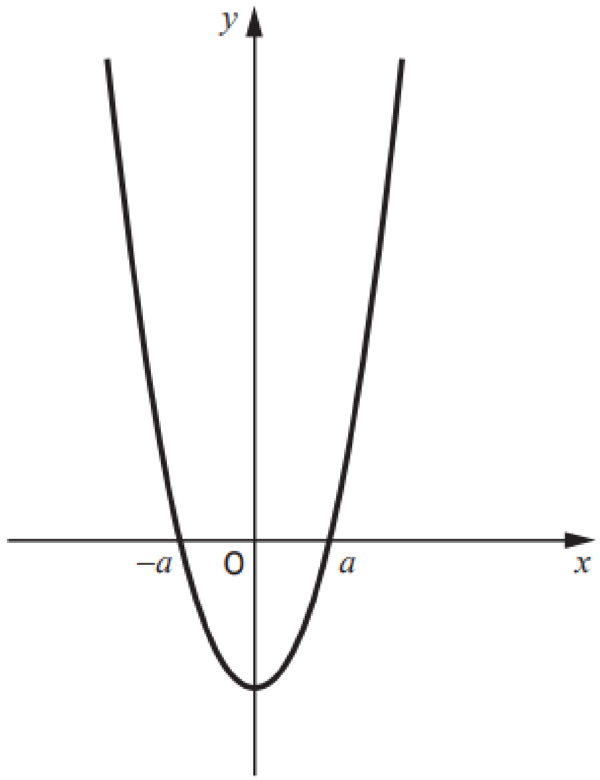
Answers:

(a)	state equations of vertical asymptotes	•1 $x=\sqrt{2},\ x=-\sqrt{2}$
	strategy for non-vertical asymptote	•2 $f(x)=3+\dfrac{8}{x^2-2}$ OR $f(x)=\dfrac{3+\frac{2}{x^2}}{1-\frac{2}{x^2}}$
	equation of non-vertical asymptote	•3 $y=3$, since $\dfrac{8}{x^2-2}\to 0$ as $x\to\pm\infty$ OR $f(x)\to\dfrac{3+0}{1-0}$ as $x\to\pm\infty$
(b)	first derivative	•4 $f'(x)=\dfrac{-16x}{(x^2-2)^2}$
	start second derivative	•5 $\dfrac{-16(x^2-2)^2-\ldots}{(x^2-2)^4}$ OR $\dfrac{\ldots-16x\times 2(x^2-2)\times 2x}{(x^2-2)^4}$
	complete second derivative and simplify	•6 $\dfrac{-16(x^2-2)^2+64x^2}{(x^2-2)^3}$
	show second derivative is never zero	•7 $32+48x^2\neq 0$
	consider the case where $f''(x)$ is undefined and state conclusion	•8 $f''(x)$ is undefined only when $x=\pm\sqrt{2}$. Since $f(x)$ is undefined at these values there is no point of inflection.

Source: 2019 Q3 AH Maths

(2)

The function $f(x)$ is defined by $f(x) = x^2 - a^2$. The graph of $y = f(x)$ is shown in the diagram.



(a) State whether $f(x)$ is odd, even or neither. Give a reason for your answer.

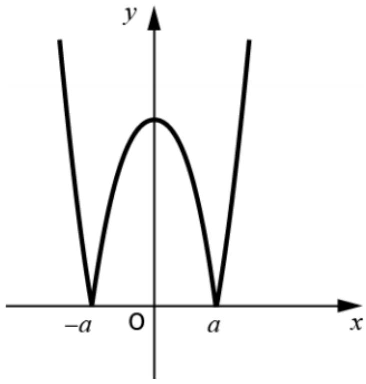
(b) Sketch the graph of $y = |f(x)|$.

