SCHOLAR Study Guide

SQA Advanced Higher Mathematics Unit 1

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Acknowledgements

Contents

Topic 1

Algebra

Contents

Prerequisite knowledge

A sound knowledge of the following techniques is required for this topic:

- *Expansion of brackets in algebraic terms.*

- *Manual fraction manipulation.*

Learning Objectives

- *Use algebraic skills.*

Minimum Performance Criteria:

- *Expand an expression of the form $(x + y)^n$*

- *Express a proper rational fraction as a sum of partial fractions where the denominator is a quadratic in factorised forms.*

1.1 Revision exercise

This exercise should help to identify any areas of weakness in techniques which are required for the study of this unit. Some revision may be necessary if any of the questions seem difficult.

Revision exercise

There is a similar exercise on the web if you would like to try it.

10 min

Q1: Expand $(2x + 3y)(x - 2y)$

Q2: Expand $(x + 4y)(2x - y)^2$

Q3: Factorise $2x^4 - 5x^2 - 12$

Q4: Express as a single fraction $^7/_8 + {}^3/_5$

Q5: Express as a single fraction $^2/_5 \times {}^3/_7 \times {}^{15}/_{16}$

Q6: Without a calculator using long division divide 8802 by 27

Q7: Without a calculator using long division divide 5344 by 16

1.2 Introduction to Binomial theorem

How easy is it to expand the expression $(x + y)^6$?

This section will demonstrate methods of making this task easier. It will also provide the techniques required to give the coefficient of a particular term in the expansion of the expression.

1.3 Factorials

Learning Objective

Use factorial notation

The following definition will be useful for this section.

n factorial

n! (called n factorial) is the product of the integers n, n - 1, n - 2, ..., 2, 1

That is, $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 2 \times 1$ for $n \in \mathbb{N}$

Example : Factorial value

What is the value of 6!

Answer:

6! = 6 $\times$ 5 $\times$ 4 $\times$ 3 $\times$ 2 $\times$ 1 = 720

Q8: What is 3!

Q9: What is 4!

Note that as n increases n! rapidly increases.

Calculator activity

5 min

Find the largest value of n that can be entered in a calculator as n! without giving an error message.

What is the value of this factorial?

Is this an accurate answer?

15! equals 1307674368000

Take this number and multiply it manually by 16

(use $\times$ 4 and then $\times$ 4 again if need be).

This should be equal to 16!

Check this carefully on a calculator. Are they exactly equal?

On my calculator the reading is 2.092278989 $\times$ 10^{13}. This is 20922789890000 but the calculation of multiplying 1307674368000 by 16 gives 209922789888000

These may be very large numbers with only a small difference but note that they are not the same.

This illustrates the need to be aware of the limitations of the calculator.

As n! is the product of the integers n, n - 1, . . ., 2, 1 and

(n - 1)! is the product of the integers (n - 1), (n - 2), . . ., 2, 1 it follows that

$$\frac{n!}{(n-1)!} = \frac{n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}{(n-1) \times (n-2) \times \cdots \times 2 \times 1} = n$$

This gives the following definition.

Factorial n formula one

n! = n $\times$ (n - 1)!

Example : (n+1)! in terms of n!

What is 9! in terms of 8!

Answer:

using n! = n × (n - 1)!

$$9! = 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 9 \times (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)$$
$$= 9 \times 8!$$

Q10: What is 8! in terms of 7!

Q11: What is 12! in terms of 11!

n! has been defined for n ∈ ℕ

By convention 0! is given the value 1

Zero factorial

0! = 1

Note that this still fits the rule n! = n × (n - 1)! as 1! = 1 and 1 × 0! = 1

1.4 Binomial coefficients

Learning Objective

Manipulate binomial coefficients

For integers n ∈ ℕ and $0 \leq r \leq n$,

the number given by $\frac{n!}{r!(n-r)!}$ is called a *Binomial Coefficient.*

It is denoted by $\begin{pmatrix} n \\ r \end{pmatrix}$

Binomial coefficient formula 1

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}$$

Example : Value of a binomial coefficient

What is the value of $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$?

Answer:

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \frac{7!}{2!(7 - 2)!} = \frac{7!}{2! \times 5!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{7 \times 6}{2 \times 1} = 21$$

This binomial coefficient $\begin{pmatrix} n \\ r \end{pmatrix}$ is also denoted ^nC_r and is used in another related maths topic called combinatorics.

The topic is not covered in this course but the term indicates the number of ways of choosing r elements from a set of n elements.

Example : $_nC^r$

How many ways can two chocolates be chosen from a box containing 20 chocolates?

$^{20}C_2 = \begin{pmatrix} 20 \\ 2 \end{pmatrix} = \frac{20 \times 19}{2 \times 1} = 190$

That is, 190 ways.

Q12: What is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$?

Q13: What is $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$?

Q14: What is $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$?

Q15: How many ways can four pupils be chosen from a group of 7 ?

Notice that $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$ and so $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

This illustrates the first rule for binomial coefficients.

Binomial coefficient formula 2
$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n-r \end{pmatrix}$

The proof of this can be found as proof (1) in the proof section near the back of the topic.

Q16: Find another binomial coefficient equal to $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

Q17: Find another binomial coefficient equal to $\begin{pmatrix} 21 \\ 17 \end{pmatrix}$

From the earlier questions $\begin{pmatrix} 5 \\ 3 \end{pmatrix} = 10$, $\begin{pmatrix} 5 \\ 4 \end{pmatrix} = 5$ and $\begin{pmatrix} 6 \\ 4 \end{pmatrix} = 15$

Therefore $\begin{pmatrix} 5 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

This illustrates the second rule for binomial coefficients.

Binomial coefficient formula 3

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

The proof of this can be found at proof (2) in the proof section near the back of this topic.

Q18: Write down $\binom{8}{6} + \binom{8}{7}$ as a binomial coefficient.

Q19: Write down $\binom{14}{11} + \binom{14}{12}$ as a binomial coefficient.

It is also possible to find 'n' given the value of a binomial coefficient and 'r'

Example If n is a positive integer such that $\binom{n}{2} = 15$ find n

Answer:

$$15 = \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n \times (n-1)}{2}$$

Hence $30 = n(n-1)$

$n^2 - n - 30 = 0$

$(n-6)(n+5) = 0$

The only solution which is a positive integer is $n = 6$

Q20: Find n when n is a positive integer such that $\binom{n}{2} = 10$

Q21: Find n when n is a positive integer such that $\binom{n}{2} = 21$

Q22: Find n when n is a positive integer such that $\binom{n}{2} = 36$

For further practice try the following selected questions which are based on the preceding work.

Q23: Evaluate 5!

Q24: Evaluate 0!

Q25: Evaluate 7!

Q26: What is 7! as a product of an integer and a factorial?

Q27: What is 100! as a product of an integer and a factorial?

Q28: What is 1! as a product of an integer and a factorial?

Q29: Evaluate $\begin{pmatrix} 8 \\ 4 \end{pmatrix}$

Q30: Evaluate $^6C_2$

Q31: Evaluate $\begin{pmatrix} 9 \\ 5 \end{pmatrix}$

Q32: Evaluate $\begin{pmatrix} 7 \\ 4 \end{pmatrix}$

Q33: Find another binomial coefficient equal to $\begin{pmatrix} 12 \\ 5 \end{pmatrix}$

Q34: Find another binomial coefficient equal to $\begin{pmatrix} 13 \\ 12 \end{pmatrix}$

Q35: Find another binomial coefficient equal to $\begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Q36: Write down $\begin{pmatrix} 3 \\ 2 \end{pmatrix} + \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ as a binomial coefficient.

Q37: Write down $\begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ as a binomial coefficient.

Q38: Write down $\begin{pmatrix} 12 \\ 9 \end{pmatrix} + \begin{pmatrix} 12 \\ 10 \end{pmatrix}$ as a binomial coefficient.

1.5 Pascal's triangle

Learning Objective

Complete entries in Pascal's triangle and relate them to binomial coefficients

Blaise Pascal was credited with discovering 'Pascal's triangle'.

It is made up of integers set out as a triangle.

The number 1 appears at the top and at each end of subsequent rows.

The numbers in the body of the triangle follow the rule:

'To find a number add the two numbers above it to the right and to the left of it'.

The rows are numbered from row 0.

Binomial coefficients interactive exercise

There is also a web version of this exercise.

10 min

Here are the first four rows of the triangle.

Q39: Complete rows 4 to 7 of the triangle.

row 0			1			
row 1		1		1		
row 2	1		2		1	
row 3	1	3		3		1
row 4						
row 5						
row 6						
row 7						

Q40: Create a table of the same design as Pascal's triangle only this time the entries are the binomial coefficients as shown.

Under this table create another table.

Enter the calculated values of these binomial coefficients.

binomial coefficients

row 0 $\binom{0}{0}$

row 1 $\binom{1}{0}$ $\binom{1}{1}$

row 2 $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

row 3 $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

values of binomial coefficients

row 0
row 1
row 2
row 3

Now complete the next four rows of the two tables.

binomial coefficients

row 4
row 5
row 6
row 7

values of coefficients

row 4
row 5
row 6
row 7

Now if row 9 is required, instead of writing out Pascal's triangle it is simply a matter of taking the binomial coefficients of row 9, namely,

$$\binom{9}{0}, \binom{9}{1}, \binom{9}{2}, \binom{9}{3}, \binom{9}{4}, \binom{9}{5}, \binom{9}{6}, \binom{9}{7}, \binom{9}{8} \text{ and } \binom{9}{9}$$

Either method can be used.

The entries in Pascal's triangle and the corresponding binomial coefficients are equal.

Recall that the second rule for binomial coefficients is $\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$

This is the definition of how to construct the entries in Pascal's triangle.

If the numbers in the n*th* row of Pascal's triangle are the binomial coefficients, then the r*th* entry in the next row, the row (n + 1), is the sum of the (r - 1)*th* entry and the r*th* entry in row n.

The r*th* entry in the (n + 1)*th* row is however $\binom{n+1}{r}$

Hence the entries in the (n + 1)*th* row of Pascal's triangle are also the binomial coefficients.

An example of this computation follows:

row 6	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
row 7	$\cdots$	$\cdots$	$\binom{7}{2}$	$\leftarrow sum \rightarrow$	$\binom{7}{3}$	$\cdots$
row 8	$\cdots$	$\cdots$	$\cdots$	$\binom{8}{3}$	$\cdots$	$\cdots$
row 9	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

1.6 Binomial theorem

Learning Objective

Use the binomial theorem in expansions

Look at the following expansions of $(x + y)^n$ for n = 0, 1, 2, 3

row 0	$(x + y)^0$	1
row 1	$(x + y)^1$	$x + y$
row 2	$(x + y)^2$	$x^2 + 2xy + y^2$
row 3	$(x + y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$

The coefficients still follow the same pattern as Pascal's triangle and the equivalent table of binomial coefficients.

coefficient of terms

$$1 \qquad\qquad 1$$
$$1x \; + \; 1y \qquad\qquad 1 \qquad 1$$
$$1x^2 \; + \; 2xy \; + \; 1y^2 \qquad\qquad 1 \qquad 2 \qquad 1$$
$$1x^3 \; + \; 3x^2y + \; 3xy^2 + \; 1y^3 \quad 1 \qquad 3 \qquad 3 \qquad 1$$

Q41: Expand $(x + y)^4$ by multiplying the $(x + y)^3$ expansion by $(x + y)$

Look carefully at the answer for the expansion of $(x + y)^4$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

The term $4x^3y$ can be written as $\binom{4}{1} x^{4-1}y^1$

The term y^4 can be written as $\binom{4}{4} x^{4-4}y^4$

This is an example of the binomial theorem and leads to two definitions: one for the expansion and one for any term within the expansion.

Binomial theorem

The Binomial Theorem states that if x, y $\in \mathbb{R}$ and n $\in \mathbb{N}$ then

$$(x + y)^n$$
$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + ... + \binom{n}{r} x^{n-r}y^r + ... + \binom{n}{n} y^n$$

General term of $(x + y)^n$

The general term of $(x + y)^n$ is given by

$$\binom{n}{r} x^{n-r}y^r$$

The proof of the Binomial Theorem can be found as proof (3) in the proof section near the back of this unit.

Examples

1. Use the binomial theorem to expand $(x + y)^5$

Answer:

$$(x + y)^5 = \binom{5}{0} x^5 + \binom{5}{1} x^4 y + \binom{5}{2} x^3 y^2 + \binom{5}{3} x^2 y^3 + \binom{5}{4} xy^4 + \binom{5}{5} y^5$$
$$= x^5 + 5x^4 y + 10x^3 y^2 + 10x^2 y^3 + 5xy^4 + y^5$$

2. Using the binomial theorem expand $(x + y)^7$

Answer:

$(x + y)^7$

$$= \binom{7}{0} x^7 + \binom{7}{1} x^6 y + \binom{7}{2} x^5 y^2 + ... + \binom{7}{5} x^2 y^5 + \binom{7}{6} xy^6 + \binom{7}{7} y^7$$
$$= x^7 + 7x^6 y + 21x^5 y^2 + 35x^4 y^3 + 35x^3 y^4 + 21x^2 y^5 + 7xy^6 + y^7$$

Binomial expansion exercise

There is a web exercise if you wish to try it.

20 min

Q42: Use the binomial theorem to expand $(x + y)^6$

Q43: Use the binomial theorem to expand $(x + y)^9$

Q44: Use the binomial theorem to expand $(x + y)^5$

Q45: Use the binomial theorem to expand $(x + y)^7$

Q46: Use the binomial theorem to expand $(x + y)^8$

The binomial theorem works also with multiples of x and y and other symbols (such as a, b or α, β).

Example Use the binomial theorem to expand $(2x + 3y)^4$

Answer:

$(2x + 3y)^4$

$$= \binom{4}{0} (2x)^4 + \binom{4}{1} (2x)^3 (3y) + \binom{4}{2} (2x)^2 (3y)^2 + \binom{4}{3} (2x)(3y)^3 + \binom{4}{4} (3y)^4$$
$$= 16x^4 + 96x^3 y + 216x^2 y^2 + 216xy^3 + 81y^4$$

Further binomial expansion exercise

There is a web exercise if you wish to try it.

15 min

Q47: Use the binomial theorem to expand $(-3x + y)^4$

Q48: Use the binomial theorem to expand $(-x - y)^5$

Q49: Use the binomial theorem to expand $(x - 2y)^2$

Rational coefficients are sometimes found as in the next example.

Example Use the binomial theorem to expand $(^1/_2x - y)^3$

Answer:

$$(^1/_2x - y)^3 = (^1/_2x + (-y))^3$$

$$= \binom{3}{0}(^1/_2x)^3 + \binom{3}{1}(^1/_2x)^2(-y) + \binom{3}{2}(^1/_2x)(-y)^2 + \binom{3}{3}(-y)^3$$

$$= {}^1/_8x^3 - {}^3/_4x^2y + {}^3/_2xy^2 - y^3$$

Harder binomial expansion exercise

There is also a web exercise if you prefer to try it.

15 min

Q50: Use the binomial theorem to expand $(^3/_4x + y)^3$

Q51: Use the binomial theorem to expand $(x - {}^1/_4y)^2$

Q52: Use the binomial theorem to expand $(x + {}^1/_3y)^4$

Mixed binomial expansion exercise

The following questions provide a mix of the types of expansions covered earlier.

20 min

Q53: Use the binomial theorem to expand $(2x - y)^3$

Q54: Use the binomial theorem to expand $(-x + 3y)^4$

Q55: Use the binomial theorem to expand $(x - y)^5$

Q56: Use the binomial theorem to expand $(^1/_2x - y)^3$

Q57: Use the binomial theorem to expand $(x + {}^2/_3y)^4$

Q58: Use the binomial theorem to expand $(^1/_4x + {}^1/_3y)^3$

1.7   Finding coefficients

Learning Objective

Find coefficients of particular terms in expansions

Sometimes only one power in an expansion is required.

By using the general term formula given earlier the coefficient of any term in an expansion can be found.

Examples

1. Find the coefficient of x^3y^4 in the expansion of $(x + y)^7$

Answer:

Start with the general term of $(x + y)^n$ which is $\binom{n}{r} x^{n-r}y^r$

This has coefficient $\binom{n}{r}$

In this problem n = 7 so the general term takes the form $\binom{7}{r} x^{7-r}y^r$

To give x^3y^4 requires 7 - r = 3 and so r = 4

The coefficient is therefore $\binom{7}{4} = 35$

2. Find the coefficient of xy^4 in the expansion of $(x - y)^5$

Answer:

The general term in this problem is $\binom{5}{r} x^{5-r}(-y)^r$

To give 5 - r = 1 requires r = 4

The coefficient is then $(-1)^4 \binom{5}{1} = 5$

3. Find the coefficient of x^5 in the expansion of $(x - \frac{3}{x})^7$

Answer:

The general term of $(x - \frac{3}{x})^7$ is

$$\binom{7}{r} x^{7-r} \left(\frac{-3}{x}\right)^r = \binom{7}{r} x^{7-r}(-3)^r(x^{-1})^r$$

This however can be simplified since both terms in this expression involve x.

This gives $\binom{7}{r} x^{7-r-r}(-3)^r = (-3)^r \binom{7}{r} x^{7-2r}$

To obtain x^5 requires n = 7 and 7 - 2r = 5, that is, r = 1

The coefficient is therefore $(-3)^1 \begin{pmatrix} 7 \\ 1 \end{pmatrix}$ = -21

4. Find the coefficient of x^2 in the expansion of $\left(x + \frac{2}{x}\right)^4$

Answer:

The general term in this problem is $\begin{pmatrix} 4 \\ r \end{pmatrix} x^{4-r}\left(\frac{2}{x}\right)^r = \begin{pmatrix} 4 \\ r \end{pmatrix} x^{4-2r}(2)^r$

To give 4 - 2r = 2 requires r = 1

The coefficient is then $(2) \begin{pmatrix} 4 \\ 1 \end{pmatrix}$ = 8

5. Find the coefficient of x^6 in the expansion of $(1 + x^2)^8$

Answer:

The general term is given by $\begin{pmatrix} 8 \\ r \end{pmatrix} (1)^{8-r}(x^2)^r = \begin{pmatrix} 8 \\ r \end{pmatrix} x^{2r}$

To obtain x^6 requires 2r = 6 that is r = 3

Therefore the coefficient is $\begin{pmatrix} 8 \\ 3 \end{pmatrix}$ = 56

6. Find the coefficient of x^4 in the expansion $(1 + 2x^2)^3$

Answer:

The general term in this problem is $\begin{pmatrix} 3 \\ r \end{pmatrix} (1)^{3-r}(2x^2)^r = \begin{pmatrix} 3 \\ r \end{pmatrix} (1)^{3-r})(2)^r x^{2r}$

To give 2r = 4 requires r = 2

The coefficient is then $(2)^2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ = 12

Finding coefficients exercise

There is a web exercise if you prefer to try it.

20 min

Q59: Find the coefficient of x^2y^4 in the expansion of $(x + 2y)^6$

Q60: Find the coefficient of x^9 in the expansion of $(1 + 3x^3)^4$

Q61: Find the coefficient of x^2 in the expansion of $\left(x + \frac{1}{x^2}\right)^5$

Q62: Find the coefficient of x^4y^3 in the expansion of $(x - y)^7$

Q63: Find the coefficient of x^2y^2 in the expansion of $(2x - y)^4$

Q64: Find the coefficient of x^6 in the expansion of $(1 + x^2)^8$

Q65: Find the coefficient of y^5 in the expansion of $\left(y - \frac{1}{y}\right)^5$

1.8   Sigma notation and binomial theorem applications

Learning Objective

Use the sigma notation and non algebraic applications of the binomial theorem

If a_0, a_1, . . . , a_n are real numbers, then the sum $a_0 + a_1 + . . . + a_n$ is sometimes written in shorthand form as $\sum\limits_{r=0}^{n} a_r$

Here the symbol $\sum$, called sigma, means ' take the sum of'.

The equation r = 0 at the bottom of the sigma sign means 'starting from r = 0'.

The letter n, at the top of the sigma sign means 'until r = n'.

The term a_r is the sequence of terms to be added together. Using this notation the binomial expansion can be written as

$(x + y)^n$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + . . . + \binom{n}{r} x^{n-r}y^r + . . . + \binom{n}{n} y^n$$

$$= \sum\limits_{r=0}^{n} \binom{n}{r} x^{n-r}y^r \text{ for r, n } \in N$$

Here is an example of the binomial theorem using the sigma notation.

Example Using the binomial theorem expand $(x + y)^5$

Answer:

$$(x + y)^5 = \sum\limits_{r=0}^{5} \binom{5}{r} x^{5-r}y^r$$

$$= \binom{5}{0} x^5 + \binom{5}{1} x^4y + \binom{5}{2} x^3y^2 + \binom{5}{3} x^2y^3 + \binom{5}{4} xy^4 + \binom{5}{5} y^5$$

$$= x^5 + 5x^4 + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$$

The binomial theorem can be used to find powers of a real number z.

The technique is to split z into two parts x and y where x is the closest integer to z and y is the remaining part.

For example, if z = 1.9 then it can be split as z = (2 - 0.1)

Examples

1. Using the binomial theorem find 1.04^3

Answer:

$1.04^3 = (1 + 0.04)^3$ (Here x = 1 and y = 0.04)

$$(1.04)^3 = \sum_{r=0}^{3} \binom{3}{r} 1^{3-r}(0.04)^r$$

$$= \binom{3}{0} 1^3 + \binom{3}{1} 1^2(0.04) + \binom{3}{2} 1^1(0.04)^2 + \binom{3}{3} (0.04)^3$$

$$= 1 + (3 \times 1 \times 0.04) + (3 \times 1 \times 0.0016) + (1 \times 0.000064)$$

$$= 1 + 0.12 + 0.0048 + 0.000064$$

$$= 1.124864$$

2. Using the binomial theorem find 0.7^4

Answer:

$0.7^4 = (1 + (-0.3))^4$

$$(0.7)^4 = \sum_{r=0}^{4} \binom{4}{r} 1^{4-r}(-0.3)^r$$

$$= \binom{4}{0} 1^4(-0.3)^0 + \binom{4}{1} 1^3(-0.3) + \binom{4}{2} 1^2(-0.3)^2 + \binom{4}{3} 1^1(0.3)^3 + \binom{4}{4} 1^0(-0.3)^4$$

$$= 1 + (4 \times 0.3) + (6 \times 0.09) + (4 \times 0.027) + 0.00811$$

$$= 0.2401$$

Binomial applications exercise

There is a web exercise if you wish to try it.