Answers:

(a)

•[1] state why function is even [1,2,3,4,5,6]	•[1] graph is symmetrical about the y-axis $\therefore$ even OR $f(-x) = (-x)^2 - a^2 = x^2 - a^2 = f(x) \therefore$ even

(b)

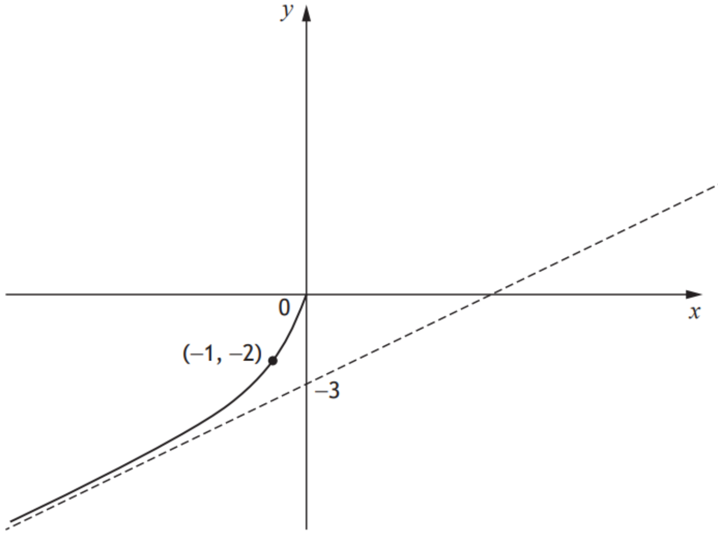


Source: 2017 Q12 AH Maths

(3)

In the diagram below part of the graph of $y = f(x)$ has been omitted.

The point (–1, –2) lies on the graph and the line $y = \frac{1}{2}x - 3$ is an asymptote.



(a) Copy and complete the diagram, including any asymptotes and any points you know to be on the graph.

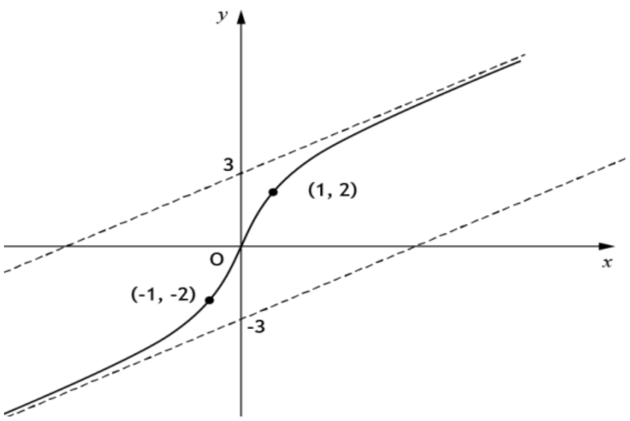
(b) $g(x) = |f(x)|$. On a separate diagram, sketch $g(x)$.

Include known asymptotes and points.

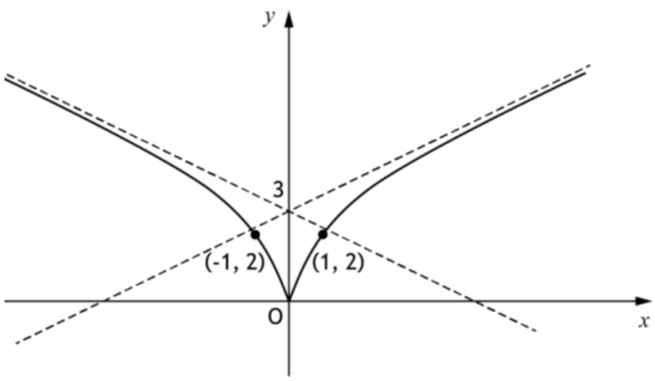
(c) State the range of values of $f'(x)$ given that $f'(0) = 2$.

Answers:

(a)



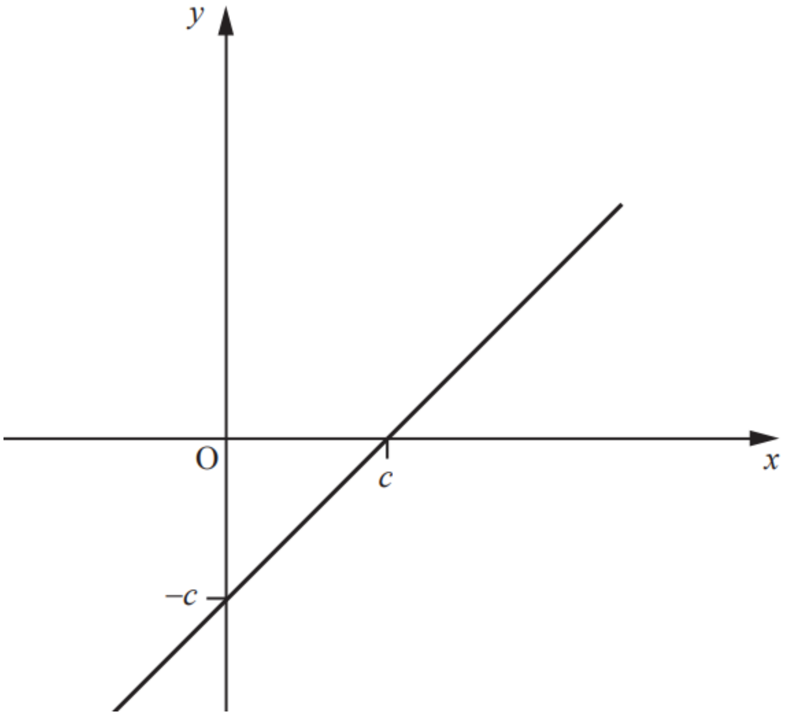
(b)



(c) $\frac{1}{2} < f'(x) \leq 2$

Source: 2016 Q12 AH Maths

(4) Below is a diagram showing the graph of a linear function, $y = f(x)$.



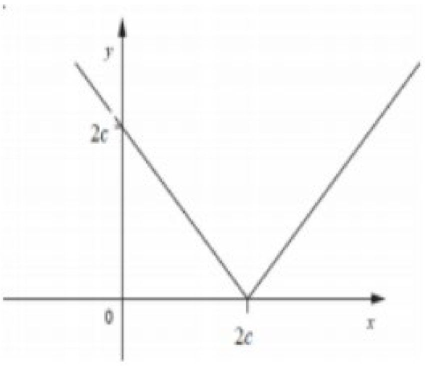
On separate diagrams show:

(a) $y = |f(x) - c|$

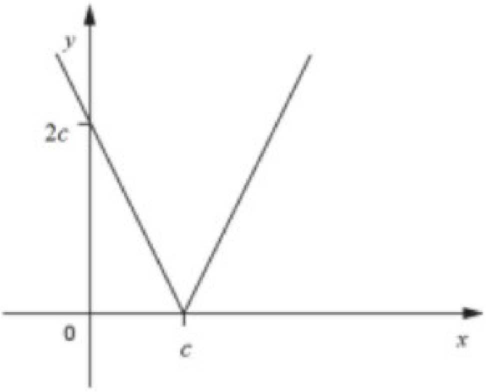
(b) $y = |2f(x)|$

Answers:

(a)



(b)



Source: 2015 Q14 AH Maths	
(5)	For some function, f, define $$g(x) = f(x) + f(-x) \quad \text{and}$$ $$h(x) = f(x) - f(-x).$$ Show that $g(x)$ is an even function and that $h(x)$ is an odd function. Hence show that $f(x)$ can be expressed as the sum of an even and an odd function.

Answers:

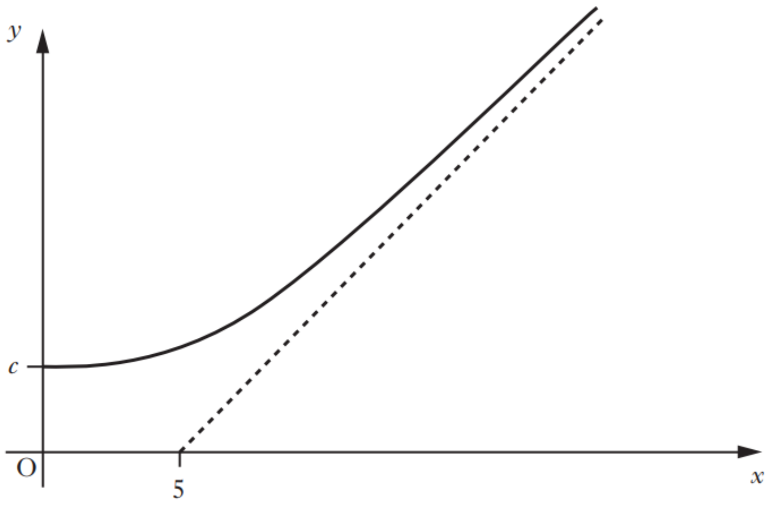
$g(x) = f(x) + f(-x)$ $g(-x) = f(-x) + f(x)$ $= f(x) + f(-x) = g(x)$ $\therefore$ since $g(-x) = g(x)$ function is even.	4	•[1] communicating knowledge of an even and an odd function[2]. •[2] showing that $g(x)$ is even.
$h(x) = f(x) - f(-x)$ $h(-x) = f(-x) - f(x)$ $= -f(x) + f(-x)$ $= -[f(x) - f(-x)]$ $= -h(x)$ $\therefore$ since $h(-x) = -h(x)$ function is odd.		•[3] showing that $h(x)$ is odd.
$g(x) + h(x) = 2f(x)$ by adding initial equations $f(x) = \frac{1}{2}g(x) + \frac{1}{2}h(x)$ $\therefore$ Since g even and h odd, $f(x)$ is the sum of an even and an odd functions.		•[4] correct expression *and* conclusion.