15 min

Q66: Using the binomial theorem expand and evaluate 2.03^3

Q67: Using the binomial theorem expand and evaluate 0.6^4

1.9 Introduction to partial fractions

> **Learning Objective**
>
> Use the terminology required for partial fractions

There are some very complex looking algebraic equations. To try to integrate them as they stand would be very difficult. In this section methods for splitting them into manageable terms are investigated.

The following definitions will help to make this section clearer.

> **Polynomial of degree n.**
>
> If $P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + ... + a_2 x^2 + a_1 x^1 + a_0$
>
> where $a_0, ..., a_n \in \mathbb{R}$
>
> then P is a polynomial of degree n.

Examples:

x + 3 has degree 1; x^2 - 2x + 3 has degree 2; $4x^3$ + $2x^2$ - 5 has degree 3 and of course a constant such as 7 has degree 0

Rational function

If P (x) and Q (x) are polynomials then $\frac{P(x)}{Q(x)}$ is called a rational function.

Proper rational function

Let P (x) be a polynomial of degree n and Q (x) be a polynomial of degree m.

If n < m then $\frac{P(x)}{Q(x)}$ is a proper rational function. For example $\frac{x^2 + 2x - 1}{x^3 - 3x + 4}$

Improper rational function

Let P (x) be a polynomial of degree n and Q (x) be a polynomial of degree m.

If n ≥ m then $\frac{P(x)}{Q(x)}$ is an improper rational function.

For example $\frac{x^3 + 2x - 1}{x^2 + 4}$ or $\frac{x^2 + 2x - 1}{x^2 + 4}$

An improper rational function, however, can always be expressed as a polynomial plus a proper fraction (by using long division).

Algebraic long division -

For practice in long division go now to the section headed 'Algebraic long division'.

1.10   Types of partial fractions

Learning Objective

Identify the types of partial fractions

Putting fractions over a common denominator,

for example,

$\frac{1}{x + 1}$ + $\frac{1}{x + 2}$ = $\frac{(x + 2) + (x + 1)}{(x + 1)(x + 2)}$ = $\frac{2x + 3}{(x + 1)(x + 2)}$ is a familiar process.

The opposite process, for example,

of expressing $\frac{2x + 3}{(x + 1)(x + 2)}$ as $\frac{1}{x + 1}$ + $\frac{1}{x + 2}$ is called putting a proper rational function into partial fractions.

Partial fractions

The process of taking a proper rational function and splitting it into separate terms each with a factor of the original denominator as its denominator is called expressing the function in partial fractions.

The way in which the rational function splits up depends on whether the denominator is a quadratic equation or a cubic equation.

It also depends on whether this denominator has linear, repeated linear or quadratic factors (with no real roots).

The different ways in which a proper rational function with a denominator of degree at most three can be split into partial fractions are now explained.

TYPE 1: $\frac{\text{linear or constant}}{\text{quadratic}}$

- Type 1a $\frac{\dots}{(x - a)(x + b)} = \frac{A}{x - a} + \frac{B}{x + b}$

This has a denominator of a quadratic with two distinct factors.

- Type 1b $\frac{\dots}{(x - a)^2} = \frac{A}{x - a} + \frac{B}{(x - a)^2}$

This has a denominator of a quadratic with repeated factors.

TYPE 2: $\frac{\text{quadratic, linear or constant}}{\text{cubic}}$

- Type 2a $\frac{\dots}{(x - a)(x - b)(x - c)} = \frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{x - c}$

This has a denominator of cubic with three distinct factors.

- Type 2b $\frac{\dots}{(x - a)(x - b)^2} = \frac{A}{x - a} + \frac{B}{x - b} + \frac{C}{(x - b)^2}$

This has a denominator of a cubic with one distinct and one repeated factor.

- Type 2c $\frac{\dots}{(x - a)^3} = \frac{A}{x - a} + \frac{B}{(x - a)^2} + \frac{C}{(x - a)^3}$

This has a denominator of a cubic with one repeated factor.

- Type 2d $\frac{\dots}{(x - a)(x^2 + bx + c)} = \frac{A}{x - a} + \frac{Bx + C}{x^2 + bx + c}$

This has a denominator of a cubic with two factors one a linear and one a quadratic (with no real roots).

The following section will examine each type with a worked example followed by an exercise.

Partial fraction formation

There are animations of the formation of each type of partial fraction on the web.

20 min

1.10.1   Partial fractions type 1a

Examples

1. Express $\dfrac{x+4}{(x^2-7x+10)}$ in partial fractions

Answer:

$\dfrac{x+4}{(x^2-7x+10)} = \dfrac{x+4}{(x-2)(x-5)}$

Let $\dfrac{x+4}{(x-2)(x-5)} = \dfrac{A}{x-2} + \dfrac{B}{x-5}$

$= \dfrac{A(x-5)}{(x-2)(x-5)} + \dfrac{B(x-2)}{(x-2)(x-5)}$

$x + 4 = A(x-5) + B(x-2)$

$1 = A + B$ (equating coefficients of x)

$4 = -5A - 2B$ (equating constants)

$A = -2$ and $B = 3$

STEP 1: Factorise the denominator

STEP 2: Identify the type of partial fraction. In this case it is type 1a

STEP 3: Obtain the fractions with a common denominator as at LHS

STEP 4: Equate numerators since the denominators are equal

STEP 5: Equate coefficients of the powers of x

STEP 6: Solve the equations simultaneously

Alternatively:

instead of STEP 5 use special values to remove one of the unknowns.

These are chosen so that $A(x-5) = 0$ and that $B(x-2) = 0$

Hence the values taken are $x = 5$ and $x = 2$

Let $x = 5$ then $9 = 3B$ so $B = 3$

Let $x = 2$ then $6 = -3A$ so $A = -2$

By either method the result follows $\dfrac{x+4}{(x^2-7x+10)} = \dfrac{-2}{x-2} + \dfrac{3}{x-5}$

2. Express $\dfrac{x+7}{x^2-x-2}$ in partial fractions.

Answer:

$\dfrac{x+7}{x^2-x-2} = \dfrac{x+7}{(x-2)(x+1)}$

STEP 1

Let $\dfrac{x+7}{(x-2)(x+1)} = \dfrac{A}{x-2} + \dfrac{B}{x+1}$

STEP 2

$= \dfrac{A(x+1)}{(x-2)(x+1)} + \dfrac{B(x-2)}{(x-2)(x+1)}$

STEP 3

$x + 7 = A(x+1) + B(x-2)$

STEP 4

$1 = A + B$

STEP 5

$7 = A - 2B$

$A = 3$ and $B = -2$

STEP 6

Alternatively:

Let x = -1   then 6 = -3B so B = -2

Let x = 2   then 9 = 3A so A = 3

therefore $\frac{x + 7}{(x^2 - x - 2)} = \frac{3}{x - 2} + \frac{-2}{x + 1}$

Partial fractions type 1a exercise

There is a web exercise if you prefer it.

20 min

Q68: Express $\frac{2x + 18}{x^2 + 2x - 15}$ in partial fractions.

Q69: Express $\frac{-7x + 11}{-x^2 + x + 6}$ in partial fractions.

Q70: Express $\frac{-x - 5}{x^2 - 1}$ in partial fractions.

Q71: Express $\frac{-4x - 10}{x^2 + 2x - 8}$ in partial fractions.

Q72: Express $\frac{5x - 13}{x^2 - 5x + 6}$ in partial fractions.

Extra Help: Partial Fractions 1

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

1.10.2 Partial fractions type 1b

Examples

1. Express $\frac{x + 3}{(x - 2)^2}$ in partial fractions.

Answer:

Let $\frac{x + 3}{(x - 2)^2} = \frac{A}{x - 2} + \frac{B}{(x - 2)^2}$ STEP 2   Note: STEP1 was not needed.

$= \frac{A(x - 2)}{(x - 2)^2} + \frac{B}{(x - 2)^2}$ STEP 3

x + 3 = A (x - 2) + B   STEP 4

    (equating powers of x)

1 = A

     STEP 5

3 = -2A + B   (equating constants)

A = 1 and B = 5   STEP 6

Alternatively:

let x = 2 then 5 = B

In this case there is only one special value.

This will eliminate A and give the value of B.

One of the equations in STEP 5 (in this case the second equation) can then be used to find A.

Namely 3 = -2A + 5 so A = 1

therefore $\frac{x + 3}{(x - 2)^2} = \frac{1}{x - 2} + \frac{5}{(x - 2)^2}$

2. Express $\frac{2x + 2}{(x + 3)^2}$ in partial fractions.

Answer:

Let $\frac{2x + 2}{(x + 3)^2} = \frac{A}{x + 3} + \frac{B}{(x + 3)^2}$ STEP 2

$= \frac{A(x + 3)}{(x + 3)^2} + \frac{B}{(x + 3)^2}$ STEP 3

$2x + 2 = A(x + 3) + B$ STEP 4

$2 = A$

                STEP 5

$2 = 3A + B$

$A = 2$ and $B = -4$ STEP 6

Alternatively:

Let $x = -3$ then $-4 = B$

Now use equation 2 in STEP 5 to obtain $A = 2$

Therefore $\frac{2x + 2}{(x + 3)^2} = \frac{2}{x + 3} + \frac{-4}{(x + 3)^2}$

Partial fraction type 1b exercise

There is a web exercise similar to this but with randomised questions if you prefer it.

20 min

Q73: Express $\frac{3x + 1}{(x + 1)^2}$ in partial fractions.

Q74: Express $\frac{2x - 7}{(x - 2)^2}$ in partial fractions.

Q75: Express $\frac{x - 3}{(x - 1)^2}$ in partial fractions.

Q76: Express $\frac{3x + 5}{(x + 2)^2}$ in partial fractions.

Q77: Express $\frac{2x + 11}{(x + 4)^2}$ in partial fractions.

1.10.3   Partial fractions type 2a

Examples

1. Express $\frac{x^2 - 13}{x^3 - 7x + 6}$ in partial fractions.

Answer:

$\frac{x^2 - 13}{x^3 - 7x + 6} = \frac{x^2 - 13}{(x - 1)(x - 2)(x + 3)}$ STEP 1

Let $\frac{x^2 - 13}{(x - 1)(x - 2)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x + 3}$ STEP 2

$= \frac{A(x - 2)(x + 3)}{(x - 1)(x - 2)(x + 3)} + \frac{B(x - 1)(x + 3)}{(x - 1)(x - 2)(x + 3)} + \frac{C(x - 1)(x - 2)}{(x - 1)(x - 2)(x + 3)}$

$x^2 - 13$

                STEP 4

$= A (x - 2)(x + 3) + B (x - 1)(x + 3) + C (x - 1)(x - 2)$

$1 = B + C$

$0 = A + 2B - 3C$ STEP 5

$-13 = -6A - 3B + 2C$

A = 3,

B = $-9/5$ STEP 6

and C = $-1/5$

Alternatively:

let x = 2   then -9 = 0 + 5B + 0 = 5B so B = $-9/5$

let x = 1   then -12 = -4A so A = 3

let x = -3   then -4 = 20C so C = $-1/5$

therefore $\frac{x^2 - 13}{x^3 - 7x + 6} = \frac{3}{x - 1} - \frac{9}{5(x - 2)} - \frac{1}{5(x + 3)}$

Note how much easier the alternative method is in this case.

It should be clear now that equating coefficients always works. Choosing special values to eliminate one unknown is quicker but not always possible.

In the example of type 1b a special value only existed for one of the unknowns.

The best method is to choose any special values that eliminate an unknown and work with equating coefficients for the rest.

This will reduce the work involved in solving simultaneous equations.

2. Express $\frac{-11x - 34}{x^3 + 5x^2 + 2x - 8}$ in partial fractions.

Answer:

$\frac{-11x - 34}{x^3 + 5x^2 + 2x - 8} = \frac{-11x - 34}{(x + 2)(x - 1)(x + 4)}$ STEP 1

Let $\frac{-11x - 34}{(x + 2)(x - 1)(x + 4)} = \frac{A}{x - 2} + \frac{B}{x - 1} + \frac{C}{x + 4}$ STEP 2

$= \frac{A(x - 1)(x + 4)}{(x + 2)(x - 1)(x + 4)} + \frac{B(x + 2)(x + 4)}{(x + 2)(x - 1)(x + 4)} + \frac{C(x + 2)(x - 1)}{(x + 2)(x - 1)(x + 4)}$

-11x - 34 =

STEP 4

A (x - 1)(x + 4) + B (x + 2)(x + 4) + C (x + 2)(x - 1)

Using the alternative method the special values can be used to eliminate each unknown.

let x = 1   then -45 = 0 + 15B + 0 = 15B so B = -3

let x = -2   then -12 = -6A so A = 2

let x = -4   then 10 = 10C so C = 1

therefore $\frac{-11 - 34}{x^3 + 5x^2 + 2x - 8} = \frac{2}{x + 2} - \frac{3}{x - 1} + \frac{1}{x + 4}$

Partial fractions type 2a exercise

There is a web exercise with random questions if you prefer it.

20 min

Q78: Express $\frac{5x^2 + 6x + 7}{x^3 - 2x^2 - x + 2}$ in partial fractions.

Q79: Express $\frac{6x^2 + 4x - 6}{x^3 - 7x - 6}$ in partial fractions.

Q80: Express $\frac{-7x + 9}{x^3 - 2x^2 - 9x + 18}$ in partial fractions.

Q81: Express $\frac{2x^2 - 3}{x^3 + 7x^2 + 14x + 8}$ in partial fractions.

1.10.4   Partial fractions type 2b

Examples

1. Express $\dfrac{4x^2 + 9}{x^3 + 4x^2 - 3x - 18}$ in partial fractions.

Answer:

$\dfrac{4x^2 + 9}{x^3 + 4x^2 - 3x - 18} = \dfrac{4x^2 + 9}{(x - 2)(x + 3)^2}$ STEP 1

Let $\dfrac{4x^2 + 9}{(x - 2)(x + 3)^2} = \dfrac{A}{x - 2} + \dfrac{B}{x + 3} + \dfrac{C}{(x + 3)^2}$ STEP 2

$= \dfrac{A(x + 3)^2}{(x - 2)(x + 3)^2} + \dfrac{B(x - 2)(x + 3)}{(x - 2)(x + 3)^2} + \dfrac{C(x - 2)}{(x - 2)(x + 3)^2}$ STEP 3

$4x^2 + 9 = A (x + 3)^2 + B (x - 2)(x + 3) + C (x - 2)$ STEP 4

Using a mix of methods

let x = -3   then 45 = -5C so C = -9
let x = 2   then 25 = 25A so A = 1
equate coefficients of x^2 4 = A + B so B = 3

therefore $\dfrac{4x^2 + 9}{x^3 + 4x^2 - 3x - 18} = \dfrac{1}{x - 2} + \dfrac{3}{x + 3} - \dfrac{9}{(x + 3)^2}$

2. Express $\dfrac{- 3x^2 + 6x + 20}{x^3 - x^2 - 8x + 12}$ in partial fractions.

Answer:

$\dfrac{- 3x^2 + 6x + 20}{x^3 - x^2 - 8x + 12} = \dfrac{- 3x^2 + 6x + 20}{(x + 3)(x - 2)^2}$ STEP 1

Let $\dfrac{- 3x^2 + 6x + 20}{(x + 3)(x - 2)^2} = \dfrac{A}{x + 3} + \dfrac{B}{x - 2} + \dfrac{C}{(x - 2)^2}$ STEP 2

$= \dfrac{A(x - 2)^2}{(x + 3)(x - 2)^2} + \dfrac{B(x + 3)(x - 2)}{(x + 3)(x - 2)^2} + \dfrac{C(x + 3)}{(x + 3)(x - 2)^2}$ STEP 3

$-3x^2 + 6x + 20 = A (x - 2)^2 + B (x + 3)(x - 2) + C(x + 3)$ STEP 4

Using a mix of methods

let x = -3   then -25 = 25A so A = -1
let x = 2   then 20 = 5C so C = 4
equate coefficients of x^2 - 3 = A + B so B = -2

therefore $\dfrac{- 11x - 34}{x^3 + 5x^2 + 2x - 8} = \dfrac{2}{x + 2} - \dfrac{3}{x - 1} + \dfrac{1}{x + 4}$

Partial fractions type 2b exercise

There is a web exercise which you may like to try.

20 min

Q82: Express $\dfrac{5x^2 + 6x + 7}{x^3 + 3x^2 - 4}$ in partial fractions.

Q83: Express $\dfrac{2x - 7}{x^3 - 3x^2 + 4}$ in partial fractions.

Q84: Express $\dfrac{6x^2 + 5x + 2}{x^3 - 3x - 2}$ in partial fractions.

Q85: Express $\dfrac{x^2 + 11x + 15}{x^3 + 3x^2 - 4}$ in partial fractions.

Extra Help: Partial Fractions 2

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

1.10.5   Partial fractions type 2c

Examples

1. Express $\frac{x^2 - 7x + 2}{x^3 - 3x^2 + 3x - 1}$ in partial fractions.

Answer:

$$\frac{x^2 - 7x + 2}{x^3 - 3x^2 + 3x - 1} = \frac{x^2 - 7x + 2}{(x - 1)^3} \qquad \text{STEP 1}$$

Let $\frac{x^2 - 7x + 2}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$ $\qquad$ STEP 2

$$= \frac{A(x - 1)^2}{(x - 1)^3} + \frac{B(x - 1)}{(x - 1)^3} + \frac{C}{(x - 1)^3} \qquad \text{STEP 3}$$

$x^2 - 7x + 2 = A (x - 1)^2 + B (x - 1) + C$ $\qquad$ STEP 4

Using a mix of methods

let x = 1 $\qquad\qquad$ then - 4 = C

equating coefficients of x^2 $\quad$ 1 = A

equating coefficients of x $\quad$ -7 = -2A + B

so -5 = B $\qquad\qquad$ A = 1, B = -5 and C = -4

therefore $\frac{x^2 - 7x + 2}{x^3 - 3x^2 + 3x - 1} = \frac{1}{x - 1} - \frac{5}{(x - 1)^2} - \frac{4}{(x - 1)^3}$

2. Express $\frac{2x^2 + 5x + 3}{x^3 + 6x^2 + 12x + 8}$ in partial fractions.

Answer:

$$\frac{2x^2 + 5x + 3}{x^3 + 6x^2 + 12x + 8} = \frac{2x^2 + 5x + 3}{(x + 2)^3} \qquad \text{STEP 1}$$

Let $\frac{2x^2 + 5x + 3}{(x + 2)^3} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2} + \frac{C}{(x + 2)^3}$ $\qquad$ STEP 2

$$= \frac{A(x + 2)^2}{(x + 2)^3} + \frac{B(x + 2)}{(x + 2)^3} + \frac{C}{(x + 2)^3} \qquad \text{STEP 3}$$

$2x^2 + 5x + 3 = A (x + 2)^2 + B (x + 2) + C$ $\qquad$ STEP 4

Using a mix of methods

let x = -2 $\qquad\qquad$ then 1 = C

equating coefficients of x^2 $\quad$ 2 = A

equating coefficients of x $\quad$ 5 = 4A + B

$\qquad\qquad\qquad$ so B = -3

therefore $\frac{2x^2 + 5x + 3}{x^3 + 6x^2 + 12x + 8} = \frac{2}{x + 2} - \frac{3}{(x + 2)^2} + \frac{1}{(x + 2)^3}$

Partial fractions type 2c exercise

There is a randomised exercise on the web if you prefer it.

20 min

Q86: Express $\frac{x^2 + 5x - 3}{(x - 2)^3}$ in partial fractions.

Q87: Express $\frac{x^2 - 2x + 3}{(x + 1)^3}$ in partial fractions.

Q88: Express $\frac{2x^2 + 7x + 8}{x^3 + 6x^2 + 12x + 8}$ in partial fractions.

Q89: Express $\frac{x^2 - x + 2}{(x - 1)^3}$ in partial fractions.

1.10.6   Partial fractions type 2d

Examples

1. Express $\frac{4x + 1}{x^3 - x^2 + x - 6}$ in partial fractions.

Answer:

$\frac{4x + 1}{x^3 - x^2 + x - 6} = \frac{4x + 1}{(x - 2)(x^2 + x + 3)}$ STEP 1

Let $\frac{4x + 1}{(x - 2)(x^2 + x + 3)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + x + 3}$ STEP 2

$= \frac{A(x^2 + x + 3)}{(x - 2)(x^2 + x + 3)} + \frac{(Bx + C)(x - 2)}{(x - 2)(x^2 + x + 3)}$ STEP 3

$4x + 1 = A(x^2 + x + 3) + (Bx + C)(x - 2)$ STEP 4

Using a mix of methods

let x = 2   then 9 = 9A so A = 1

equate coefficients of x^2 0 = A + B so B = -1

equate constants   1 = 3A - 2C so C = 1

therefore $\frac{4x + 1}{x^3 - x^2 + x - 6} = \frac{1}{x - 2} + \frac{-x + 1}{x^2 + x + 3}$

2. Express $\frac{-7x + 5}{x^3 - x^2 + x + 3}$ in partial fractions.

Answer:

$\frac{-7x + 5}{x^3 - x^2 + x + 3} = \frac{-7x + 5}{(x + 1)(x^2 - 2x + 3)}$ STEP 1

Let $\frac{-7x + 5}{(x + 1)(x^2 - 2x + 3)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 - 2x + 3}$ STEP 2

$= \frac{A(x^2 - 2x + 3)}{(x + 1)(x^2 - 2x + 3)} + \frac{(Bx + C)(x + 1)}{(x + 1)(x^2 - 2x + 3)}$ STEP 3

$-7x + 5 = A(x^2 - 2x + 3) + (Bx + C)(x + 1)$ STEP 4

Using a mix of methods

let x = -1   then 12 = 6A so A = 2

equate coefficients of x^2 0 = A + B so B = -2

equate constants   5 = 3A + C so C = -1

therefore $\frac{-7x + 5}{x^3 - x^2 + x + 3} = \frac{2}{x + 1} + \frac{-2x - 1}{x^2 - 2x + 3}$

Partial fractions type 2d exercise

There is a randomised exercise on the web.

20 min

Q90: Express $\frac{5x^2 - 4x + 1}{x^3 - 2x^2 - 2x + 1}$ in partial fractions.

Q91: Express $\frac{x + 4}{x^3 - 4x^2 + 7x - 6}$ in partial fractions.

Q92: Express $\frac{5x^2 + 9x + 14}{x^3 + x - 10}$ in partial fractions.

Q93: Express $\frac{x^2 + 9x}{2x^3 - 3x^2 - x - 2}$ in partial fractions.

Extra Help: Partial Fractions 3

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

1.10.7 Partial fractions type improper rational fractions

A variation on the previous problems occurs when the initial expression is an improper rational function.

In these circumstances it is necessary to divide through and obtain a polynomial and a rational function first.

This rational function can then be expressed as a sum of partial fractions using the appropriate method from the types given above.

Example Express $\frac{2x^3 - 7x^2 - 11x + 10}{(x + 2)(x - 4)}$ in partial fractions.

Answer:

Divide through as in the extended examples of long division.

[use $(2x^3 - 7x^2 - 11x + 10) \div (x^2 - 2x - 8)$] to give $2x - 3 - \frac{x + 14}{(x + 2)(x - 4)}$

Now express $\frac{x + 14}{(x + 2)(x - 4)}$ in partial fractions as before.

It is type 1a and gives $\frac{-2}{x + 2} + \frac{3}{x - 4}$

Hence $\frac{2x^3 - 7x^2 - 11x + 10}{(x + 2)(x - 4)} = 2x - 3 - \frac{-2}{x + 2} + \frac{3}{x - 4}$

Note carefully that in this example the minus sign after $2x - 3$ changes the sign of the second fraction.

Q94: Express $\frac{x^3}{(x + 1)(x + 2)}$ in partial fractions.

Q95: Express $\frac{x^3 + 3x^2 + 5x + 4}{(x + 1)(x + 2)}$ in partial fractions.

Extra Help: Partial Fractions 4

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

Partial fractions mixed exercise

To round off this section the following set of questions consists of a mix of the types that have been explained.

20 min

Q96: Express $\frac{10x^2 + 16x + 9}{2x^3 + 7x^2 + 5x + 6}$ in partial fractions.

Q97: Express $\frac{-7x - 26}{-x^2 - 8x - 12}$ in partial fractions.

Q98: Express $\frac{3x^2 - 8x + 1}{x^3 - 2x^2 - x + 2}$ in partial fractions.

Q99: Express $\frac{x^2 + x + 1}{x^3 + 4x^2 + 5x + 2}$ in partial fractions.

Q100: Express $\frac{4x - 6}{(x - 1)^2}$ in partial fractions.

Q101: Express $\frac{x^2 - 3x - 3}{(x - 2)^3}$ in partial fractions.

Q102: Express $\frac{2x^3 - 5x^2 - x + 3}{(x - 1)(x - 2)}$ in partial fractions.

Extra Help: Partial Fractions 5

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

1.11   Algebraic long division

Learning Objective

Use algebraic long division

Dividend

The dividend in a long division calculation is the expression which is being divided. As a fraction it is the numerator.

Divisor

The divisor is the expression which is doing the dividing.

It is the expression outside the division sign. As a fraction it is the denominator.

Quotient

The quotient is the answer to the division **but not including** the remainder.

Example 35 ÷ 8 = 4 r 3

The dividend is 35

The divisor is 8

The quotient is 4 and the remainder is 3

In long division style this is written as

$$8 \overline{)\begin{array}{l} 4\,3 \\ \overline{3\,5\,1} \\ 3\,2\downarrow \\ \overline{3\,1} \\ \underline{2\,4} \\ 7 \end{array}}$$ This would be written in fraction terms as 43 $^7/_8$

The same technique can be used for dividing polynomials.

Long Division strategy

An example of algebraic long division is available on the web.

10 min

Examples

1. Divide $x^3 - 2x + 5$ by $x^2 + 2x - 3$

Answer:

STEP 1: Lay out the division leaving gaps for 'missing terms'. Start with the highest power of x.

$$x^2 + 2x - 3 \overline{) x^3 \qquad - 2x + 5}$$

STEP 2: Divide the first term of the divisor (x^2) into the first of the dividend (x^3) and write the answer at the top (x).

$$x^2 + 2x - 3 \overline{) \overset{x}{x^3 \qquad - 2x + 5}}$$

STEP 3: Multiply each of the terms in the divisor by the first term of the quotient and write underneath the dividend.

$$x^2 + 2x - 3 \overline{) \overset{x}{x^3 \qquad - 2x + 5}}$$
$$x^3 + 2x^2 - 3x$$

STEP 4: Subtract to give a new last line in the dividend.

$$x^2 + 2x - 3 \overline{) \overset{x}{x^3 \qquad - 2x + 5}}$$
$$\underline{x^3 + 2x^2 - 3x}$$
$$- 2x^2 + x + 5$$

STEP 5: Divide the first term of the divisor (x^2) into the first term of the last line ($-2x^2$) and write the answer at the top.

$$x^2 + 2x - 3 \overline{) \overset{x - 2}{x^3 \qquad - 2x + 5}}$$
$$\underline{x^3 + 2x^2 - 3x}$$
$$- 2x^2 + x + 5$$

STEP 6: Multiply each of the terms in the divisor by the 2nd term of the quotient (-2) and write underneath the divisor.

$$x^2 + 2x - 3 \overline{) \overset{x - 2}{x^3 \qquad - 2x + 5}}$$
$$\underline{x^3 + 2x^2 - 3x}$$
$$- 2x^2 + x + 5$$
$$- 2x^2 - 4x + 6$$

STEP 7: Subtract to give a new last line in the dividend (5x-1). The division stops here in this case as the degree of the divisor (2) is greater than the degree of the last line (1).

$$x^2 + 2x - 3 \overline{) \overset{x - 2}{x^3 \qquad - 2x + 5}}$$
$$\underline{x^3 + 2x^2 - 3x}$$
$$- 2x^2 + x + 5$$
$$\underline{- 2x^2 - 4x + 6}$$
$$5x - 1$$

Therefore $\frac{x^3 - 2x + 5}{x^2 + 2x - 3} = x - 2 + \frac{5x - 1}{x^2 + 2x - 3}$

The (x - 2) is the quotient and the (5x - 1) is the remainder.

2. Divide $3x^3 - 2x^2 + 6$ by $x^2 + 4$.

$$x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,}$$

STEP 1: Lay out the division leaving gaps for 'missing terms'. Start with the highest power of x.

$$\begin{array}{r} 3x \\ x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,} \end{array}$$

STEP 2: Divide the first term of the divisor into the first of the dividend and write the answer at the top.

$$\begin{array}{r} 3x \\ x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,} \\ 3x^3 \qquad + 12x \end{array}$$

STEP 3: Multiply each of the terms in the divisor by the first term of the quotient and write underneath the dividend.

$$\begin{array}{r} 3x \\ x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,} \\ \underline{3x^3 \qquad + 12x} \\ - 2x^2 - 12x + 6 \end{array}$$

STEP 4: Subtract to give a new last line in the dividend.

$$\begin{array}{r} 3x - 2 \\ x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,} \\ \underline{3x^3 \qquad + 12x} \\ - 2x^2 - 12x + 6 \end{array}$$

STEP 5: Divide the first term of the divisor into the first term of the last line and write the answer at the top.

$$\begin{array}{r} 3x - 2 \\ x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,} \\ \underline{3x^3 \qquad + 12x} \\ - 2x^2 - 12x + 6 \\ - 2x^2 \qquad - 8 \end{array}$$

STEP 6: Multiply each of the terms in the divisor by the 2nd term of the quotient and write underneath the divisor.

$$\begin{array}{r} 3x - 2 \\ x^2 + 4 \overline{)\, 3x^3 - 2x^2 \qquad + 6\,} \\ \underline{3x^3 \qquad + 12x} \\ - 2x^2 - 12x + 6 \\ \underline{- 2x^2 \qquad - 8} \\ - 12x + 14 \end{array}$$

STEP 7: Subtract to give a new last line in the dividend. The division stops here in this case as the degree of the divisor (2) is greater than the degree of the last line (1).

Q103: Divide $3x^6 - 4x^3 + 9$ by $x^3 + 4$

Q104: Divide $x^5 - 2x^4 + 5x + 3$ by $x^2 - 2$

Q105: Divide $x^4 - 2x + 5$ by $x^2 + 4$

Q106: Find the remainder when $x^5 - 4x^3$ is divided by $x - 4$

Q107: Divide $x^3 + 3x^2 + 7$ by $x^2 - 2x$

1.12 Summary

At this stage the following topics and techniques should be known:

- The use of factorials.

- Pascal's Triangle construction.

- The use of binomial coefficients and related formula.

- The Binomial Theorem and its use in expanding expressions.

- How to find a particular coefficient given an expression.

- The degree of a polynomial.

- Proper and improper rational functions.

- Long division of polynomials by polynomials.

- The different types of partial fractions.

- How to express various rational functions as partial fractions.

1.13 Proofs

Proof 1

Prove that $\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n \\ n - r \end{pmatrix}$

This is the same as proving that $\begin{pmatrix} n \\ n - r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix}$

$$\text{LHS} = \begin{pmatrix} n \\ n - r \end{pmatrix}$$

$$= \frac{n!}{(n-r)!(n-(n-r))!}$$

$$= \frac{n!}{(n-r)!(n-n+r)!}$$

$$= \frac{n!}{(n-r)!(r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

$$= \begin{pmatrix} n \\ r \end{pmatrix}$$

$$= \text{RHS}$$

Proof 2

Prove that $\begin{pmatrix} n \\ r - 1 \end{pmatrix} + \begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} n + 1 \\ r \end{pmatrix}$

In this proof the following facts are needed:

1. $r! = r \times (r - 1)!$

2. $(n - r + 1)! = (n - r + 1) \times (n - r)!$

3. Two fractions can be combined over a common denominator.

$$\text{LHS} = \binom{n}{r-1} + \binom{n}{r}$$

$$= \frac{n!}{(r-1)!(n-r+1)!} + \frac{n!}{r!(n-r)!}$$

$$= \frac{r \times n!}{r(r-1)!(n-r+1)!} + \frac{(n-r+1) \times n!}{r!(n-r+1)(n-r)!}$$

$$= \frac{r \times n! + (n-r+1) \times n!}{r! \times (n-r+1)!}$$

$$= \frac{n! \times (r+n-r+1)}{r! \times (n-r+1)!}$$

$$= \frac{n! \times (n+1)}{r!(n+1-r)!}$$

$$= \frac{(n+1)!}{r!(n+1-r)!}$$

$$= \binom{n+1}{r}$$

$$= \text{RHS}$$

Proof 3

Prove the Binomial Theorem that if x, y $\in \mathbb{R}$ and n $\in \mathbb{N}$

$(x+y)^n$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1}y + \binom{n}{2} x^{n-2}y^2 + \ldots + \binom{n}{r} x^{n-r}y^r + \ldots + \binom{n}{n} y^n$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^{n-r}y^r \text{ for r, n } \in \text{N}$$

This proof is by induction.

It is shown for interest and the method of proof by induction will be covered later in the course.

Let n =1 then

LHS = $(x + y)^n = (x + y)^1 = x + y$

RHS = $\binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x + y$

So LHS = RHS and the theorem holds for n = 1

Now suppose that the result is true for n = k (where k $\geq$ 1)

Then

$$(x + y)^k = \binom{k}{0} x^k + \binom{k}{1} x^{k-1}y + \binom{k}{2} x^{k-2}y^2 + \ldots + \binom{k}{k-1} xy^{k-1} + \binom{k}{k} y^k$$

Consider

$$(x + y)^{k+1} = (x + y)(x + y)^k = x(x + y)^k + y(x + y)^k$$

$$= \binom{k}{0} x^{k+1} + \binom{k}{1} x^k y + \binom{k}{2} x^{k-1}y^2 + \ldots + \binom{k}{k-1} x^2 y^{k-1} + \binom{k}{k} xy^k$$

$$+ \binom{k}{0} x^k y + \binom{k}{1} x^{k-1}y^2 + \binom{k}{2} x^{k-2}y^3 + \ldots + \binom{k}{k-1} xy^k + \binom{k}{k} y^{k+1}$$

$$= \binom{k}{0} x^{k+1} + \left[\binom{k}{1} + \binom{k}{0}\right] x^k y + \left[\binom{k}{2} + \binom{k}{1}\right] x^{k-1}y^2 + \ldots$$

$$+ \left[\binom{k}{k-1} + \binom{k}{k-2}\right] x^2 y^{k-1} + \left[\binom{k}{k} + \binom{k}{k-1}\right] xy^k + \binom{k}{k} y^{k+1}$$

$$= 1x^{k+1} + \binom{k+1}{1} x^k y + \ldots + \binom{k+1}{k} xy^k + 1y^{k+1}$$

$$= \sum_{r=0}^{k+1} \binom{k+1}{r} x^{k+1-r}y^r$$

Hence if it is true for n = k it is also true for n = k + 1.

However it was also true for n = 1

So it is true for all values of n

Remember that $\binom{r}{0} = \binom{r}{r} = 1$ for all r and n

and that $\binom{k}{j+1} + \binom{k}{j} = \binom{k+1}{j}$

(See binomial coefficient formula 3.)

1.14 Extended information

> **Learning Objective**
>
> Show an awareness of additional information available on this subject

The following may be of interest for further study.

PASCAL

Blaise Pascal (1632-1662) was a French mathematician, physicist and philosopher. He published the triangle in 'Traite du Triangle Arithmetique' in 1665 but did not claim recognition for it. His interest in the triangle arose from his study of the theory of probabilities linked to his gambling with his friend Fermat. He was the inventor of the first mechanical calculator and a programming language (pascal) is named after him.

CHU SHIH - CHIEH

This Chinese mathematician (1270-1330) published a version of the triangle in his 'Precious Mirror of the Four Elements' in 1303.

OPEN-ENDED CHALLENGE

Try to create a 3-d version of Pascal's Triangle using a triangular pyramid.

1.15 Review exercise

Review exercise

There are two web exercises if you prefer them.

30 min

Q108: Evaluate $\begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 4 \end{pmatrix}$

Q109: Using the binomial theorem expand $(x - 2y)^4$

Q110: What is the coefficient of x^2y^2 in the expansion of $(2x - 3y)^4$?

Q111: What is the coefficient of x^6 in the expansion of $(3x^2 + 1)^4$?

Q112: Express $\frac{5x - 8}{x^2 - 3x + 2}$ in partial fractions.

Q113: Express $\frac{-6x - 16}{x^2 + 4x + 3}$ in partial fractions.

Q114: Express $\frac{x^2 + 7x + 19}{x^3 + 3x^2 - 4}$ in partial fractions.

Q115: Express $\frac{2x^2 - 5x + 6}{(x - 3)^3}$ in partial fractions.

1.16 Advanced review exercise

Advanced review exercise

There are two web exercises if you prefer it.

30 min

Q116: Using the binomial theorem expand $\left(x - \frac{2}{x}\right)^6$

Q117: Find the term independent of x in the expansion of $\left(\frac{3}{x^2} - 2x\right)^6$

Q118: What is the fifth term in the expansion of $\left(2x - \frac{1}{x}\right)^9$

Q119: Express $\frac{4x^3 - 16x^2 + 20x - 11}{x^2 - 4x + 4}$ in partial fractions.

Q120: Express $\frac{2x^4 - 2x^3 - 2x^2 - 15x + 20}{x^2 - 3x + 2}$ in partial fractions.

Q121: Express $\frac{-3x^4 + 11x^3 - 17x^2 - 4x + 18}{x^3 - 2x^2 + 2x + 5}$ in partial fractions.

1.17   Set review exercise

Set review exercise

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers.

10 min

Q122: In the expansion of $(2x^3 - y^2)^7$, what value of 'r' should be used in the general term formula to find the coefficient of x^6y^{10}?

What is the value of this coefficient?

Q123: Evaluate $(1.93)^5$ by expanding using the Binomial Theorem.

Q124: Express $\frac{2x + 8}{x^2 + 6x + 9}$ in partial fractions.

Q125: Express $\frac{4x^2 + 15x + 23}{x^3 + 5x^2 + 11x + 7}$ in partial fractions.

Topic 2

Differentiation

Contents

Learning Objectives

Use the rules of differentiation on the elementary functions

x^n, ($n \in \mathbb{Q}$), sin x, cos x, exp x, ln x and their composites.

Minimum Performance Criteria:

- *Differentiate a product.*

- *Differentiate a quotient.*

- *Differentiate a simple composite function using the chain rule.*

2.1   Prerequisites

> **Learning Objective**
>
> Revision of previous knowledge.

You should recall some elementary properties of differentiation such as

1. If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

    If $f(x) = \sin x$ then $f'(x) = \cos x$

    If $f(x) = \cos x$ then $f'(x) = -\sin x$

2. If $f(x) = k\, g(x)$ then $f'(x) = k\, g'(x)$: (k is a constant)

3. If $f(x) = h(x) + g(x)$ then $f'(x) = h'(x) + g'(x)$

4. You should recognise the alternative Leibniz notation for the derivative i.e. that $\frac{dy}{dx} = f'(x)$. You should also know that the notation $\frac{d}{dx}(f(x))$ indicates that you should find the first derivative of $f(x)$

5. You should remember the chain rule. It is useful when finding the derivative of a more complicated expression. For example

    When $y(x) = f(g(x))$

    then $y'(x) = f'(g(x))\, g'(x)$

    In Leibniz notation we can write

    When $y = f(u)$ where $u = g(x)$ then

    $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

6. You should also be familiar with some fundamental applications of differentiating:

    *   Finding the equation of a tangent to a curve.
    *   Curve sketching and the nature of stationary points.
    *   Optimisation problems.

The English mathematician **Sir Isaac Newton** and the German mathematician **Gottfried Leibniz** each independantly developed theories regarding Differentiation. You can read more about these famous mathematicians in the Extended Information chapter near the end of this topic.

Study the following examples and make sure you understand the working. Note that you should always try to give answers in a similar form to the way the question is posed. Compare examples 1 and 2; the answer to example 1 is left as a negative power whereas the answer to example 2 is rewritten as a fraction.

Examples

1. $f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

2. $f(x) = \frac{1}{3x} = -\frac{1}{3} \times^{-2} =$

$-\frac{1}{3x^2}$

3.

$f(x) = \dfrac{3x^3 + 5}{x^2} = 3x + 5x^{-2}$

$f'(x) = 3 - 10x^{-3} = 3 - \dfrac{10}{x^3}$

4. Differentiate $y(x) = \sin(2x + 1)$

Answer

We use the chain rule with

$y = \sin(u) \Rightarrow \frac{dy}{du} = \cos(u)$

$u = 2x + 1 \Rightarrow \frac{du}{dx} = 2$

Hence

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\qquad = \cos(u) \times 2$

$\qquad = 2\cos(2x + 1)$

5. Differentiate $y(x) = \frac{1}{\sqrt{3x - 1}} = (3x - 1)^{-1/2}$

Answer

We use the chain rule with

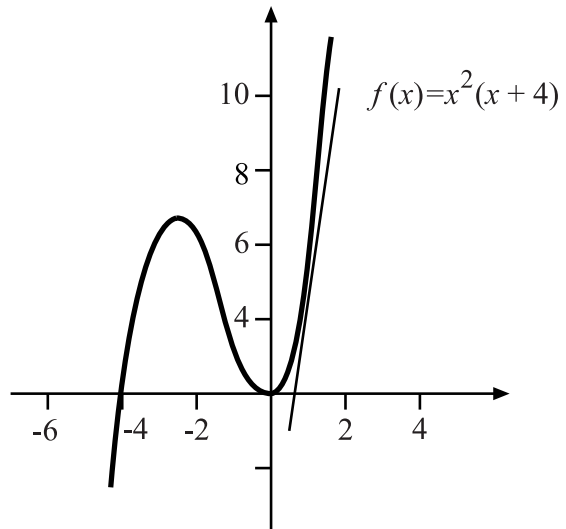$y = u^{-1/2} \Rightarrow \dfrac{dy}{du} = -\dfrac{1}{2}u^{-3/2}$

$u = 3x - 1 \Rightarrow \dfrac{du}{dx} = 3$

Hence

$\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}(3x - 1)^{-3/2}.3$

$\qquad = -\dfrac{3}{2}(3x - 1)^{-3/2}$

$\qquad = \dfrac{-3}{2\sqrt{3x - 1}^3}$

6. The equation of a tangent to the curve

Find the equation of the tangent to the
curve $f(x) = x^2(x + 4)$ at $x = 1$

$f(x) = x^2(x + 4)$

Answer

To do this we need to find a point on the tangent, and the gradient of the tangent.

At $x = 1$, $y = f(1) = 1 \times 5 = 5$

So the point (1, 5) lies on the tangent.

When $f(x) = x^2(x + 4) = x^3 + 4x^2$

then $f'(x) = 3x^2 + 8x$

Thus the gradient of the tangent at $x = 1$ is calculated from $f'(1) = 3 \times 1^2 + 8 \times 1 = 11$

Hence using the straight line equation in the form $y - b = m(x - a)$a, the line through

(1, 5) with gradient 11 has equation

$y - 5 = 11(x - 1)$

$y = 11x - 6$

Therefore the equation of the tangent to $f(x) = x^2(x + 4)$ at $x = 1$ is $y = 11x - 6$

7. Curve sketching

Sketch the curve $f(x) = x^3 + 3x^2$

Answer

In order to sketch this curve we need to find

- the y-intercept,
- the x-intercept,
- the stationary points and their nature,
- the behaviour of the curve for large positive and negative values of x.

The y-intercept

The curve cuts the y-axis when $x = 0$ and $f(0) = 0$

Therefore the curve cuts the y-axis at (0, 0)

The x-intercept

The curve cuts the x-axis when y = 0

so when $x^3 + 3x^2 = 0$ then

$x^2 (x + 3) = 0$

and x = 0 or x = -3

Therefore the curve cuts the x-axis at (0, 0) and (-3, 0)

Stationary points

When $f(x) = x^3 + 3x^2$ then $f'(x) = 3x^2 + 6x$

At stationary values $f'(x) = 0$

therefore $3x^2 + 6x = 0$

$3x(x + 2) = 0$

x = 0 or x = -2

When x = 0, f(0) = 0 and when x = -2, f(-2) = 4

So there are stationary points at (0, 0) and (-2, 4)

x	-2^-	-2	-2^+
f'(x)	+	0	-
slope	↗	→	↘

x	-2^-	-2	-2^+
f'(x)	-	0	+
slope	↘	→	↗

From the tables shown here we can see that there is a maximum turning point at (-2,4) and there is a minimum turning point at (0, 0)

Large values of x

As x ⇒ -∞, $x^3 + 3x^2$ ⇒ -∞

As x ⇒ ∞, $x^3 + 3x^2$ ⇒ ∞

The sketch of the curve

Notice that stationary points and points where the curve cuts the axes are clearly labelled. The graph is then as shown here:

Now try the questions in Exercise 1

Revision Exercise 1

30 min

An on-line version of this exercise is available. It contains randomised questions which are similar to those below.