Source: 2014 Q11 AH Maths

(6) The function $f(x)$ is defined for all $x \geq 0$.

The graph of $y = f(x)$ intersects the y-axis at $(0, c)$, where $0 < c < 5$.

The graph of the function and its asymptote, $y = x - 5$, are shown below.



(*a*) Copy the above diagram.

On the same diagram, sketch the graph of $y = f^{-1}(x)$.

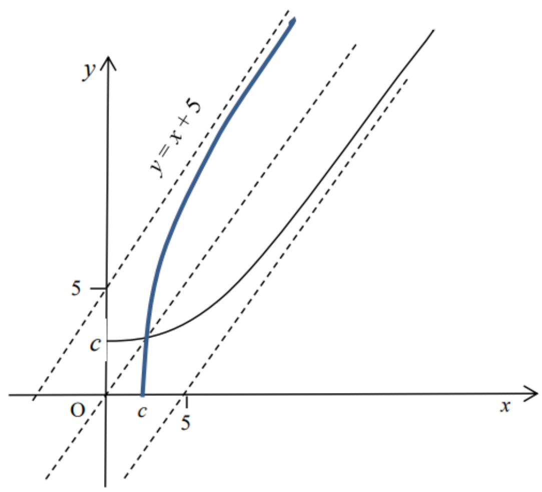
Clearly show any points of intersection and any asymptotes.

(*b*) What is the equation of the asymptote of the graph of $y = f(x + 2)$?

(*c*) Why does your diagram show that the equation $x = f(f(x))$ has at least one solution?

Answers:

(a)



(b) $y = x - 3$

(c) From the diagram, the two **curves/graphs** intersect

OR

$y = f(x)$ intersects $y = x$

OR

$y = f^{-1}(x)$ intersects $y = x$

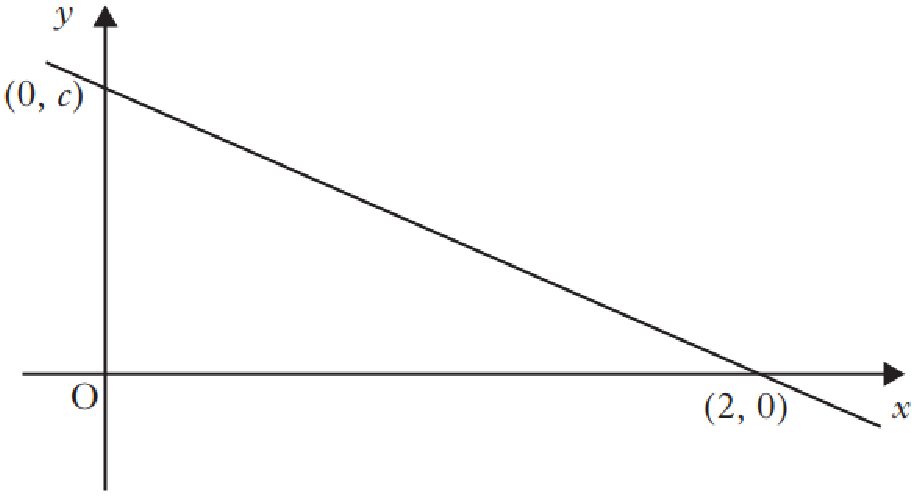
OR

$f^{-1}(x) = f(x)$

So $x = f(f(x))$

Source: 2013 Q13 AH Maths

(7) Part of the straight line graph of a function $f(x)$ is shown.



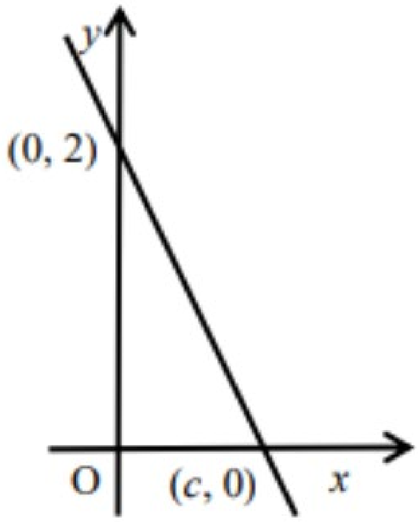
(a) Sketch the graph of $f^{-1}(x)$, showing points of intersection with the axes.

(b) State the value of k for which $f(x) + k$ is an odd function.

(c) Find the value of h for which $|f(x + h)|$ is an even function.

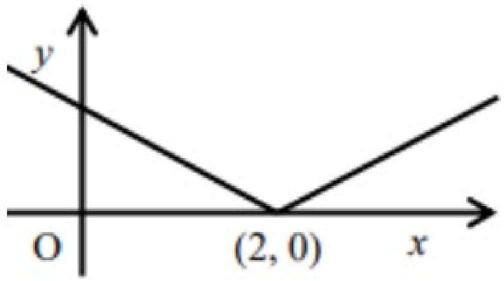
Answers:

(a)



(b) $y = f(x) - c$ is odd, therefore $k = -c$

(c)



$y = |f(x+2)|$ is even $\therefore$ $\boldsymbol{h = 2}$

Source: 2012 Q7 AH Maths

(8)

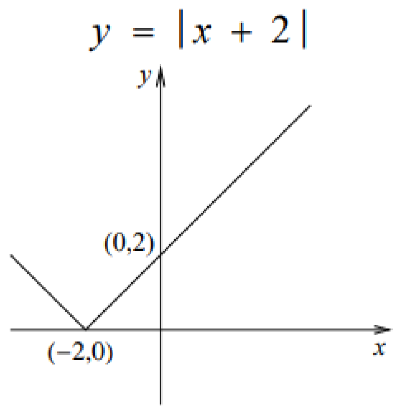
A function is defined by $f(x) = |x + 2|$ for all x.

(a) Sketch the graph of the function for $-3 \leq x \leq 3$.

(b) On a separate diagram, sketch the graph of $f'(x)$.

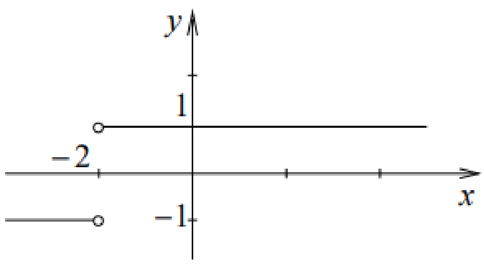
Answers:

(a)



1 for shape
1 for coordinates

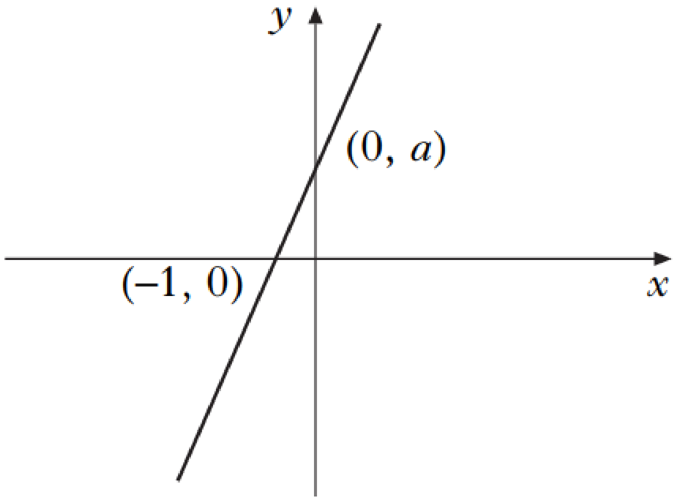
(b)