Q1:   Differentiate the following with respect to the relevant variable, simplifying where possible.
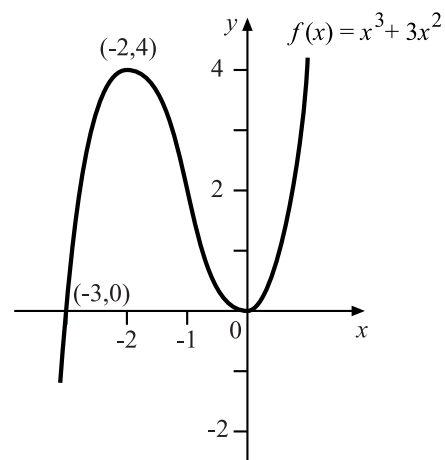
a) $f(t) = 3t^{-4} + 2t - 5$

b) $f(x) = 5x^2 + 3\cos x$

c) $f(x) = \dfrac{5}{3\sqrt{x}}$

d) $f(w) = \sqrt{w}(w + \sqrt{w})$

e) $f(u) = \dfrac{u^4 - 3u^2 + 7}{u^2}$

f) $f(x) = (4x + 3)^5$

g) $f(\theta) = \sin\left(5\theta + \dfrac{\pi}{4}\right)$

Q2:   Find the equation of the tangent to the curve $y = \cos 3\theta$ at $\theta = \pi/6$

Q3:   Sketch and clearly annotate the curve $f(x) = x^3 - 6x^2 + 9x$
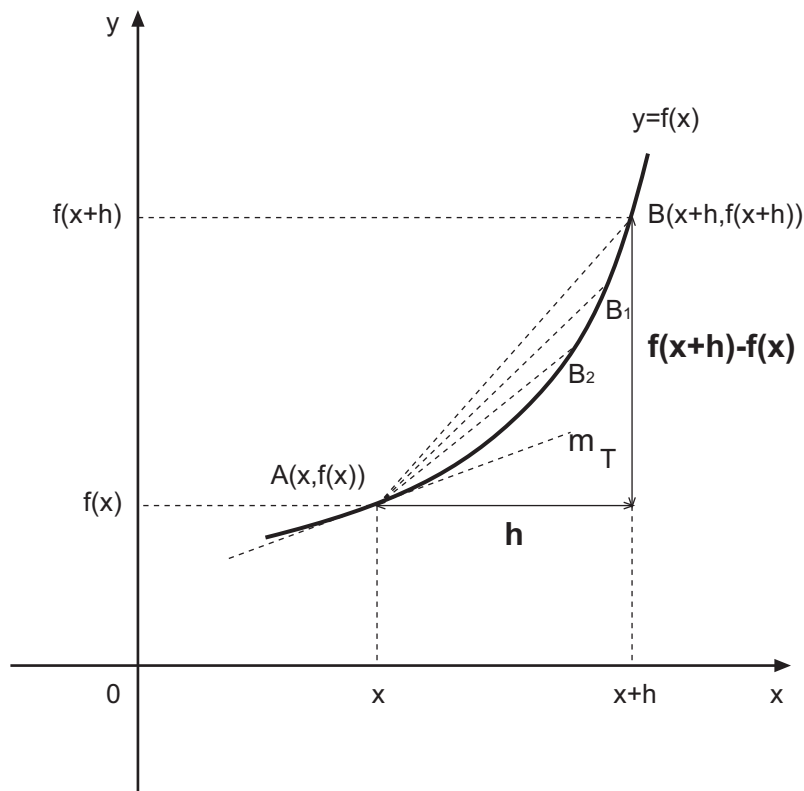
2.2   The Derivative

Learning Objective

Understand that the derivative represents the gradient of the tangent of a function at a point, and so the rate of change of the function

A reminder:

Suppose we wish to find the gradient of a curve $y = f(x)$ at a point A $(x, f(x))$

You should remember that the gradient of the curve at A = gradient of the tangent of the curve at A = m_T

An estimation of the gradient at A can be found by taking a second point

B $(x + h, f(x + h))$ and calculating m_{AB}, the gradient of the chord AB. Consider h to be small then

$m_{AB} = \frac{y_B - y_A}{x_B - x_A} = \frac{f(x + h) - f(x)}{h}$

If we move B closer and closer to A, say to points B_1, B_2 and B_3 then the gradient of the chords AB_1, AB_2 and AB_3 will give better and better approximations for the gradient of the curve at A.

If m_{AB_n} tends to a limit value as B_n approaches A then this value is denoted by f $'$ (x) and we write

$$\text{f' } (x) = \lim_{B_n \to A} m_{AB_n} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

where $\lim_{h \to 0}$ means the limit value as h approaches 0.

Differentiation The process of finding the derivative f $'$ (x) is called **differentiation**

f $'$ (x) is the derivative of f at x. It represents the gradient of the tangent of the function at a point and so the rate of change of the function.

2.3   Differentiation 'from first principles'

> **Learning Objective**
>
> Understand the method for differentiating 'from first principles'

Example

Find, 'from first principles', (i.e. from the definition) the derivative of f with respect to x, when $f(x) = x^2$

Answer

Note that when $f(x) = x^2$ then $f(x + h) = (x + h)^2$

hence

$$
\begin{aligned}
\text{hence} \quad f'(x) &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{(x + h)^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h \to 0} \frac{h(2x + h)}{h} \\
&= \lim_{h \to 0} 2x + h \\
&= 2x
\end{aligned}
$$

Therefore when $f(x) = x^2$ then $f'(x) = 2x$ (as you knew already).

Now try the questions in Exercise 2

Exercise 2

10 min

There is an on-line version of this exercise , which you might find helpful.

Find, from 'first principles', the derivatives of these functions.

Q4: $f(x) = x^3$

Q5: $f(x) = 3x^2$

Q6: $f(x) = \sqrt{x}$

Hint: Multiply the formula for f'(x) by $\dfrac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$

2.3.1   Leibniz Notation

> **Learning Objective**
>
> Appreciation of an alternative notation for the derivative

The gradient of the curve as the limit of the gradients of the chords can also be expressed using Leibniz notation.

Here δx denotes a small change in x and δy denotes a corresponding change in y.

Then $m_{AB} = \frac{\Delta y}{\Delta x}$

The gradient of the tangent at A is

$m_T = \lim\limits_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

This is usually written as $\frac{dy}{dx}$
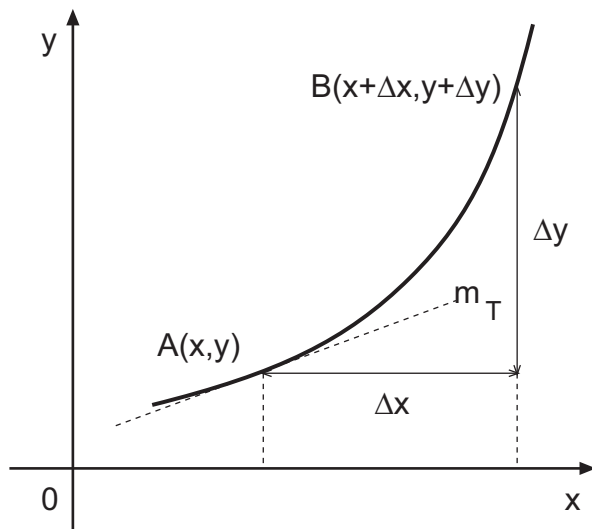
The Sky's The Limit

Learning Objective

Gain an appreciation of limits

15 min

Solve an equation involving a limit.

A simulation is available in the on-line materials to complement this section

2.4   Differentiability at a Point

Learning Objective

Understand when a function may or may not be differentiable at a point

In general a function f (x) is differentiable at x = a if the curve y = f (x) has a tangent at the point (a, f (a))

Look at these diagrams:

(a)   (b)   (c)

A Tangent   No Tangent   No Tangent
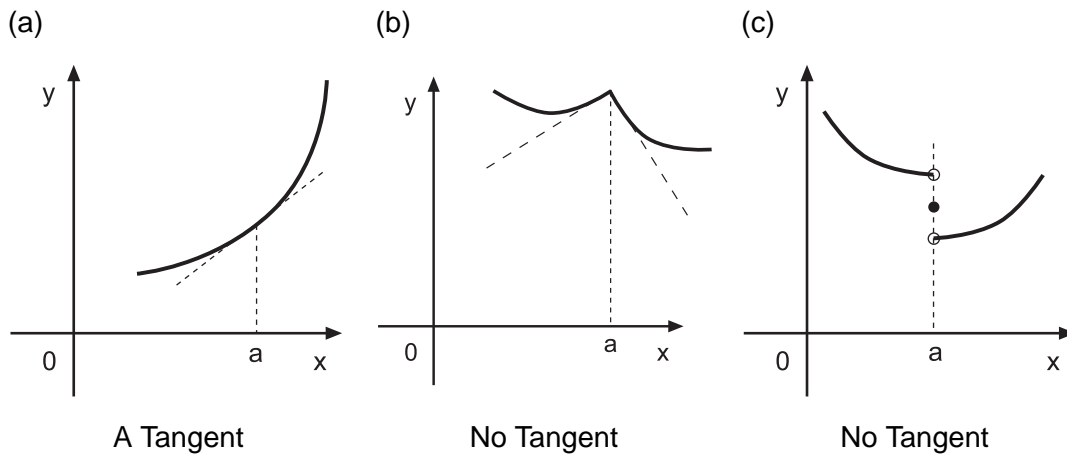
For the diagrams (a), (b) and (c) above, at x = a:

(a) is smooth,   (b) has a corner,   (c) is discontinuous.

If a function is not continuous at x = a then the function is not differentiable at x = a

A definition of continuity at a is that $f(a) = \lim_{x \to a} f(x)$

(See Proof 1 in the Proofs section for a justification of the connection between differentability and continuity.)

In our examples above only graph (a) is differentiable at x = a

However, look at the following examples.

Examples

1.

Notice that the graph of $f(x) = x^{1/3}$ is smooth and continuous but is not differentiable at x = 0

$f(x) = x^{1/3}$ is not differentiable at x = 0 because as $x \Rightarrow 0$ then $m_T \Rightarrow \infty$

2.

Consider the modulus function

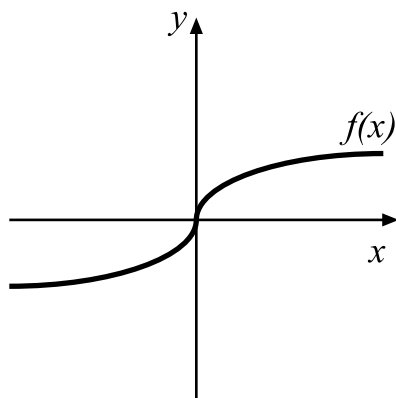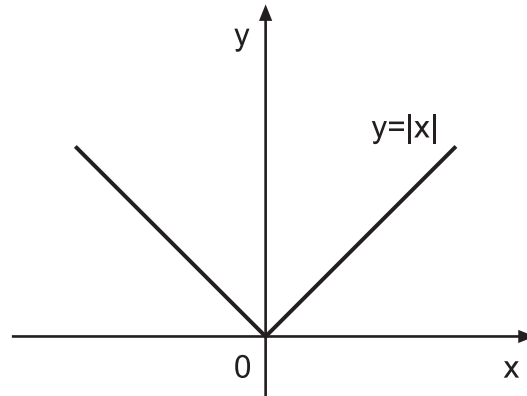f (x) = | x |

This defines a function on $\mathbb{R}$ where

$$f(x) = |x| = \begin{cases} x \text{ when } x \geqslant 0 \\ -x \text{ when } x < 0 \end{cases}$$

Notice that the graph has a sharp point at x = 0 and so does not have a tangent and as a consequence the function is not differentiable at x = 0.

Now try the questions in Exercise 3

Exercise 3

There is an on-line version of this exercise, which you might find helpful.

15 min

Q7: a) Make a sketch of the function

$$g(x) = \begin{cases} 5 \text{ when } x \leqslant 0 \\ 5 + 4x - x^2 \text{ when } 0 \leqslant x \leqslant 5 \\ 0 \text{ when } x \geqslant 5 \end{cases}$$

b) Write down the values of x where the function is not differentiable.

Q8: a) Write down the largest possible domain for x when $f(x) = \sqrt{x + 4}$

b) Sketch the function f (x)

c) Write down the values of x where the function is not differentiable.

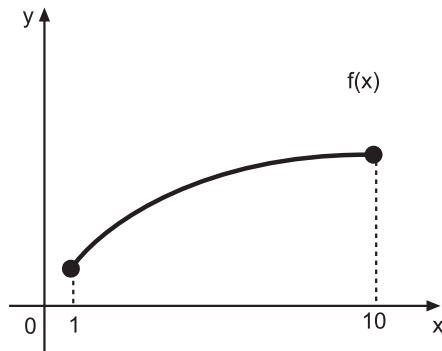## 2.5 Differentiability over an Interval

Learning Objective

Decide when a function is differentiable over an interval

The closed interval [a, b]

Note that the notation $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ means **the closed interval** with end points a and b.

We can say that f (x) is differentiable over the interval [a, b] if f $'$ (x) exists for each $x \in$ [a, b]

Example

The graph shown here is

f (x) = $\sqrt{x}$ for x $\in$ [1, 10]

We can say that f (x) is differentiable in this interval because f $'$ (x) exists for each x $\in$ [1, 10]

Whereas for the interval [0, 10], f (x) is not differentiable at x = 0 because as x $\Rightarrow$ 0, f $'$ (x) $\Rightarrow$ ∞

2.6   The Second Derivative

Learning Objective

Calculate the second derivative

If y = f (x) then its derivative f $'$ (x) is also a function of x and may itself have a derivative.

The derivative of f $'$ (x) is denoted by f $'$ (x) or f $^{(2)}$(x)

This is read as 'f double dash x' or ' f two of x'

In Leibniz notation the derivative of $^{dy}/_{dx}$ is denoted by

$\dfrac{d^2y}{dx^2}$

This is read as ' d two y by d x squared'.

> f"(x) or $\dfrac{d^2y}{dx^2}$ is called the second derivative of the function.

Examples

1. Find the first and second derivative of the function y = $3x^3$ - 6x + 4

Answer

The first derivative is $^{dy}/_{dx}$ = $9x^2$ - 6

and the second derivative is

$\dfrac{d^2y}{dx^2} = 18x$

the result of finding the derivative of $^{dy}/_{dx}$.

2. Find f $'$ (x) and f $''$ (x) for the function f (x) = sin 3x

Answer

f $'$ (x) = 3 cos 3x

f $''$ (x) = -9 sin 3x by differentiating f $'$ (x) again

Exercise 4

There is an on-line version of this exercise, which you might find helpful.

20 min

Q9: Find the first and second derivatives of the following functions:

a) $y = 6x^2 + 4x - 3$

b) $y = (2x + 9)^4$

c) $y = \cos(4x + \dfrac{\pi}{4})$

Q10: If $f(x) = 2x^3 + 5x$ find the value of a so that $f''(a) = 36$

Q11: If $y = 4x^2 - 3x + 1$ show that

$$y\frac{d^2y}{dx^2} + \frac{dy}{dx} - 8y + 3 = 8x$$

2.7 Rate Of Change

> **Learning Objective**
>
> Understand the relationship between distance, speed and acceleration

The rate of change of a differentiable function, dy/dx, can be calculated for any value of x.

In particular, the rate of change of displacement or distance (s) with respect to time (t) is speed (v).

Therefore $v = ds/dt$

Also, the rate of change of speed (v) with respect to time (t) is acceleration (a).

Therefore $a = dv/dt$

Example

The distance (in metres) that a rocket has travelled at time t (seconds) in the initial stages of lift-off is calculated using the formula $s(t) = 2t^3$

a) Find expressions for the speed and acceleration of the rocket.

b) Calculate the velocity and acceleration of the rocket after 10 seconds.

Answer

a) Speed $= v = \dfrac{ds}{dt} = 6t^2$

Acceleration $= a = \dfrac{dv}{dt} = 12t$

b) After 10 seconds:

Speed $= 6 \times 10^2 = 600$ m/s

Acceleration $= 12 \times 10 = 120$ m/s^2

Now try the questions in Exercise 5

Exercise 5

25 min

There is an on-line version of this exercise, which you might find helpful.

Q12:

The displacement of a particle s metres from a certain point at time t seconds is given by $s = 8 - 75t + t^3$.

a) Find the acceleration of the particle in terms of t.

b) Evaluate the acceleration after 5 seconds.

Q13:

The displacement of the weight on a spring, after t seconds, is given by f (t) = 3 sin t

a) Calculate the velocity after 1 second (remember angles are in radians).

b) After what time is the velocity first zero?

c) Calculate the acceleration after 1 second.

Q14:

The distance, d metres, travelled on a roller-coaster is calculated using the formula
d (t) = $12t^2$ - 6t, where t is the time in seconds after the start of the ride.

Calculate the speed and acceleration of a car on the roller-coaster 5 seconds after the start of the ride.

Q15:

A particle is moving so that its distance, s metres, from the origin at time t seconds is given by $s(t) = \sin\left(3t - \frac{\pi}{6}\right), \quad 0 \leqslant t \leqslant 2\pi$

a) Find an expression for the velocity of the particle and calculate its velocity at t = 0

b) Find an expression for the acceleration of the particle

c) At what times is the acceleration a maximum ? Give your answers as multiples of π

2.8 Higher Derivatives

Learning Objective

Be able to calculate higher derivatives

When f ′ (x) and f ′ ′ (x) exist and are differentiable, then the derivative of f ′ ′ (x) is denoted by

f ′ ′ ′ (x) or $f^{(3)}(x)$

Similarily in Leibniz notation when $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ exist and are differentiable then

$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3}$ and more generally, for n = 2, 3, 4, ... the nth derivative of f (x), when it exists, is denoted by $f^{(n)}$ or $\frac{d^n y}{dx^n}$

Example Find all the derivatives of $y = 5x^3 + 2x^2 + 3x - 1$

Answer

$\frac{dy}{dx} = 15x^2 + 4x + 3$

$\frac{d^2y}{dx^2} = 30x + 4$

$\frac{d^3y}{dx^2} = 30$

$\frac{d^4y}{dx^4} = 0$ and $\frac{d^n y}{dx^n} = 0$ for   n = 5, 6, 7 ...

Now try the questions in Exercise 6

Exercise 6

There is an on-line version of this exercise, which you might find helpful.

20 min

Q16: Find all the derivatives of $f(x) = 3x^4 + 5x^2 + 2x + 7$

Q17: Find the first four derivatives for $y = 3\sin(2x)$

Also try to write down the tenth derivative (without working out all the previous derivatives).

2.9   Discontinuities

Learning Objective

Understand when discontinuities can occur in a function

Derivatives can have discontinuities.

Study the following example.

Example

Consider f (x) = $x^{4/3}$

It has a graph as shown here.
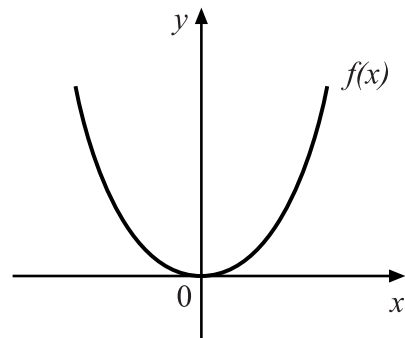
It is possible to find f ' (x) = $^4/_3$ $x^{1/3}$

The graph is as shown here.

Notice that as x $\Rightarrow$ 0, m_T $\Rightarrow$ ∞

and so f ' ' (x) is undefined at x = 0

Therefore f ' (x) is not differentiable at x = 0
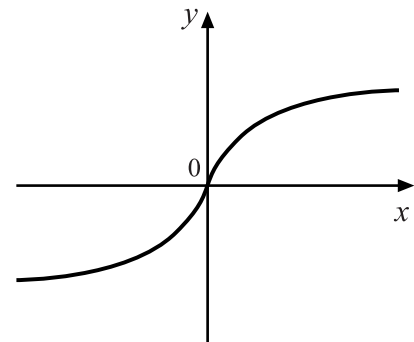
It is also possible to find f"(x)

and f"(x) = $\frac{4}{9}x^{-2/3}$

It has a graph as shown here.

Notice that the limit does not exist for
f"(x) as x $\rightarrow$ 0

The graph is discontinuous at x = 0

So in this example f (x) and f ' (x) are continuous but f ' ' (x) is discontinuous.
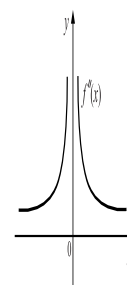
Exercise 7

There is an on-line version of this exercise, which you might find helpful.

30 min

Q18: Consider   f(x) = $\begin{cases} -x^2 & \text{for} \quad x < 0 \\ x^2 & \text{for} \quad x \geqslant 0 \end{cases}$

a) Make a sketch of this graph.

b) Calculate f ' (x) and sketch this graph.

c) For what value of x is f ′ (x) not differentiable?

Q19: Let f (x) = 3 (x + 2)$^{5/3}$

a) Calculate f ′ (x) and f ′ ′ (x)

b) Write doen the value of x where f ′ ′ (x) is discontinuous and where f ′ (x) is not differentiable

2.10   The Product Rule

> **Learning Objective**
>
> Understand how the product rule can help to differentiate the product of two or more functions

Product rule

The **product rule** gives us a method to differentiate the product of two or more functions.