1 for both horizontal lines
1 for values: 1, −1, −2

Source: 2011 Q6 AH Maths

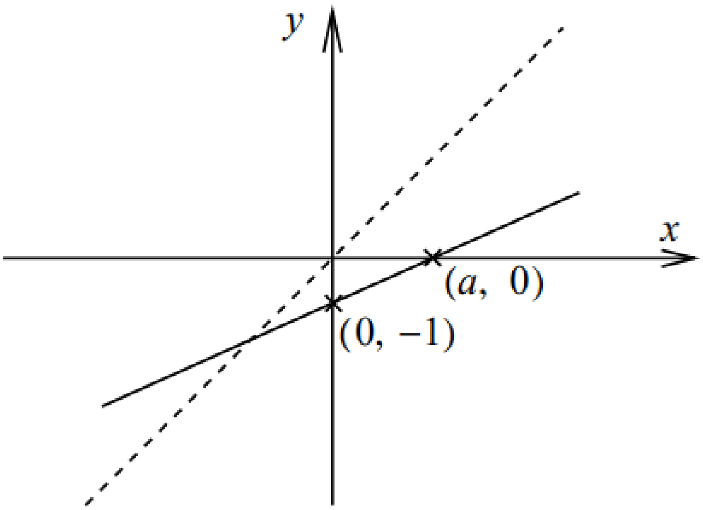
(9)



The diagram shows part of the graph of a function $f(x)$. Sketch the graph of $|f^{-1}(x)|$ showing the points of intersection with the axes.

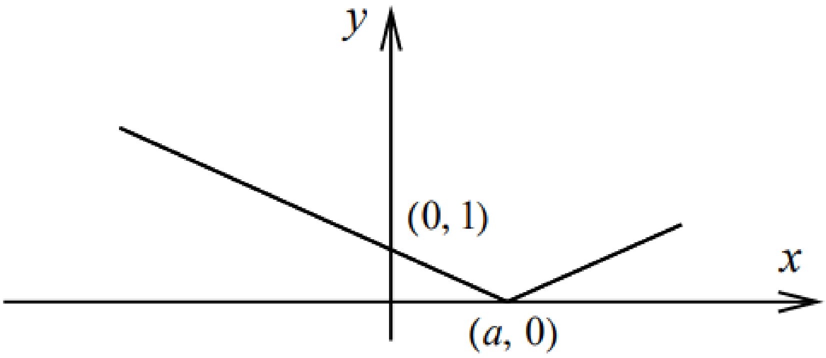
Answer:

Reflect in the line $y = x$ to get



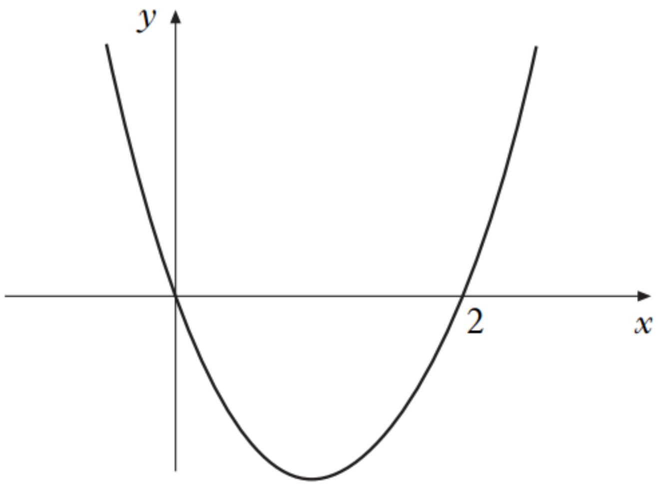
1 for position
1 for coordinates

Now apply the modulus function



1 for shape
1 for coordinates

Source: 2010 Q10 AH Maths	
(10)	The diagram below shows part of the graph of a function $f(x)$. State whether $f(x)$ is odd, even or neither. Fully justify your answer.



Answer:

The graph is not symmetrical about the y-axis (or $f(x) \neq f(-x)$) so it is not an even function.	**1**	
The graph does not have half-turn rotational symmetry (or $f(x) \neq -f(-x)$) so it is not an odd function.	**1**	
The function is neither even nor odd.	**1**	{*apply follow through*}

Source: 2009 Q13 AH Maths	
(11)	The function $f(x)$ is defined by $$f(x)=\frac{x^2+2x}{x^2-1} \qquad (x \neq \pm 1).$$ Obtain equations for the asymptotes of the graph of $f(x)$. Show that $f(x)$ is a strictly decreasing function. Find the coordinates of the points where the graph of $f(x)$ crosses (i) the x-axis and (ii) the horizontal asymptote. Sketch the graph of $f(x)$, showing clearly all relevant features.

Answer:

$$f(x) = \frac{x^2+2x}{x^2-1} = \frac{x^2+2x}{(x-1)(x+1)}$$

Hence there are vertical asymptotes at $x = -1$ and $x = 1$. **1**

$$f(x) = \frac{x^2+2x}{x^2-1} = \frac{1+\frac{2x}{x^2}}{1-\frac{1}{x^2}} = \frac{1+\frac{2}{x}}{1-\frac{1}{x^2}} \qquad \textbf{1}$$

$$\rightarrow 1 \text{ as } x \rightarrow \infty.$$

So $y = 1$ is a horizontal asymptote. **1**

$$f(x) = \frac{x^2+2x}{x^2-1}$$

$$f'(x) = \frac{(2x+2)(x^2-1)-(x^2+2x)2x}{(x^2-1)^2} \qquad \textbf{1}$$

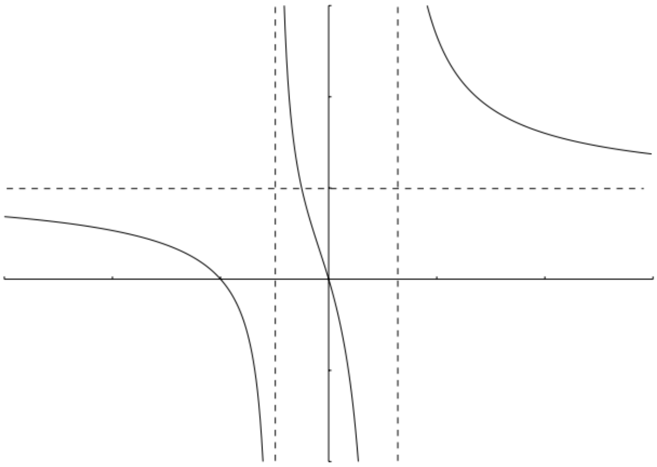
$$= \frac{2x^3-2x+2x^2-2-2x^3-4x^2}{(x^2-1)^2} = \frac{-2(x^2+x+1)}{(x^2-1)^2} \qquad \textbf{1}$$

$$= \frac{-2\left((x+\frac{1}{2})^2+\frac{3}{4}\right)}{(x^2-1)^2} < 0 \qquad \textbf{1}$$

Hence $f(x)$ is a strictly decreasing function.

$$f(x) = \frac{x^2+2x}{x^2-1} = 0 \Rightarrow x = 0 \text{ or } x = -2 \qquad \textbf{1}$$

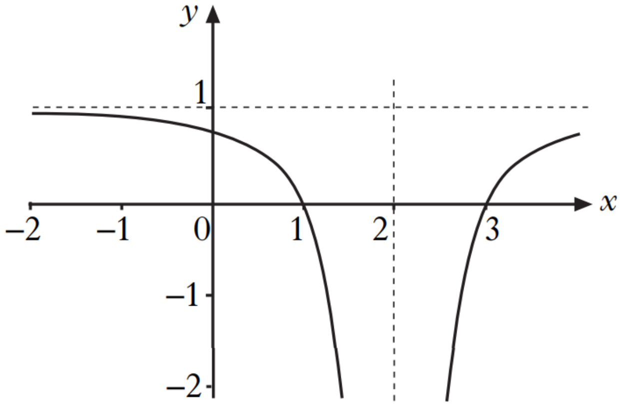
$$f(x) = \frac{x^2+2x}{x^2-1} = 1 \Rightarrow x^2+2x = x^2-1 \Rightarrow x = -\frac{1}{2} \qquad \textbf{1}$$