It states that when k (x) = f (x) g (x)

then k ′ (x) = f ′ (x) g (x) + f (x) g ′ (x)

Proof

To prove this result we differentiate from 'first principles'.

$$
\begin{aligned}
\text{Let} \quad & k(x) = f(x)g(x) \\
\text{then} \quad k'(x) &= \lim_{h \to 0} \frac{k(x + h) - k(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x + h) + f(x)g(x + h) - f(x)g(x)}{h} \\
&= \lim_{h \to 0} \frac{[f(x + h) - f(x)]g(x + h) + f(x)[g(x + h) - g(x)]}{h} \\
&= \lim_{h \to 0} \left\{ \frac{f(x + h) - f(x)}{h} \times g(x + h) \right\} + \lim_{h \to 0} \left\{ \frac{g(x + h) - g(x)}{h} \times f(x) \right\} \\
&= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \times \lim_{h \to 0} g(x + h) + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \times \lim_{h \to 0} f(x) \\
&= f'(x)g(x) + g'(x)f(x) \\
&= f'(x)g(x) + f(x)g'(x)
\end{aligned}
$$

In short

$$\frac{d}{dx}(f\, g) = f'\, g + f\, g'$$

See Proof (1) in the Proofs section for a justification of $\lim_{h \to 0} g\,(x + h) = g\,(x)$

This can be extended to more than two factors.

So when k (x) = f (x) g (x) h (x)

then k ' (x) = f ' (x) g (x) h (x) + f (x) g ' (x) h (x) + f (x) g (x) h ' (x)

In short

$$\frac{d}{dx}(f\,g\,h) = f\,'\,g\,h + f\,g\,'\,h + f\,g\,h\,'$$

Leibniz Notation

If y = uv where u and v are functions of x

then $^{dy}/_{dx} = {}^{du}/_{dx}$ v + u $^{dv}/_{dx}$

In short

$$y\,' = u\,'v + u\,v'$$

Examples

1. Calculate k ' (x) = x^2 (x + 2)^3

Answer

Let f = x^2 and g = (x + 2)^3

then f ' = 2x and g ' 3 (x + 2)^2

Now $^d/_{dx}$ (f g) = f ' g + f g '

So k ' (x) = 2x (x + 2)^3 + x^2 . 3(x + 2)^2

= x (x + 2)^2[2(x + 2) + 3x] (Notice that a common factor is taken out here.)

= x (x + 2)^2 (5x + 4)

2. Using Leibniz notation calculate $^{dy}/_{dx}$ when y = $\sqrt{x}$(2x + 3)^2

Answer

Let u = $\sqrt{x}$ = $x^{1/2}$ and v = (2x + 3)^2

then $^{du}/_{dx}$ = $^1/_2$ $x^{-1/2}$ and $^{dv}/_{dx}$ = 2 (2x + 3). 2 = 4 (2x + 3)

$$\text{Now} \quad \frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

$$\text{So} \quad \frac{dy}{dx} = \frac{1}{2}x^{-1/2}(2x + 3)^2 + x^{1/2}4(2x + 3)$$

$$= \frac{1}{2}x^{-1/2}(2x + 3)\left[(2x + 3) + 8x\right] \quad \text{(Notice that common factors are taken out here)}$$

$$= \frac{1}{2}x^{-1/2}(2x + 3)\left[10x + 3\right]$$

3. Calculate k ' (x) = when k (x) = $2x^2$ cos x

Answer

Let f = $2x^2$ and g = cos x

then f ' = 4x and g ' = - sin x

Now $^d/_{dx}$ (f g) = f ' g + f g '

So k ' (x) = 4x cos x - $2x^2$ sin x

= 2x (2 cos x - x sin x)

Now try the questions in Exercise 8

Exercise 8

An on-line version of this exercise is available. It contains randomised questions which are similar to those below.

45 min

Differentiate the following with respect to x, simplifying your answers where possible.

Q20: $y = x^3(3x + 1)^2$

Q21: $y = x \sin x$

Q22: $y = x^2 (2x + 3)^6$

Q23: $y = x^2 \cos (2x + 1)$

Q24: $y = (x^2 - 3)(x^3 + 9)^2$

Q25: $y = x\sqrt{x^2 + 4}$

Q26: $y = \sin 3x \cos x$

Q27: $y = x^3 \sin x \cos 2x$

Q28: Find the equation of the tangent to the curve $y = x \sin \pi x$ at $x = {}^1/_2$

Q29: Find the coordinates of any stationary points on the curve $y = x^2(x - 3)^4$ and state the nature of these stationary points.

2.11   The Quotient Rule

Learning Objective

Use the quotient rule to differentiate algebraic fractions

Quotient rule

The **quotient rule** gives us a method that allows us to differentiate algebraic fractions.

It states that when $k(x) = \frac{f(x)}{g(x)}$

then

$k'(x) = \frac{f'(x)\, g(x) - f(x)\, g'(x)}{[g(x)]^2}$

Proof

Let $k(x) = \frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$

Now using the product rule

Let $f = f(x)$ and $g = [g(x)]^{-1}$

then $f' = f'(x)$ and $g' = -[g(x)]^{-2} \cdot g'(x)$ (By the chain rule)

Since $\dfrac{d}{dx}(fg) = f'g + fg'$

then $k'(x) = f'(x)[g(x)]^{-1} + f(x)(-1)[(g(x)]^{-2}g'(x)$

$= [g(x)]^{-2}[f'(x)g(x) - f(x)g'(x)]$

$= \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

In short

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

Leibniz Notation

If $y = \dfrac{u}{v}$ where u and v are functions of x

then $\dfrac{dy}{dx} = \dfrac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$

or more simply

$$y' = \frac{u'v - uv'}{v^2}$$

Example

Find   k'(x)   when $k(x) = \dfrac{(2x-3)^2}{(x+2)^2}$

Answer

Let $f = (2x - 3)^2$ and $g = (x + 2)^2$

then $f' = 2(2x - 3).2 = 4(2x - 3)$ and $g' = 2(x + 2)$

Now $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{f'g - fg'}{g^2}$

So $k'(x) = \dfrac{4(2x - 3)(x + 2)^2 - (2x - 3)^2.2(x + 2)}{(x + 2)^4}$

$= \dfrac{2(2x - 3)(x + 2)[2(x + 2) - (2x - 3)]}{(x + 2)^4}$

$= \dfrac{2(2x - 3)(7)}{(x + 2)^3}$

$= \dfrac{14(2x - 3)}{(x + 2)^3}$

Definitions

Before we look at the next example we need to define some new functions.

secant
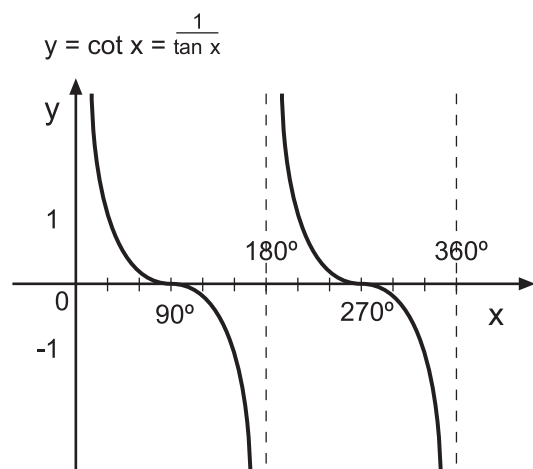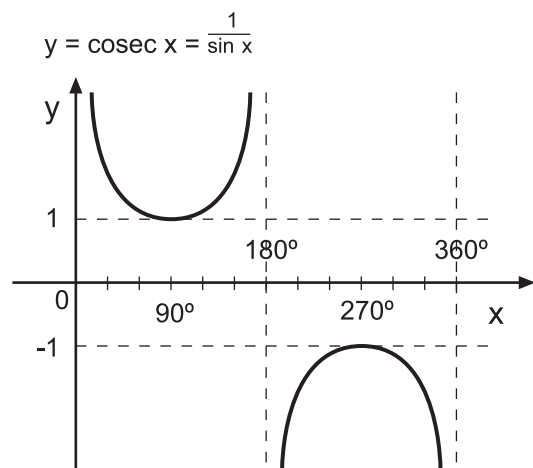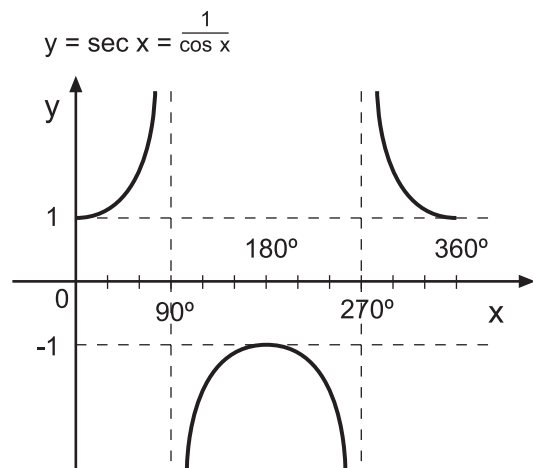
The **secant** of $x = \sec x = {}^1/_{\cos x}$

cosecant

The **cosecant** of x = cosec x = $^1/_{\sin x}$

cotangent

The **cotangent** of x = cot x = $^1/_{\tan x}$

The graphs of these functions are as shown here:

$y = \sec x = \dfrac{1}{\cos x}$

$y = \operatorname{cosec} x = \dfrac{1}{\sin x}$

$y = \cot x = \dfrac{1}{\tan x}$

Example : The Derivative of Tan x

Find k $'$ (x) when k (x) = tan x

Answer

We rewrite k (x) = tan x as k (x) = $^{\sin x}/_{\cos x}$ and use the quotient rule

Let f = sin x and g = cos x

then f $'$ = cos x and g $'$ = - sin x

Since $\quad \dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{f'g - fg'}{g^2}$

then $\qquad$ k'(x) $= \dfrac{\cos x . \cos x - \sin x . (-\sin x)}{\cos^2 x}$

$\qquad\qquad\qquad = \dfrac{\cos^2 x + \sin^2 x}{\cos^2 x}$

$\qquad\qquad\qquad = \dfrac{1}{\cos^2 x} \quad \left(\text{Since } \cos^2 x + \sin^2 x = 1\right)$

$\qquad\qquad\qquad = \sec^2 x$

Hence $\quad \frac{d}{dx}(\tan x) = \sec^2 x$

Now try the questions in Exercise 9

Exercise 9

45 min

There are randomised questions on-line which you can use for further practice.

Differentiate the following with respect to x, simplifying your answer where possible.

Q30: $y = \frac{3x}{x+2}$

Q31: $y = \frac{2x-1}{3x+2}$

Q32: $y = \frac{3x-1}{x^2-x+1}$

Q33: $y = \frac{\sin 2x}{x^2}$

Q34: $y = \frac{2x+1}{\sqrt{2x-1}}$

Q35: $y = \frac{\cos^3 x}{x^2}$

Q36: $y = \frac{2x^2}{\cos^2 x}$

Q37: Show that when $y = \cot x \left(= \frac{\cos x}{\sin x}\right)$ then $\frac{dy}{dx} = -\cosec^2 x$. Use the same method as for tan x

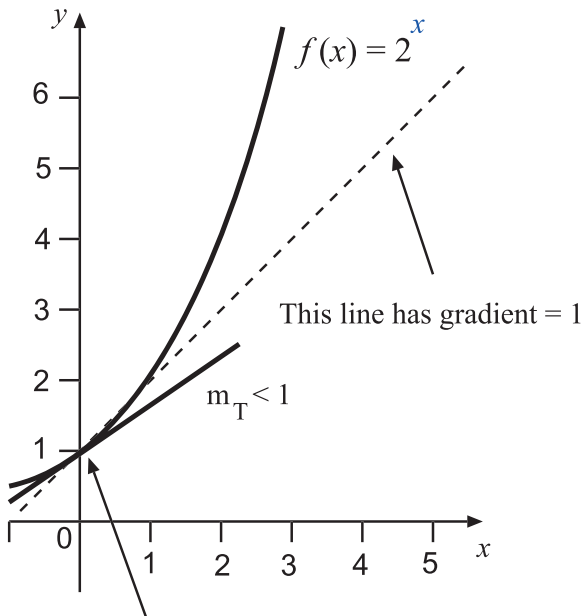**Q38:** Show that when $\;y = \sec x\;$ then $\;\frac{dy}{dx} = \sec x \tan x$

Q39: Show that when $\;y = \cosec x\;$ then $\;\frac{dy}{dx} = -\cosec x \cot x$

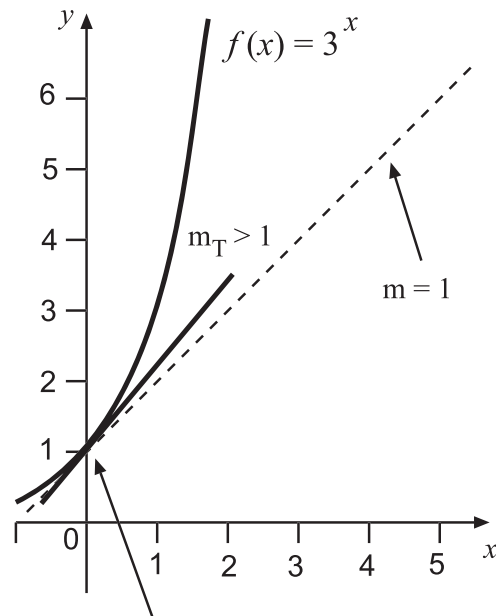## 2.12   The Derivative of exp(x) or e^x

> **Learning Objective**
>
> Understand how to find the derivative of the exponential function

Consider the graphs for f (x) = 2^x and f (x) = 3^x as shown here.

Notice that at $x = 0$ the gradient
of the tangent to the curve, m < 1

Notice that at $x = 0$ the gradient
of the tangent to the curve, m > 1

From 'first principles' the gradient of a tangent to these curves is given by these formulae:

When   f(x) = 2^x then   f'(x) $= \lim_{h \to 0} \frac{2^{x+h} - 2^x}{h}$

When   f(x) = 3^x then   f'(x) $= \lim_{h \to 0} \frac{3^{x+h} - 3^x}{h}$

Thus for f (x) = 2^x the gradient of the tangent to the curve at x = 0 is given by

f'(0) $= \lim_{h \to 0} \frac{2^h - 1}{h}$

and for f (x) = 3^x the gradient of the tangent to the curve at x = 0 is given by

f'(0) $= \lim_{h \to 0} \frac{3^h - 1}{h}$

For very small h, $\frac{2^h - 1}{h} < 1$ and $\frac{3^h - 1}{h} > 1$

This means that at x = 0 the gradient of the tangent to f (x) = 2^x is less than 1

whereas at x = 0 the gradient of the tangent to f (x) = 3^x is greater than 1

(as we have already observed from the graphs for f (x) = 2^x and f (x) = 3^x)

You can check this for some small values of h, e.g. h = 0.5, 0.2, etc.

So there must be a value, which we denote as e, so that

$2 < \ e \ < 3$ and $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

$e \approx 2.71828...$ is a real number constant that occurs in many mathematical problems.

Now from 'first principles'

When $f(x) = e^x$

then $\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h \to 0} \frac{e^x(e^h - 1)}{h} \\ &= \lim_{h \to 0} e^x \lim_{h \to 0} \frac{e^h - 1}{h} \\ &= e^x \times 1 \\ &= e^x \end{aligned}$

So when $f(x) = e^x$ then $f'(x) = e^x$

---

e^x is also denoted by exp (x) and is called the exponential function..

An effective way to approximate the value of e is to use the following infinite sum.

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots\ldots\ldots = \sum_{r=0}^{\infty} \frac{1}{r!}$$

(The concept of an infinite sum is studied in more detail in Unit 3: Further sequences and series.)

Note: On your calculator the constant $e = \exp(1) = e^1$

Now try the questions in Exercise 10

Exercise 10

There is an on-line version of this exercise, which you might find helpful.

15 min

Evaluate these finite sums (giving your answer to 6 decimal places):

Q40: $\displaystyle\sum_{r=0}^{4} \frac{1}{r!}$

Q41: $\displaystyle\sum_{r=0}^{8} \frac{1}{r!}$

Q42: $\displaystyle\sum_{r=0}^{10} \frac{1}{r!}$

Examine how accurate your answers are compared to the value that your calculator gives for *e*

The derivative of e^x can be combined along with other rules for differentiation.

Examples

1. Differentiate y = exp (4x)

Answer

We use the chain rule with u = 4x then y = exp (4x) = exp (u).

Hence

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= exp(u).4$$
$$= 4exp(4x)$$

2. Differentiate y = 2xe$^{x^2}$

Answer

We use the product rule with

f = 2x   and   g = e$^{x^2}$

f' = 2   and   g' = 2xe$^{x^2}$ (chain rule)

Therefore

$$\frac{dy}{dx} = f' g + f g'$$
$$= 2e^{x^2} + 2x.2xe^{x^2}$$
$$= 2e^{x^2}(1 + 2x^2)$$

Exercise 11

There is an on-line version of this exercise, which contains randomised questions.

30 min

Differentiate the following with respect to x

Q43: y = e^{3x}

Q44: y = exp (x^2 + 4)

Q45: $y = 3exp(x^2) + \frac{5}{exp(2x)}$

Q46: y = 2x^3 exp (2x)

Q47: y = e^{-x} sin x

Q48: $y = \frac{e^x + 2}{x^3}$

Q49: $y = \frac{cos4x}{e^{3x}}$

Q50: Find the gradient of the curve y = x exp (x - 4) at the point (4, 4)

Q51: Find the coordinates and nature of any turning points on the curve y = x exp (x)

2.13   The Derivative of ln(x)

> **Learning Objective**
>
> Understand how to find the derivative of the function ln(x)

You should remember that for $f(x) = e^x$ an inverse function $f^{-1}(x)$ exists.

$f^{-1}(x)$ is the logarithmic function with base e and is written as $y = \log_e x = \ln x$

In the graph shown here, $y = \ln x$ is the reflection of $y = e^x$ in the line $y = x$
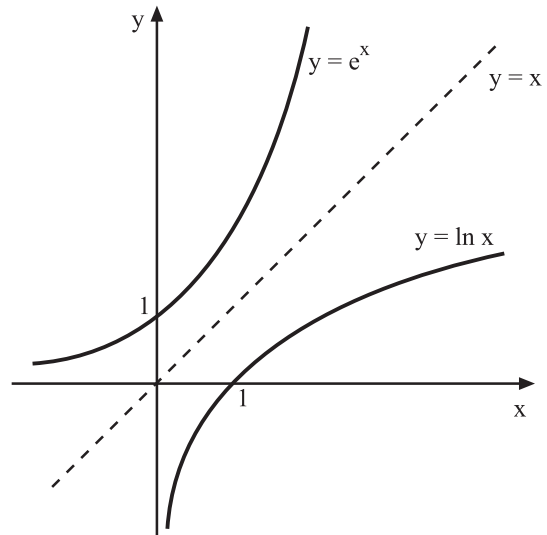
Thus if $y = \ln x$

then $x = e^y$

Now if we have $y = \ln x$ and $x = e^y$

Then we can say that $^{dx}/_{dy} = e^y$ (differentiating with respect to y)

and therefore $\dfrac{dy}{dx} = 1 \left/ \dfrac{dx}{dy} \right. = \dfrac{1}{e^y}$

$$= \dfrac{1}{x}$$

> So when $y = \ln x$
>
> then $^{dy}/_{dx} = {}^1/_x$

(See Proof (2) in the Proofs section for a justification that $\frac{dy}{dx} = 1 \left/ \frac{dx}{dy} \right.$)

Examples

1. Differentiate $y = \ln(4x + 1)$

Answer

$\dfrac{dy}{dx} = \dfrac{1}{4x + 1} \cdot 4$ (chain rule)

$\quad = \dfrac{4}{4x + 1}$

2. Differentiate $f(x) = \ln(\sin x)$

Answer

$$f'(x) = \frac{1}{\sin x}.\cos x \quad \text{(chain rule)}$$
$$= \cot x$$

Before looking at the next example it is useful to remember some facts:

Laws of Logarithms
$\log_a xy = \log_a x + \log_a y$
$\log_a(^x/_y) = \log_a x - \log_a y$
$\log_a x^n = n \log_a x$

3. Differentiate $f(x) = \ln\left(\frac{x+2}{x+3}\right)$

Answer

First note that

$f(x) = \ln\left(\frac{x+2}{x+3}\right) = \ln(x+2) - \ln(x+3)$

Thus

$$f'(x) = \frac{1}{x+2} - \frac{1}{x+3}$$
$$= \frac{(x+3) - (x+2)}{(x+2)(x+3)}$$
$$= \frac{1}{(x+2)(x+3)}$$

Now try the questions in Exercise 12

Exercise 12

There are randomised questions on-line which you can use for further practice.

Differentiate the following with respect to x

40 min

Q52: $y = \ln 7x$

Q53: $y = -5 \ln(x^2 + 6x)$

Q54: $y = \frac{1}{2} \ln(\sin 2x)$

Q55: $y = \ln(x^2 e^x)$. Notice that $\ln(x^2 e^x) = \ln x^2 + \ln e^x = 2\ln x + x)$

Q56: $y = \ln(3x \cos x)$

Q57: $y = \ln\left(\frac{5x}{2x-3}\right)$

Q58: $y = e^x \ln x$

Q59: $y = \frac{\ln x}{e^x}$
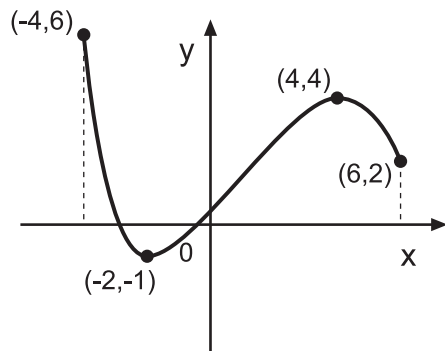
Q60: $y = \ln(\sec x)$

Q61: $y = x^3 \ln 2x \sin 3x$

2.14   Curve Sketching on a Closed Interval

Learning Objective

Be able to identify the extrema of functions within a closed interval

Study the following graphs and notice that the maximum and minimum values can occur at end points or at stationary points.

For the interval [-4, 6]

The minimum value is -1 at (-2,-1)

The minimum occurs at a stationary point.
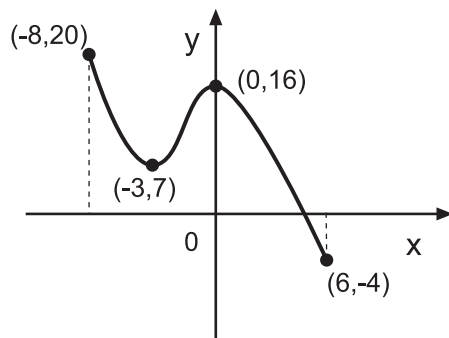
For the interval [-8, 6]

The minimum value is -4 at (6,-4)

The minimum occurs at an end point.

Now study this example:

This graph is defined as

$$f(x) = \begin{cases} x^2 & \text{for} \quad 0 \leqslant x \leqslant 1 \\ 2 - x & \text{for} \quad 1 \leqslant x \leqslant 2 \end{cases}$$

It has maximum value 1

However, at x = 1 the graph has a corner, it is not smooth, and $f'(x)$ does not exist.

So (1, 1) is not a stationary point.
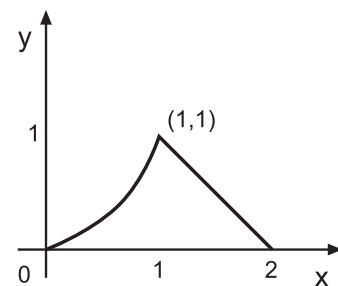
In a closed interval the maximum and minimum values of a function can either be at a stationary point, at an end point of the interval or where $f'(x)$ does not exist.

Now try the questions in Exercise 13

Exercise 13

40 min

There is a question on-line, which you can use for further practice.

Look at these graphs and identify the maximum and minimum values within the given closed intervals.

Q62:
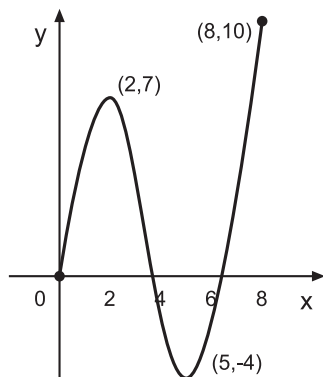
a) $0 \leqslant x \leqslant 8$
b) $0 \leqslant x \leqslant 4$
c) $2 \leqslant x \leqslant 6$

Q63:
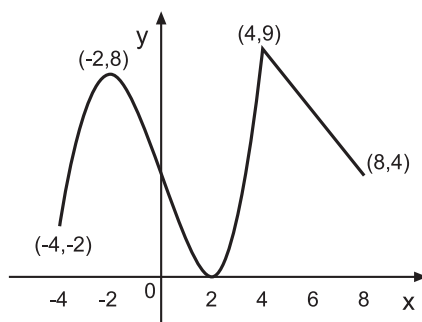
a) $-4 \leqslant x \leqslant 8$
b) $-4 \leqslant x \leqslant 4$

Q64:
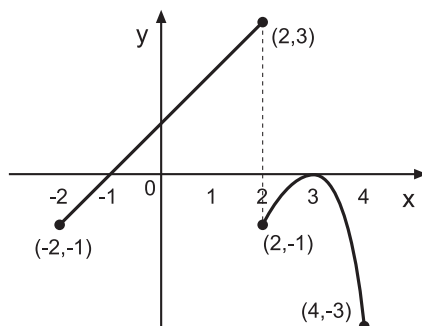
a) $-2 \leqslant x \leqslant 4$
b) $-2 \leqslant x \leqslant 2$

For the following questions find the maximum and minimum values within the given interval.

(You will need to make a sketch of the graph.)

Q65: f (x) = x^3 + $4x^2$ - 3x - 18 for $-4 \leq x \leq 3$

Q66: $y = 3\cos\left(2x - \frac{\pi}{4}\right)$ for $0 \leqslant x \leqslant \frac{\pi}{2}$

Q67: f (x) = $(x + 3)^2(x - 1)^3$ for $-4 \leq x \leq 2$

Q68: $f(x) = \begin{cases} x^2 \text{ - } 2x + 2 & \text{for} \quad 0 \leqslant x \leqslant 1 \\ x & \text{for} \quad 1 \leqslant x \leqslant 3 \end{cases}$

Q69: $f(x) = \begin{cases} 2x + 1 & \text{for} \quad -3 \leqslant x < -2 \\ x^2 + 2x \text{ - } 3 & \text{for} \quad -2 \leqslant x < 0 \end{cases}$

Q70: $f(x) = \begin{cases} -3x + 1 & \text{for} \quad 0 \leqslant x \leqslant 1 \\ x^2 - 2x + 1 & \text{for} \quad 1 < x \leqslant 3 \end{cases}$

Q71: $f(x) = \begin{cases} x(4 - x) & \text{for} \quad 0 \leqslant x \leqslant 2 \\ \frac{1}{2}x & \text{for} \quad 2 \leqslant x \leqslant 4 \end{cases}$

2.15 Applications

Learning Objective

Apply differentiation techniques to problems in context

Many problems often concern a maximum or a minimum and it may help to follow a particular strategy as listed here.

Some or all of these steps may be necessary:

1. Label any unknown quantities.

2. Identify the quantity that is to be maximised or minimised, in terms of the variables.

3. Use the information in the question to establish equations connecting the variables.

4. Express the unknown quantity in terms of only one variable.

5. Calculate the extrema of the function i.e. calculate the turning points and their nature, and if appropriate consider any end points.

Example

Two workers A and B share an 8 hour shift. A works for x hours and B works for y hours.

Their work output varies with time (in hours) from the start of the shift.

A has work output $= x - \frac{x^2}{16}$ and B has work output $= \frac{1}{2}\ln(2y)$

Calculate the hours each should be employed for maximum output.

Answer

Since they share the eight hour shift x + y = 8 and so y = 8 - x

The total output of the two workers is

$$T = x - \frac{x^2}{16} + \ln(2y)$$
$$= x - \frac{x^2}{16} + \ln(16 - 2x) \quad \text{(replacing 2y with 16 - 2x)}$$

Now that T is written in terms of one variable we can determine any stationary points.

$$\frac{dT}{dx} = 1 - \frac{2x}{16} - \frac{1}{16 - 2x} = 0 \quad \text{(at a stationary point)}$$

$$16(16 - 2x) - (16 - 2x)2x - 16 = 0 \quad \text{(multiplied by } 16(16 - 2x))$$

$$(16 - 2x)(16 - 2x) = 16$$

$$(16 - 2x)^2 = 16$$

$$16 - 2x = \pm 4$$

$$2x = 12 \text{ or } 20$$

$$x = 6 \text{ or } 10$$

x = 10 is impossible since the shift has a maximum of eight hours, so the stationary point occurs at x = 6

From the table shown here we can see that a maximum occurs at x = 6

x	6^-	6	6^+
$\frac{dT}{dx}$	$+$	0	$-$
slope	↗	→	↘

So for maximum total output the shift should be split so that A works 6 hours and B works 2 hours.

Now try the questions in Exercise 14

Exercise 14

There is a randomised version of this exercise on-line, which you can use for further practice.

30 min

Q72:
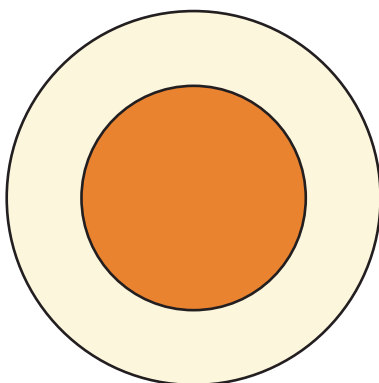
A communications cable has a copper core with a concentric sheath of insulating material. If *x* is the ratio of the radius of the core to the thickness of the insulating sheath, the speed of a signal along the cable is given by

$$S = 8x^2 \ln\left(\frac{1}{2x}\right)$$

Find the value of x that gives maximum speed.

Q73:

A water container is made into the shape shown here:

a) Express the volume V (in cubic metres) of the container in terms of the angle θ where $0 \leqslant \theta \leqslant \frac{\pi}{2}$

b) Find the value of θ which gives the maximum volume for this container and justify your answer.

c) Write down how many litres of water the container will hold.
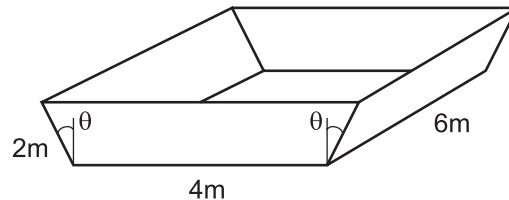
Q74:

The radius r centimetres of a circular ink spot on a piece of blotting paper t seconds after it was first observed is given by the formula

$r = \frac{1 + 3t}{1 + t}$

Calculate:

a) The radius of the ink spot when it was first observed.

b) The time at which the radius of the ink spot was 2 cm.

c) The rate of increase of the ink spot when the radius was 2 cm.

d) By considering the expression for 3 - r in terms of t show that the radius of the ink spot never reaches 3 cm.

Q75: A farmer wants to enclose some sheep in a small corner of his field. Two straight walls meet at A at an angle $\pi/4$. The farmer has a piece of fencing 20 metres long and uses it to create an enclosure ABC as shown.
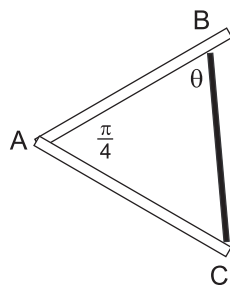
a) Show that the area ABC is given by $200\sqrt{2}\ \sin\theta\sin\left(\frac{3}{4}\pi - \theta\right)$

b) If the angle θ is varied, find, using differentiation, the value of θ which maximises the area of ABC

2.16 Proofs

Proof (1)

Prove that $\lim\limits_{h\to 0} g(x + h) - \lim\limits_{h\to 0} g(x) = 0 \Rightarrow \lim\limits_{h\to 0} g(x + h) = g(x)$

Proof

$$
\begin{aligned}
\lim_{h\to 0} [g(x + h) \text{ - } g(x)] &= \lim_{h\to 0} \left[\frac{g(x + h) \text{ - } g(x)}{h} \times h \right] \\
&= \lim_{h\to 0} \frac{g(x + h) \text{ - } g(x)}{h} \times \lim_{h\to 0} h \\
&= g'(x) \times 0 \\
&= 0
\end{aligned}
$$

Thus

$$\lim_{h\to 0} [g(x + h) \text{ - } g(x)] = 0$$

Hence

$$\lim_{h\to 0} g(x + h) - \lim_{h\to 0} g(x) = 0 \Rightarrow \lim_{h\to 0} g(x + h) = g(x)$$

This proof tells us that if g is differentiable at x then g is also continuous at x

Proof (2)

Prove that $\frac{dy}{dx} = 1 \Big/ \frac{dx}{dy}$

Proof

By definition
$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \left(\frac{\Delta y}{\Delta x} \right)$$

$$\frac{dy}{dx} = \lim_{\Delta x\to 0} \left(\frac{1}{\frac{\Delta x}{\Delta y}} \right)$$

But as $\delta x \Rightarrow 0$ then so also $\delta y \Rightarrow 0$

So we can write

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta y\to 0} \left(\frac{1}{\frac{\Delta x}{\Delta y}} \right) \\
&= 1 \Big/ \frac{dx}{dy}
\end{aligned}
$$

2.17 Summary

Learning Objective

Consolidation of new content

1. $f'(x) = \lim\limits_{h \to 0} \frac{f(x + h) - f(x)}{h}$

2. The nth derivative when it exists, is denoted by $f^{(n)}$ or $\frac{d^n y}{dx^n}$

3. The Product Rule:

 When $k(x) = f(x)g(x)$

  then $k'(x) = f'(x)g(x) + f(x)g'(x)$

4. The Quotient Rule:

 When $k(x) = \dfrac{f(x)}{g(x)}$

  then $k'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

5. $\sec x = \frac{1}{\cos x}$ $\operatorname{cosec} x = \frac{1}{\sin x}$ $\cot x = \frac{1}{\tan x}$

6. If $f(x) = \tan x$ then $f'(x) = \sec^2 x$

 If $f(x) = \sec x$ then $f'(x) = \sec x \tan x$

 If $f(x) = \operatorname{cosec} x$ then $f'(x) = -\operatorname{cosec} x \cot x$

 If $f(x) = \cot x$ then $f'(x) = -\operatorname{cosec}^2 x$

 If $f(x) = \exp x$ then $f'(x) = \exp x$

 If $f(x) = \ln x$ then $f'(x) = \dfrac{1}{x}$
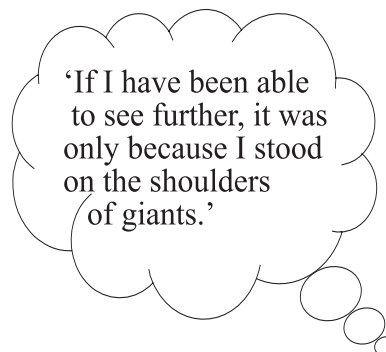
2.18 Extended Information

Learning Objective

To encourage an interest in related topics

There are links on the web which give a variety of web sites related to this topic.

Sir Isaac Newton

- Born on 4th January 1643 in Woolsthorpe, England

- Died on 31st March 1727 in London.

'If I have been able to see further, it was only because I stood on the shoulders of giants.'

Newton came from a family of farmers but never knew his father who died three months before he was born. Although a wealthy man, Newton's father was uneducated and could not sign his own name. His mother, Hannah Ayscough remarried when Newton was two years old. Newton was then left in the care of his grandmother and he had a rather unhappy childhood.

In 1653 he attended the Free Grammar School in Grantham. However, his school reports described him as idle and inattentive and he was taken away from school to manage his mother's estate. He showed little interest for this and, due to the influence of an uncle, he was allowed to return to the Free Grammar School in 1660. This time he was able to demonstrate his academic promise and passion for learning and on 5th June 1661 he entered Trinity College,Cambridge.

His ambition at Cambridge was to obtain a law degree but he also studied philosophy, mechanics and optics. His interest in mathematics began in 1663 when he bought an astrology book at a fair and found that he could not understand the mathematics in it. This spurred him on to read several mathematical texts and to make further deep mathematical studies.

Newton was elected a scholar at Cambridge on 28th April 1664 and received his Bachelor's degree in April 1665. In the summer of 1665 the University was closed due to the plague and Newton had to return to Lincolnshire. There, while still less than 25 years old, he made revolutionary advances in mathematics, physics, astronomy and optics. While at home, Newton established the foundations for differential and integral calculus, several years before the independent discovery by Leibniz. The method of fluxions as he named it was based on his crucial insight that integration is merely the inverse procedure to differentiating a function.

In 1672 he was elected a fellow of the Royal Society after donating a reflecting telescope. In that year he also published his first scientific paper on light and colour. However, he came in for some criticism from other academics who objected with some of his methods of proof and from then on Newton was torn between wanting fame and recognition and

the fear of criticism. He found the easiest way to avoid this was to publish nothing.
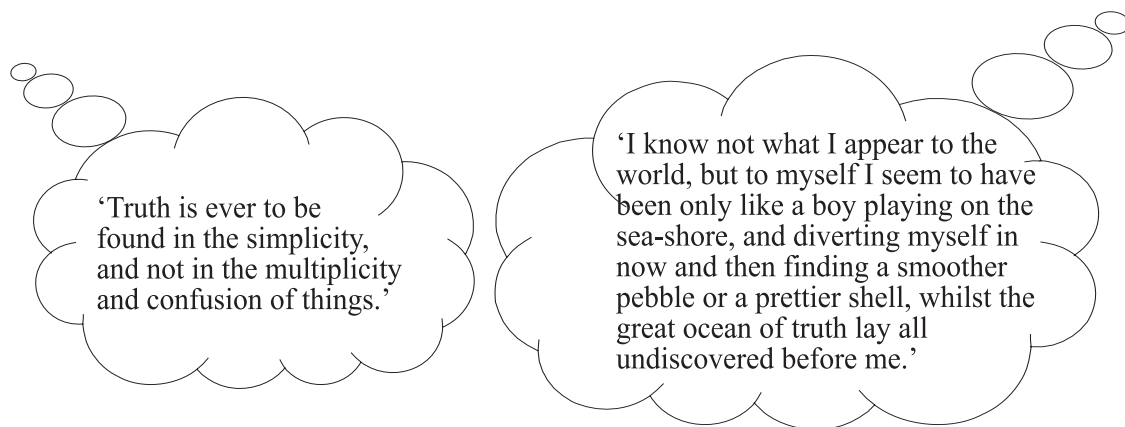
Newton's greatest achievement was his work in physics and celestial mechanics that lead to his theory of universal gravitation. He was persuaded to write a full account of his new physics and its application to astronomy. In 1687 he published the Philosophiae naturalis principia mathematica or Principia as it is always known. This is recognised as the greatest scientific book ever written. It made him an international leader in scientific research.

On 15th January Newton was elected by the University of Cambridge as one of their two members to the Convention Parliament in London. This may have led him to see that there was a life in London which might appeal more to him than that of the academic world in Cambridge.

After suffering a nervous breakdown in 1693, Newton retired from research and decided to leave Cambridge to take up a government position in London as Warden and then later as Master of the Royal Mint. He made an effective contribution to the work of the Mint particularly on measures to prevent counterfeiting of the coinage.

In 1703 he was elected as president of the Royal Society, a position he retained until his death. He was knighted by Queen Anne in 1705, the first scientist to be honoured in this way for his work.

However, his last years were not easy, dominated in many ways over the controversy with Leibniz as to who had first invented calculus.

'Truth is ever to be found in the simplicity, and not in the multiplicity and confusion of things.'

'I know not what I appear to the world, but to myself I seem to have been only like a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell, whilst the great ocean of truth lay all undiscovered before me.'

Gottfried Leibniz

• Born on 1st July 1646 in Leipzig,Germany

• Died on 14th November1716 in Hannover.

'The soul is the mirror of an indestructible universe.'
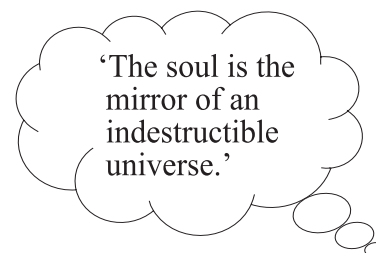
His father Friedrich was a professor of moral philosophy and his mother Catharina Schmuck was Friedrichs third wife. Friedrich died when Leibniz was only six, and so he was brought up by his mother. It was her influence that played an important role in

his life and philosophy.

In 1661 Leibniz entered the University of Leipzig. He was only fourteen, which nowadays would be considered highly unusual. However at that time there would be others of a similar age. He studied philosophy and mathematics and graduated with a Bachelor's degree in 1663. Further studies took him on to a Master's Degree in philosophy and a Bachelor's degree and doctorate in Law.

By November 1667 Leibniz was living in Frankfurt where he investigated various different projects, scientific, literary and political. He also continued his law career.

In 1672 Leibniz went to Paris with the aim of contacting the French government and dissuading them from attacking German land. While there he made contact with mathematicians and 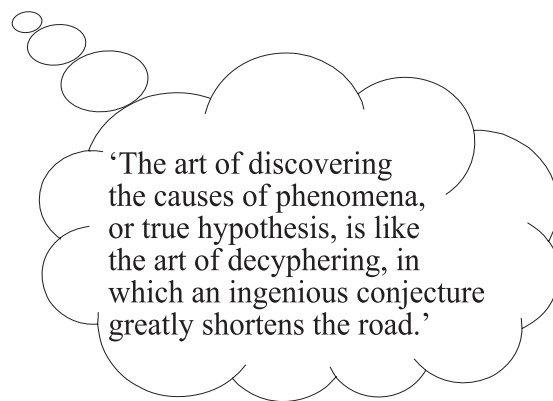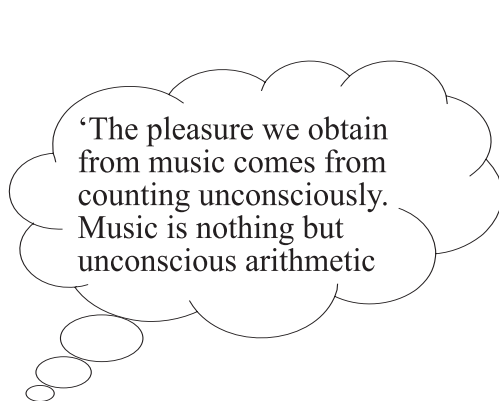philosophers and began construction of a calculating machine. In the January of the following year he went to England to try the same peace mission, the French one having failed, and while there he visited The Royal Society of London and presented his incomplete calculating machine. The Royal Society elected him as a fellow on 19th April 1673 but by 1674 he had not kept his promise to finish his mechanical calculating machine and so he fell out of favour.

It was during his time in Paris that Leibniz developed his version of calculus. However, the English mathematician Sir Isaac Newton, several years before Leibniz, had already laid the foundations for differential and integral calculus. This lead to much controversy over who had invented calculus and caused Newton to fly into an irrational temper directed against Leibniz. Neither Leibniz nor Newton thought in terms of functions; both always worked in terms of graphs. Leibniz concentrated on finding a good notation for calculus and spent a lot of time thinking about it, whereas Newton wrote more for himself and tended to use whatever notation he thought of on the day.

'The pleasure we obtain from music comes from counting unconsciously. Music is nothing but unconscious arithmetic

'The art of discovering the causes of phenomena, or true hypothesis, is like the art of decyphering, in which an ingenious conjecture greatly shortens the road.'

Amongst Leibniz's other achievements in mathematics were his development of the binary system of arithmetic and his work on determinants, which arose from his developing methods to solve systems of linear equations. He also produced an important piece of work on dynamics.

Leibniz is descibed as:

"a man of medium height with a stoop, broad shouldered but bandy-legged, as capable of thinking for several days sitting in the same chair as of travelling the roads of Europe summer and winter. He was an indefatigable worker, a universal letter writer (he had more than 600 correspondents), a patriot and cosmopolitan, a great scientist, and one

of the most powerful spirits of Western civilisation."

It was also said about him

"It is rare to find learned men who are clean, do not stink and have a sense of humour."

2.19   Review exercise

Review exercise in differentiation

10 min

There are randomised questions on-line which you can use for further practice.

Find the derivatives of the following:

Q76: $f(x) = 3x^4 \ln x$

Q77: $f(x) = \frac{3x + 2}{4x + 1}$

Q78: $f(x) = \exp(\cos x)$

2.20   Advanced review exercise

Advanced review exercise in differentiation

40 min

There are randomised questions on-line which you can use for further practice.