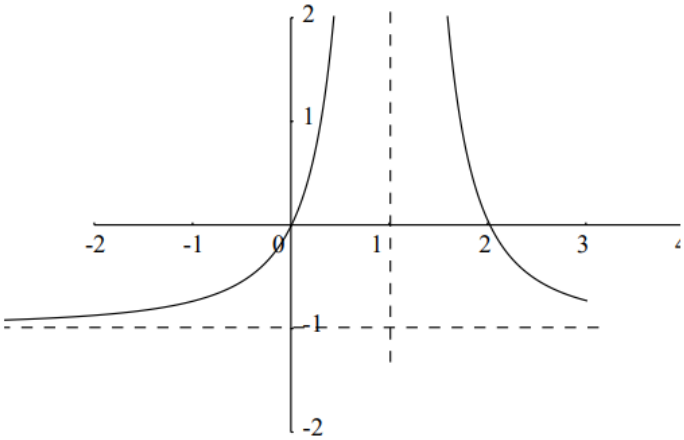
1 for shape
1 for position

Source: 2008 Q3 AH Maths

(12) Part of the graph $y = f(x)$ is shown below, where the dotted lines indicate asymptotes. Sketch the graph $y = -f(x + 1)$ showing its asymptotes. Write down the equations of the asymptotes.



Answer:



1 for inverting

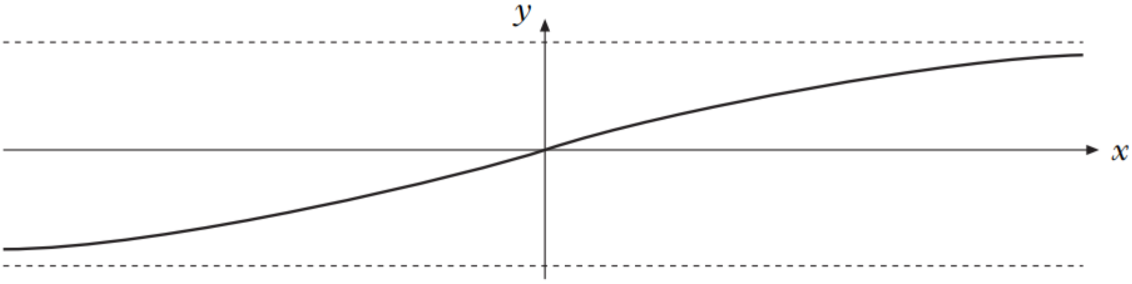
1 for translation

1 for showing asymptotes

Asymptotes are $y = -1$ and $x = 1$. **1**

Source: 2007 Q16 AH Maths

(13)



(*a*) The diagram shows part of the graph of $f(x) = \tan^{-1} 2x$ and its asymptotes. State the equations of these asymptotes.

(*b*) Use integration by parts to find the area between $f(x)$, the x-axis and the lines $x = 0$, $x = \frac{1}{2}$.

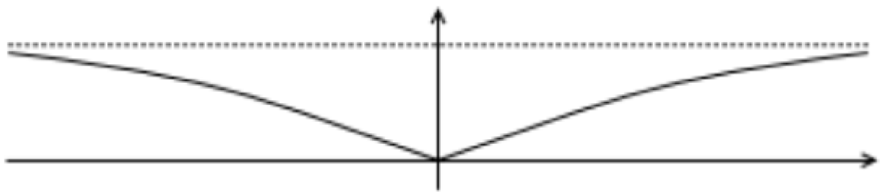
(*c*) Sketch the graph of $y = |f(x)|$ and calculate the area between this graph, the x-axis and the lines $x = -\frac{1}{2}$, $x = \frac{1}{2}$.

Answers:

(*a*) *Horizontal asymptote at* $y = \pm\frac{\pi}{2}$

(*b*) *Area* $= \frac{\pi}{8} - \frac{ln2}{4}$

(*c*)



Area $= \frac{\pi}{4} - \frac{ln2}{2}$