Q79: Differentiate the following functions, with respect to x, simplifying your answers where possible.

a) $y(x) = x^2 e^{-x^3}$

b) $g(x) = \frac{\sin x}{2 + \cos x}, \quad -\pi \leqslant x \leqslant \pi$

c) $h(x) = \cos(x^2)\sin(3x)$

d) $f(x) = \frac{\ln(x + 4)}{x + 4}, \quad x > -4$

Q80: A function f is defined by

$f(x) = \frac{3x}{x-2}, x \neq 2$.

Find f'(x) and deduce that the derivative is always negative.

Q81:

Use calculus to find all the values of x for which the function

$f(x) = (1 - x)^2 e^x, \quad x \in \mathbb{R}$

is decreasing.

Q82:

Given that $y = e^x \sin x$ find the value of x in the interval $0 < x < \pi$

such that $\frac{dy}{dx} = 0$

Q83:

Show that the function $y = \dfrac{\cos(kx)}{x}$, where $x \neq 0$ and k is a non - zero constant,

satisfies the differential equation: $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\dfrac{dy}{dx} + k^2y = 0$

Q84:

A pencil case is in the shape of a cylinder with a conical end as shown in the diagram. The cylinder has radius 3 cm and height H cm. The cone has a perpendicular height h cm and slant height l cm. The total volume of the container is 900 cm^3.
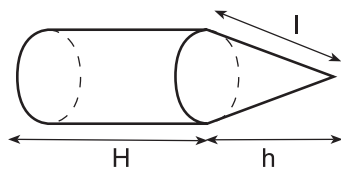
a) Find an expression for H in terms of h. (Hint: first write down an expression for V (the total volume) and rearrange for H.)

b) Show that the surface area S cm^2, of the container is given by $S = 600 - 2\pi h + 9\pi + 3\pi\sqrt{h^2 + 9}$

c) Find the value of h for which the total surface area of the pencil case is a minimum.

Justify your answer.

Note: for a cone   Volume $= \frac{1}{3}\pi r^2 h$

                      Curved surface area $= \pi r l$

2.21   Set review exercise

Set Review Exercise in differentiation

An on-line assessment is available at this point, which you will need to attempt to have these answer marked. These questions are not randomised on the web. The questions on the web may be posed in a different manner but you should have the required answers in you working.

10 min

Q85: Differentiate f (x) = 3x^{4}ln(x) with respect to x

Q86: Differentiate f (x) = $\frac{2x + 3}{3x + 5}$ with respect to x

Q87: Differentiate f (x) = exp(1 + cosx) with respect to x

Topic 3

Integration

Contents

Learning Objectives

Integrate using standard results and the substitution method.

Minimum performance Criteria:

- *Integrate an expression requiring a standard result.*

- *Integrate using the substitution method where the substitution is given.*

- *Integrate an expression requiring a simple substitution.*

3.1   Prerequisites

Learning Objective

Revision of previous knowledge

You should recall some rules for integrating:

1.

$\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$ and where C is the constant of integration.

$\int \cos x \, dx = \sin x + C$

$\int \sin x \, dx = -\cos x + C$

2.

$\int [a\,f(x) + bg(x)]\,dx = a \int f(x)\,dx + b \int g(x)dx$

3.

$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$ where $a < b < c$

4.

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx,\ a \neq b$

5.

$\int (px + q)^n dx = \frac{(px+q)^{n+1}}{p(n+1)} + C,\quad n \neq -1$

$\int p\cos(qx + r)dx = \frac{p}{q}\sin(qx + r) + C$

$\int p\sin(qx + r)\,dx = -\frac{p}{q}\cos(qx + r) + C$

6.

Know the Fundamental Theorem of Calculus;

If $f(x) = F'(x)$ then $\int_a^b f(x)dx = F(b) - F(a)$ where $a \leqslant x \leqslant b$

and $\int f(x)dx = F(x) + C$

7.

Be able to evaluate definite integrals.

8.

Area between a curve and the x-axis

Know that the area between the graph y = f (x) and the x-axis from x = a to x = b is given by

Area = $\int_{a}^{b} f(x)dx$

Recall that areas above the x-axis give positive values whereas areas below the x-axis give negative values.
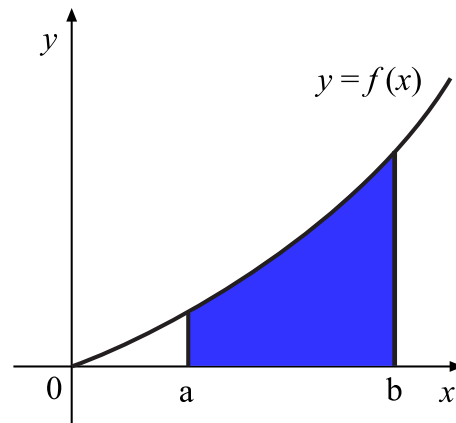
9.

Area between two curves.
Know that the area bounded by two curves y = f (x) and y = g(x) from x = a to x = b is given by

$$\int_{a}^{b} f(x)\, dx - \int_{a}^{b} g(x)\, dx$$

$$= \int_{a}^{b} (f(x) - g(x))\, dx$$

when f (x) $\geqslant$ g (x)
for a $\leqslant$ x $\leqslant$ b
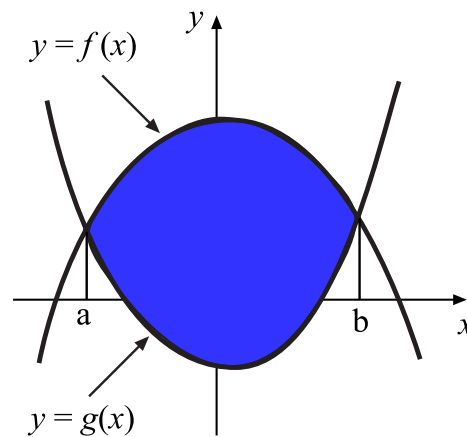
Study the following examples carefully and make sure that you understand the working.

(Notice that you can always check you have the correct answer by reversing the process, i.e. differentiating to get back to the original question.)

Examples

1. $\int x^3 dx = \frac{1}{4}x^4 + C$

$$\int \frac{8 - x}{\sqrt{x}}dx = \int \left(8x^{-1/2} - x^{1/2}\right)dx$$

2.
$$= \frac{8x^{1/2}}{1/2} - \frac{x^{3/2}}{3/2} + C$$

$$= 16x^{1/2} - \frac{2}{3}x^{3/2} + C$$

$$= 16\sqrt{x} - \frac{2}{3}\sqrt{x^3} + C$$

$$\int \frac{2dt}{(4t - 5)^3} = \int 2(4t - 5)^{-3}dt$$

3.
$$= \frac{2(4t - 5)^{-2}}{4(-2)} + C$$

$$= \frac{-1}{4(4t - 5)^2} + C$$

4.
$$\int 4\cos\left(2x + \frac{\pi}{3}\right) dx = \frac{4}{2}\sin\left(2x + \frac{\pi}{3}\right) + C$$

$$= 2\sin\left(2x + \frac{\pi}{3}\right) + C$$

5. Evaluate $\int_{1}^{3}\left(r^2 + \frac{1}{r^2}\right) dr$

Answer

$$\int_{1}^{3}\left(r^2 + \frac{1}{r^2}\right) dr = \int_{1}^{3}\left(r^2 + r^{-2}\right) dr$$

$$= \left[\frac{1}{3}r^3 + \frac{r^{-1}}{-1}\right]_{1}^{3}$$

$$= \left[\frac{1}{3}r^3 - \frac{1}{r}\right]_{r=3} - \left[\frac{1}{3}r^3 - \frac{1}{r}\right]_{r=1}$$

$$= (9 - 1/3) - (1/3 - 1)$$

$$= 9 - 1/3 - 1/3 + 1$$

$$= 9\frac{1}{3}$$

6.

Calculate the area enclosed by the graphs

$y = x^2 + 4x - 5$ and $y = 5x + 1$

Answer

Make a sketch of the curves and note that they intersect when

$x^2 + 4x - 5 = 5x + 1$

$\Rightarrow x^2 - x - 6 = 0$

$\Rightarrow (x + 2)(x - 3) = 0$

$\Rightarrow x = -2$ or $x = 3$

Notice that the graph for $y = 5x + 1$ is above that for $y = x^2 + 4x - 5$

So the area between the curves is given by

$$\int_{-2}^{3} [(5x + 1) - (x^2 + 4x - 5)]dx$$

$$= \int_{-2}^{3} (5x + 1 - x^2 - 4x + 5)dx$$

$$= \int_{-2}^{3} (-x^2 + x + 6)dx$$

$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x \right]_{-2}^{3}$$

$$= \left(-\frac{27}{3} + \frac{9}{2} + 18 \right) - \left(\frac{8}{3} + 2 - 12 \right)$$

$$= \left(-9 + 4\frac{1}{2} + 18 \right) - \left(2\frac{2}{3} + 2 - 12 \right)$$

$$= 13\frac{1}{2} - \left(-7\frac{1}{3} \right)$$

$$= 20\frac{5}{6} \quad \text{square units}$$

y = 5 x + 1

y = x² + 4 x - 5
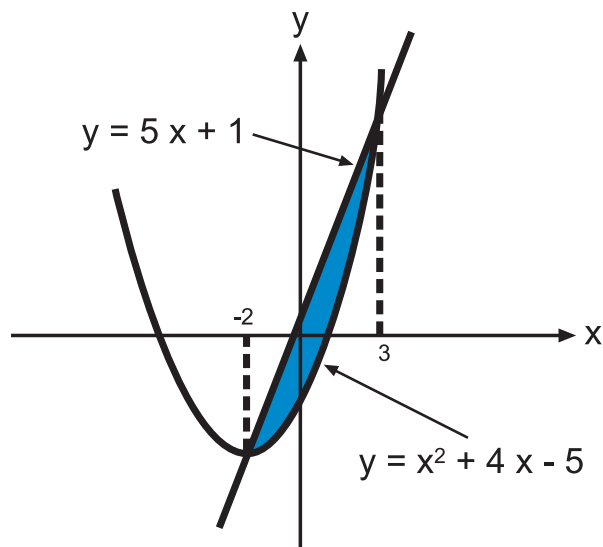
Revision exercise 1

There are randomised questions on-line which you can use for further practice.

30 min

Integrate the following with respect to the relevant variable:

Q1: $\int \left(6x^{-3} - 2 + 3x^3 \right) dx$

Q2: $\int \left(\frac{3 - x^5}{x^3} \right) dx$

Q3: $\int \frac{dt}{\sqrt{5 - 2t}}$

Q4: $\int \left(u + \frac{1}{u} \right)^2 du$

Q5: $\int 3 \cos \left(2\theta + \frac{\pi}{4} \right) d\theta$

Evaluate the following integrals:

Q6: $\int_{0}^{2} (3 + 2x) \, dx$

Q7: $\int_{-2}^{-1} \left(t - 2 + \frac{3}{t^3} \right) dt$

Q8: $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos \left(\frac{\pi}{2} - 2\theta \right) d\theta$

Q9: $\int_{4}^{1} \frac{x^2 - \sqrt{x}}{x} \, dx$ and $\int_{1}^{4} \frac{x^2 - \sqrt{x}}{x} \, dx$

Q10: Find, in its simplest form, the exact value of $\int_{0}^{1} \sqrt{3 - 2x}\ dx$

Q11: Calculate the total area between the graph of $y = x^2 - 5x + 4$, the x-axis and the lines $x = 0$ and $x = 4$.

First, make a sketch of this area and be careful as the area is in two parts, some above and some below the x-axis.

3.2   Introduction

Learning Objective

Recall integration terminology

anti-differentiation

Anti-differentiation is the reverse process of differentiation.

and the method we use to find anti-derivatives is called

integration

Integration is the method we use to find anti-derivatives.

constant of integration

In general if $\frac{d}{dx}$ (F (x)) = f (x) then F (x) is called an anti-derivative, or integral, of f (x) and we write $\int$ f (x)dx = F(x) + C and C is the **constant of integration**

general indefinite integral of f (x)

For $\int$ f (x)dx = F(x) + C, F (x) + C is the **general indefinite integral of f (x)**.

with respect to x of the integral $\int$ f (x)dx

integrand

For $\int$ f (x)dx = F(x) + C, f (x) is the **integrand**

particular integral

For $\int$ f (x)dx = F(x) + C an anti-derivative given by a particular value of C is a **particular integral**.

So, for example, F (x) = x^3 + C is the indefinite integral of f (x) = $3x^2$ whereas

F (x) = x^3 + 6 is a particular integral of f (x) = $3x^2$

definite integral

$\int_{a}^{b}$ f (x)dx is a **definite integral** because the limits of integration are known and

$\int_{a}^{b}$ f (x)dx = $[F(x)]_{a}^{b}$ = F(b) - F(a)

3.3 Standard Integrals

> **Learning Objective**
>
> Learn the standard integrals

Recall that

$^d/_{dx}$ (tan x) = sec^2 x

$^d/_{dx}$(e^x) = e^x

$^d/_{dx}$ (ln x) = $^1/_x$

It therefore follows that

$$\int \sec^2 x dx = \tan x + C$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

Notice that, in this last integral, we take the modulus of x as the ln function does not exist for negative values.

It may be useful to have a summary of all the results we have so far

1.
$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$
$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

2.
$$\int \cos x dx = \sin x + C$$
$$\int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$$

3.
$$\int \sin x \, dx = -\cos x + C$$
$$\int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C$$

4.
$$\int \sec^2 x \, dx = \tan x + C$$
$$\int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + C$$

5.
$$\int \exp(x) \, dx = \exp(x) + C$$
$$\int \exp(ax+b) dx = \frac{1}{a} \exp(ax+b) + C$$

6.
$$\int \frac{1}{x} dx = \ln|x| + C$$
$$\int \frac{dx}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

Study the following examples and make sure that you understand the working.

Examples

1.
$$\int \frac{1}{(3x + 7)^3}dx = \int (3x + 7)^{-3}dx$$
$$= \frac{(3x + 7)^{-2}}{3 \times (-2)} + C$$
$$= -\frac{1}{6(3x + 7)^2} + C$$

2. $\int 3\cos(2x - 1)dx = \frac{3}{2}\sin(2x - 1) + C$

3.
$$\int 6e^{3x + 2}dx = \frac{6}{3}e^{3x + 2} + C$$
$$= 2e^{3x + 2} + C$$

4.
$$\int_0^{\pi/8} \sec^2(2x)dx = \left[\frac{1}{2}\tan(2x)\right]_0^{\pi/8}$$
$$= \frac{1}{2}\left[\tan(\pi/4) - \tan 0\right]$$
$$= \frac{1}{2}[1 - 0]$$
$$= \frac{1}{2}$$

5.
$$\int_0^1 \frac{dx}{3x + 5} = \left[\frac{1}{3}\ln|3x + 5|\right]_0^1$$
$$= \frac{1}{3}[\ln 8 - \ln 5]$$
$$= \frac{1}{3}\ln\left(\frac{8}{5}\right)$$
$$= \frac{1}{3}\ln(1.6)$$
$$= 0.157 \text{ (to 3 decimal places)}$$

6. This question can be done in two ways

$$\int \frac{1}{3x}dx = \frac{1}{3}\int \frac{1}{x}dx \qquad\qquad \int \frac{1}{3x}dx = \frac{1}{3}\ln|3x| + C_2\int \frac{1}{3x}dx$$
$$= \frac{1}{3}\ln|x| + C_1 \qquad\qquad\qquad\qquad = \frac{1}{3}\ln|3x| + C_2$$

                                              or

Both solutions are correct even though at first sight they look different, and in fact we can determine that

$$\frac{1}{3}\ln|3x| + C_2 = \frac{1}{3}\ln|x| + \frac{1}{3}\ln 3 + C_2 \quad (\text{since } \ln|ab| = \ln|a| + \ln|b|)$$
$$= \frac{1}{3}\ln|x| + C_1 \quad (\text{where } C_1 = \frac{1}{3}\ln 3 + C_2)$$

Now try the questions in Exercise 2

Exercise 2

An on-line version of this exercise is available. It contains randomised questions which are similar to those below.

40 min

Integrate the following with respect to the relevant variable:

Q12: $\int \frac{dx}{(2x - 5)^{3/2}}$

Q13: $\int \frac{ds}{2s}$

Q14: $\int 8e^{-4u}du \int 6\cos(3x - 7)dx$

Q15: $\int 6\cos(3x - 7)dx$

Q16: $\int \sec^2(4x + 1)dx$

Q17: $\int \exp(-6t)dt$

Q18: $\int \frac{10dt}{5t + 1}$

Q19: $\int 5e^{3x + 1}dx$

Evaluate the following integrals:

Q20: $\int\limits_{-1}^{1} 3e^{(1 - 3x)}dx$

Q21: $\int\limits_{3}^{7} \frac{4}{x}dx$

Q22: $\int\limits_{0}^{\pi/4} \sec^2 3x\,dx$

Q23: $\int\limits_{2}^{5} \frac{6}{2x - 3}dx$

Q24: $\int\limits_{0}^{\pi/4} (1 - \sin 2x)dx$

Q25: $\int\limits_{0}^{1} \left(\frac{\exp(x + 3) - 2}{\exp(x)} \right) dx$

Example : A tricky integration:

Look at this and make sure that you follow the working.

$$\int \frac{x}{x+2} dx = \int \left(\frac{x+2-2}{x+2} \right) dx \qquad \text{(Rearranging to obtain proper fractions.)}$$

$$= \int \left(\frac{x+2}{x+2} - \frac{2}{x+2} \right) dx$$

$$= \int \left(1 - \frac{2}{x+2} \right) dx$$

$$= x - 2 \ln | x + 2 | + C$$

Now try the questions in Exercise 3

Exercise 3

There is an on-line version of this exercise, which you might find helpful.

10 min

Integrate the following with respect to the relevant variable.

Q26: $\int \frac{x}{x-1} dx$

Q27: $\int \frac{t+5}{t+2} dt$

Q28: $\int \frac{x}{4+2x} dx$

3.4   Integration by Substitution (substitution given)

Learning Objective

Learn to use the substitution method to integrate more complicated functions

Suppose we wish to integrate the function $x^3(2x^4 - 3)^4$. We could expand the bracket, multiply by x^3 and then integrate each term separately. Not an enviable task.

We can make this integration a lot easier if we change the variable from x to some other suitably chosen variable. To understand how this works see the following examples.

Examples

1.

Find $\int x^3(2x^4 - 3)^4 dx$

Answer

Let $u = 2x^4 - 3$

then $\frac{du}{dx} = 8x^3$ and $\frac{dx}{du} = \frac{1}{8x^3}$

Thus

$$\int x^3(2x^4 - 3)dx = \int x^3(2x^4 - 3)^4 \frac{dx}{du}du$$

$$= \int \frac{x^3}{8x^3}(2x^4 - 3)^4 du$$

$$= \int \frac{u^4}{8}du$$

$$= \frac{u^5}{40} + C$$

$$= \frac{1}{40}(2x^4 - 3)^5 + C$$

2.

Find $\int \sin^2 x \cos x dx$

Answer

Let $u = \sin x$

then $\frac{du}{dx} = \cos x$ and $\frac{dx}{du} = \frac{1}{\cos x}$

Thus

$$\int \sin^2 x \cos x\, dx = \int \sin^2 x \cos x \frac{dx}{du}du$$

$$= \int \sin^2 x \cos x \frac{1}{\cos x}du$$

$$= \int u^2 du$$

$$= \frac{1}{3}u^3 + C$$

$$= \frac{1}{3}\sin^3 x + C$$

Now try the questions in Exercise 4

Exercise 4

There are randomised questions on-line which you can use for further practice.

35 min

Integrate the following functions using the suggested substitution:

Q29: $\int 5x(x^2 + 6)^5 dx$,   using $u = x^2 + 6$

Q30: $\int 3x\sqrt{(1 + x^2)}dx$,   using $u = 1 + x^2$

Q31: $\int \cos^3 x \sin x dx$,   using $u = \cos x$

Q32: $\int 6x \sin(x^2 + 3)dx$,   using $u = x^2 + 3$

Q33: $\int \frac{8x}{x^2 + 1}dx$,   using $u = x^2 + 1$

Q34: $\int x^2 \exp(x^3 + 5)dx$,   using $u = x^3 + 5$

The following example has an extra step. Study the example carefully and make sure you understand the working.

Example

Find $\int 3x(x + 2)^5 dx$

Answer

Let $u = x + 2$ then $^{du}/_{dx} = 1$ and $^{dx}/_{du} = 1$

This time we also need to know that since $u = x + 2$ then $x = u - 2$

This substitution makes it easier to multiply out the brackets.

Thus

$$\int 3x(x + 2)^5 dx = \int 3x(x + 2)^5 \frac{dx}{du} du$$

$$= \int 3x\,u^5 du$$

$$= \int 3(u - 2)u^5 du \quad (u - 2 \text{ has been substituted for } x)$$

$$= \int 3u^6 - 6u^5 du$$

$$= \frac{3}{7}u^7 - u^6 + C$$

$$= \frac{1}{7}u^6(3u - 7) + C$$

$$= \frac{1}{7}(x + 2)^6(3(x + 2) - 7) + C$$

$$= \frac{1}{7}(x + 2)^6(3x - 1) + C$$

Now try the questions in Exercise 5

Exercise 5

There are randomised questions on-line which you can use for further practice.

15 min

Integrate the following functions using the suggested substitution:

Q35: $\int x(x + 5)^4 dx$,   using $u = x + 5$

Q36: $\int \frac{4x}{\sqrt{2x - 1}} dx$,   using $u = 2x - 1$

Q37: $\int 6x(3x - 2)^5 dx$,   using $u = 3x - 2$

3.5   Definite Integrals with Substitution

Learning Objective

Learn to use the substitution method to evaluate the integrals of functions where the substitution is given

Substitution can also be used with a definite integral.

You must be careful to pay attention to the limits of the integral as these will change with a substitution.

Look at the following example:

Example Find $\int\limits_{0}^{1} 15x^2(x^3 + 1)^4 dx$

Answer

Let $u = x^3 + 1$ then $^{du}/_{dx} = 3x^2$ and $\frac{dx}{du} = \frac{1}{3x^2}$

The limits of the original function are from $x = 0$ to $x = 1$

when $x = 0$ then $u = 0^3 = 1 = 1$

when $x = 1$ then $u = 1^3 + 1 = 2$

So the limits after substitution are from $u = 1$ to $u = 2$

Thus

$$\int\limits_{0}^{1} 15x^2(x^3 + 1)^4 dx = \int\limits_{1}^{2} 15x^2(x^3 + 1)^4 \frac{dx}{du} du$$

$$= \int\limits_{1}^{2} 15x^2(u)^4 \frac{1}{3x^2} du$$

$$= \int\limits_{1}^{2} 5u^4 du$$

$$= \left[u^5\right]_{1}^{2}$$

$$= 2^5 - 1^5$$

$$= 31$$

Now try the questions in Exercise 6

Exercise 6

There are randomised questions on-line which you can use for further practice.

30 min

Evaluate the following integrals, making sure that you change the limits of integration whenever necessary:

Q38: $\int\limits_{0}^{1} 12x(x^2 + 5)^5 dx$, using $u = x^2 + 5$

Q39: $\int\limits_{1}^{4} \frac{(1 + \sqrt{x})^3}{\sqrt{x}} dx$, using $u = 1 + \sqrt{x}$

Q40: $\int\limits_{0}^{\pi/6} 10\sin^4 x \cos x \, dx$, using $u = \sin x$

Q41: $\int\limits_{1}^{2} 4x(2x-3)^4 dx$,   using $u = 2x - 3$

Q42: $\int\limits_{1}^{6} \frac{x}{\sqrt{x+3}} dx$,   using $u = x + 3$

Q43: $\int\limits_{0}^{1} (6x-1)\exp(3x^2 - x)dx$,   using $u = 3x^2 - x$

Q44: $\int\limits_{0}^{\pi/4} \sec^2 x \tan^5 x\, dx$,   using $u = \tan x$

Q45: $\int\limits_{0}^{\pi/3} \tan x\, dx$,   using $u = \cos x$

Often a trig substitution will make an integration much easier, especially when used along with a trig identity.

Example

Find $\int\limits_{0}^{3/4} \frac{x}{\sqrt{9-4x^2}} dx$,   using $x = \frac{3}{2}\sin u$ for $0 \leqslant u \leqslant \frac{\pi}{2}$

Answer

Notice that we have a different type of substitution this time.

When $x = \frac{3}{2}\sin u$ then $\frac{dx}{du} = \frac{3}{2}\cos u$ and $x^2 = \frac{9}{4}\sin^2 u$

The limits of the integral will also change,

when x = 0 then $^3/_2 \sin u = 0$ and thus u = 0

and when x = $^3/_4$ then $^3/_2 \sin u = ^3/_4$

and so sin u = $^1/_2$

u = $\pi/6$

Therefore $\displaystyle\int_0^{3/4} \frac{x}{\sqrt{9-4x^2}}\,dx = \int_0^{\pi/6} \frac{\frac{3}{2}\sin u}{\sqrt{9-9\sin^2 u}}\frac{dx}{du}\,du$

$$= \int_0^{\pi/6} \frac{\frac{3}{2}\sin u}{\sqrt{9}\sqrt{(1-\sin^2 u)}}\frac{3}{2}\cos u\,du$$

$$= \int_0^{\pi/6} \frac{\frac{3}{2}\sin u}{3\sqrt{\cos^2 u}}\frac{3}{2}\cos u\,du \text{ (since } 1-\sin^2 u = \cos^2 u)$$

$$= \int_0^{\pi/6} \frac{\frac{3}{2}\sin u}{3\cos u}\frac{3}{2}\cos u\,du$$

$$= \int_0^{\pi/6} \frac{3}{4}\sin u\,du$$

$$= \left[-\frac{3}{4}\cos u \right]_0^{\pi/6}$$

$$= -\frac{3}{4}\frac{\sqrt{3}}{2} + \frac{3}{4}$$

$$= \frac{3}{4}\left(1 - \frac{\sqrt{3}}{2}\right)$$

Now try the questions in Exercise 7

Exercise 7

There is an on-line version of this exercise with randomised questions, which you might find helpful.

30 min

Evaluate the following integrals, using the substitution given. Remember to change the limits of the integral whenever necessary:

Q46: $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx$,   using $x = \sin u$

Q47: $\displaystyle\int_0^{5/4} \frac{1}{\sqrt{25-4x^2}}\,dx$,   using $x = \frac{5}{2}\sin u$

Q48: In this question it helps to notice that $1 = \tan^2 u = \sec^2 u$ (Check this for yourself.)

$\displaystyle\int_0^{2/3} \frac{1}{4+9x^2}\,dx$,   using $x = \frac{2}{3}\tan u$

Q49: $\displaystyle\int_0^3 \frac{1}{9+3x^2}\,dx$,   using $x = \frac{3}{\sqrt{3}}\tan u$

Q50: In this question you will need to know that $\cos 2\theta = 2\cos^2\theta - 1$ and so $\cos^2\theta = \frac{1}{2}(\cos 2\theta + 1)$

$\displaystyle\int_0^{5/\sqrt{2}} \sqrt{25-x^2}\,dx$,   using $x = 5\sin\theta$

Q51: $\int_{0}^{1/4} \sqrt{1 - 4x^2}\,dx$,   using $x = \frac{1}{2}\sin t$

Q52: $\int_{1/2}^{1} \frac{6dx}{\sqrt{2x - x^2}}$,   using $x = 1 - \sin\theta$

Q53: $\int_{0}^{3\sqrt{3}/2} \frac{x - 2}{\sqrt{9 - x^2}}\,dx$,   using $x = 3\sin t$

3.6   Integration by Substitution (substitution not given)

> **Learning Objective**
>
> Learn to use the substitution method to integrate functions when the substitution is not given

You should now begin to recognise which substitution to use, when necessary, in an integration.

This is made easier if you can recognise some of the different types of integration that you will encounter.

The following examples may help you to do this:

1. When $\int f(ax + b)dx$ substitute $u = ax + b$

Example

Calculate $\int \sin(3x + 2)dx$

Answer

Let $u = 3x + 2$ then $\frac{du}{dx} = 3$

Hence

$$\int \sin(3x + 2)dx, \; = \int \sin u \frac{dx}{du}du$$
$$= \int \frac{\sin u}{3}du$$
$$= -\frac{1}{3}\cos u + C$$
$$= -\frac{1}{3}\cos(3x + 2) + C$$

Normally you would not do this using the substitution method. It is often thought of as 'reverse chain rule'. Similarily, for example: $\int (5x - 2)^4 dx$

2. When $\int f'(x)\,[f(x)]^n\,dx$ substitute $u = f(x)$

Example

Calculate $\int x^4 (x^5 - 6)^3 dx$

Answer

Let $u = x^5 - 6$ then $^{du}/_{dx} = 5x^4$

Hence

$$
\begin{aligned}
\int x^4(x^5 - 6)^3 dx &= \int x^4 u^3 \frac{dx}{du} du \\
&= \int x^4 u^3 \frac{1}{5x^4} du \\
&= \int \tfrac{1}{5} u^3 du \\
&= \frac{u^4}{20} + C \\
&= \frac{1}{20}(x^5 - 6)^5 + C
\end{aligned}
$$

---

3. When $\int (ax + b)(cx + d)^n dx$ substitute $u = cx + d$

Example

Calculate $\int x(4x - 3)^5 dx$

Answer

In this example the substituiton makes it easier to multiply out the brackets.

Let $u = 4x - 3$ then $^{du}/_{dx} = 4$ and $x = \frac{1}{4}(u + 3)$

Hence

$$
\begin{aligned}
\int x(4x - 3)^5 dx &= \tfrac{1}{4} \int (u^6 + 3u^5) \tfrac{1}{4} du \\
&= \tfrac{1}{16} \left(\tfrac{1}{7} u^7 + \tfrac{1}{2} u^6 \right) + C \\
&= \tfrac{1}{16} u^6 \left(\tfrac{1}{7} u + \tfrac{1}{2} \right) + C \\
&= \frac{1}{224}(4x - 3)^6(8x + 1) + C
\end{aligned}
$$

---

4. When $\int f'(x) e^{f(x)} dx$ substitute $u = f(x)$

Example

Calculate $\int x e^{(x^2 - 5)} dx$

Answer

Let $u = x^2 - 5$ then $^{du}/_{dx} = 2x$

Hence

$$\int xe^{(x^2 - 5)}dx = \int xe^u \frac{dx}{du}du$$

$$= \int xe^u \frac{1}{2x}du$$

$$= \int \tfrac{1}{2}e^u du$$

$$= \tfrac{1}{2}e^u + C$$

$$= \tfrac{1}{2}e^{(x^2 - 5)} + C$$

---

5. When $\int \frac{f'(x)}{f(x)}dx$ substitute $u = f(x)$

Example

Calculate $\int \frac{6x}{x^2 + 3}dx$

Answer

Let $u = x^2 + 3$, then $^{du}/_{dx} = 2x$

Hence

$$\int \frac{6x}{x^2 + 3}dx = \int \frac{6x}{u}\frac{dx}{du}du$$

$$= \int \frac{6x}{u}\frac{1}{2x}du$$

$$= \int \frac{3}{u}du$$

$$= 3\ln|u| + C$$

$$= 3\ln\left|x^2 + 3\right| + C$$

Now try the questions in Exercise 8

Exercise 8

There is an on-line version of this exercise with randomised questions, which you might find helpful.

40 min

Perform the following integrations, using a suitable substitution if you need to.

It will help if you decide on the type of integral first, as in the previous examples.

Q54: $\int (3x^3 - 1)^4 x^2 dx$

Q55: $\int 4\cos(2x - 5)dx$

Q56: $\int 6x^2 \exp(x^3 + 5)dx$

Q57: $\int \frac{x^2 dx}{x^3 - 3}$

Q58: $\int 3x\cos(1 - x^2)dx$

Q59: $\int \frac{(2x - 3)dx}{\sqrt{x^2 - 3x + 1}}$

Evaluate the following integrals using a suitable substitution whenever necessary:

Q60: $\int\limits_{0}^{2} \frac{6x^2}{\sqrt{x^3 + 1}}dx$

Q61: $\int\limits_{2}^{3} x \exp(x^2 - 3)dx$

Q62: $\int\limits_{0}^{\pi/2} 2\cos\left(4\theta - \frac{\pi}{2}\right)d\theta$

Q63: $\int\limits_{0}^{2} x\sqrt{2x^2 + 1}\,dx$

Q64: $\int\limits_{2}^{4} \frac{2x}{1 - x^2}dx$

Q65: $\int\limits_{1}^{4} (x + 1)(x - 1)^3 dx$
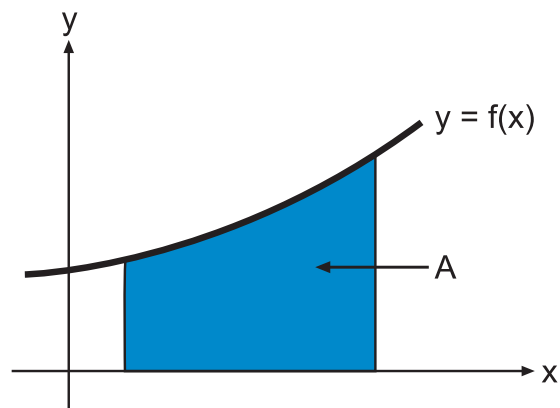
3.7   The area between a curve and the x-axis

Learning Objective

To gain an elementary understanding of the integral as the area under a curve

In the diagram A is the area under the curve
y = f (x) from x = a to x = b

We can estimate this area by dividing it into strips which approximate to rectangles and then totalling the areas of these rectangles.

The total area of the rectangles gives an approximate value for A.

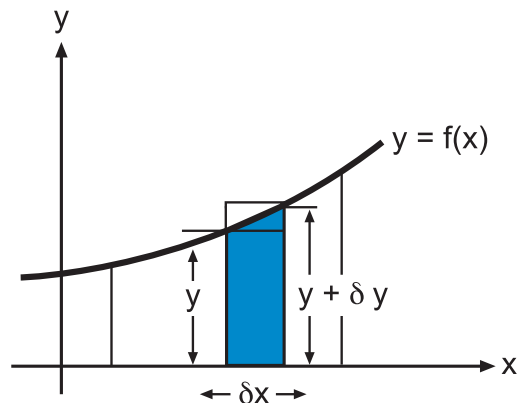Generally the more rectangles we use, the closer we get to the actual value for A.
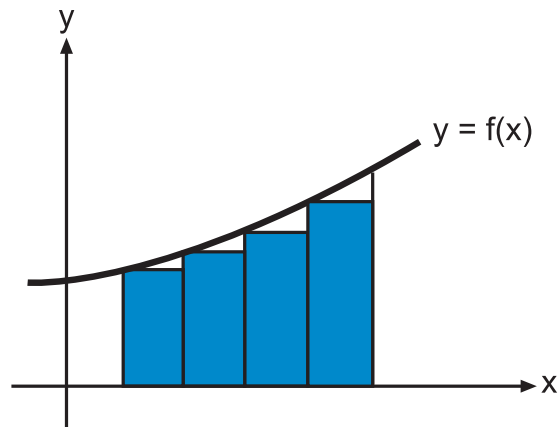
Consider one rectangle, width δx, and let δS be the shaded area under the curve $y = f(x)$

By comparing areas in the diagram we have

$$y\delta x \leqslant \delta S \leqslant (y + \delta y)\delta x$$

and then

$$y \leqslant \frac{\delta S}{\delta x} \leqslant y + \delta y$$

Now as $\delta x \to 0$ the number of rectangles increases and $\dfrac{\delta S}{\delta x} \to \dfrac{dS}{dx}$ and $\delta y \to 0$

So $\quad y \leqslant \dfrac{\delta S}{\delta x} \leqslant y + \delta y$

becomes $\quad y \leqslant \dfrac{dS}{dx} \leqslant y$

$$\Rightarrow \frac{dS}{dx} = y$$

$$\Rightarrow \quad S = \int y\,dx$$

This integration will give an area function S (x) and will involve a constant of integration C. However, if we define right- and left-hand boundaries for x we can then obtain the area function as a definite integral, and in fact the original area under the curve is

$$A = \lim_{\delta x \to 0} \sum_{x=a}^{x=b} y\delta x = \int_{a}^{b} y\,dx = S(b) - S(a)$$

We have shown this result to be true for a monotonically increasing function; however, note that this same result is also true for more complicated functions.

Area under a curve

Remember that when finding the area under a curve:

5 min

- make a sketch of the curve;

- areas above the x-axis give a positive value for the definite integral;

- areas below the x-axis give a negative value for the definite integral.

> Also, you should recall that the shaded area between two curves can be calculated
> as $\int\limits_{a}^{b} [f(x) - g(x)]\, dx$ where $f(x) \geqslant g(x)$ and $a \leqslant x < b$

(You can see an example of finding the area between two curves in Prerequisites: Example 6.)

Example

Find the area between the curve $y = 4x - 2x^2$ and the x-axis from $x = 0$ to $x = 3$

Answer

A sketch of the curve shows that the area is in two parts. One part is above the x-axis and has a positive value for the definite integral and the other part is below the x-axis and has a negative value for the definite integral.

So the two areas need to be calculated separately.
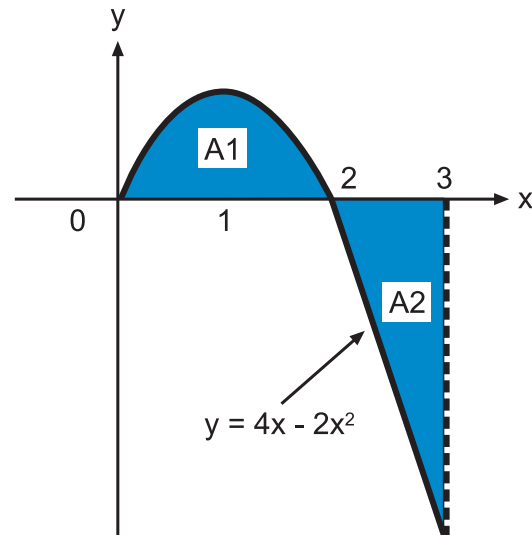
The shaded area is above the x-axis for $x = 0$ to $x = 2$

$$\int\limits_{0}^{2} (4x - 2x^2)\,dx = \left[2x^2 - \tfrac{2}{3}x^3\right]_{0}^{2}$$

$$= 8 - \frac{16}{3}$$

$$= 2\tfrac{2}{3} \int\limits_{0}^{2} (4x - 2x^2)\,dx = \left[2x^2 - \tfrac{2}{3}x^3\right]_{0}^{2}$$

$$= 8 - \frac{16}{3}$$

$$= 2\tfrac{2}{3}$$

The shaded area is below the x-axis for $x = 2$ to $x = 3$

$$\int_{2}^{3} (4x - 2x^2)dx = \left[2x^2 - \tfrac{2}{3}x^3\right]_{2}^{3}$$

$$= (18 - 18) - (8 - \tfrac{16}{3})$$

$$= -2\tfrac{2}{3}$$

The answer is negative as the area is below the x-axis. So the actual area is $2^2/_3$

The total shaded area is therefore $2^2/_3 + 2^2/_3 + 5^1/_3$ units ^{2}

Q66: What value would you obtain for $\int_{0}^{3} \left(4x - 2x^2\right) dx$? Why?

Now try the questions in Exercise 9

Exercise 9

20 min

There is an on-line version of this exercise, which you might find helpful.

Q67: Find the area enclosed by the x-axis and the curve

$y = x^2 - 6x + 8$ between $x = 0$ and $x = 4$
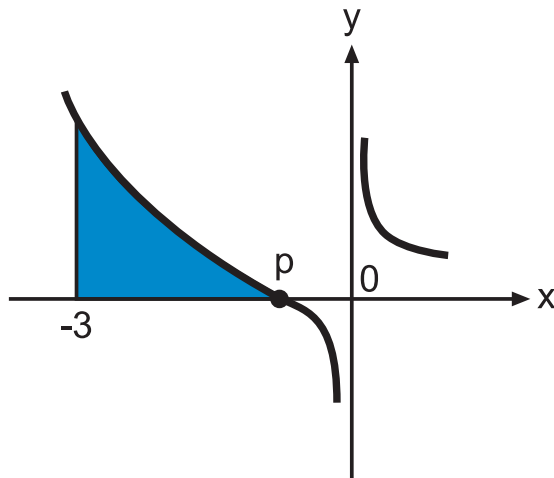
Remember to make a sketch of the curve first.

Q68:

a) Find the three roots where the curve $y = x^3 + 3x^2 - 8x - 10$ and the straight line $y = 5x + 5$ intersect.

b) Find the shaded area enclosed between the curve and the straight line. Look at the sketch and notice that the shaded area is in two parts and will need to be calculated separately.

$y = 5x+5$

$y = x^3 + 3x^2 - 8x -10$
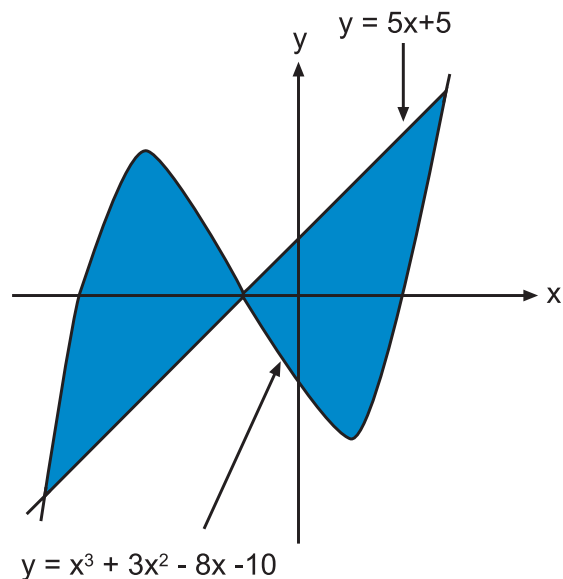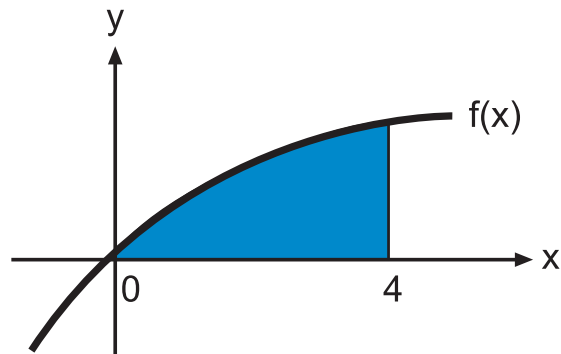
Q69:

The diagram shows part of the curve

for $y = x^2 + \frac{1}{x}$

Find the coordinates of p and calculate the shaded area.

Q70:

Part of the graph for $f(x) = \frac{4x}{\sqrt{x^2+2}}$ is shown here.

Calculate the shaded area.

3.8 The area between a curve and the y-axis

Learning Objective

Understand how to calculate the area between a curve and the y-axis

In the diagram shown here we wish to find the shaded area between the curve and the y-axis.

In a similar as before we find that

$$A = \int_a^b x \, dy$$

Notice that this time we must integrate with respect to y
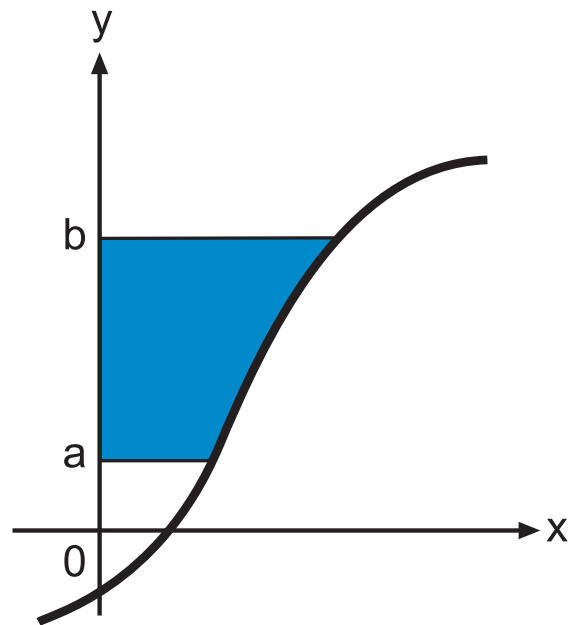
An example may help to make this clearer.

Example

Find the shaded area enclosed by the y-axis and the curve $y = (x - 4)^3$ between $y = 1$ and $y = 8$

Answer

Area = $\int_1^8 x \, dy$

We cannot integrate x with respect to y so we must rearrange the equation of the curve in terms of y

$$y = (x - 4)^3$$
$$y^{1/3} = x - 4$$
$$x = y^{1/3} + 4$$
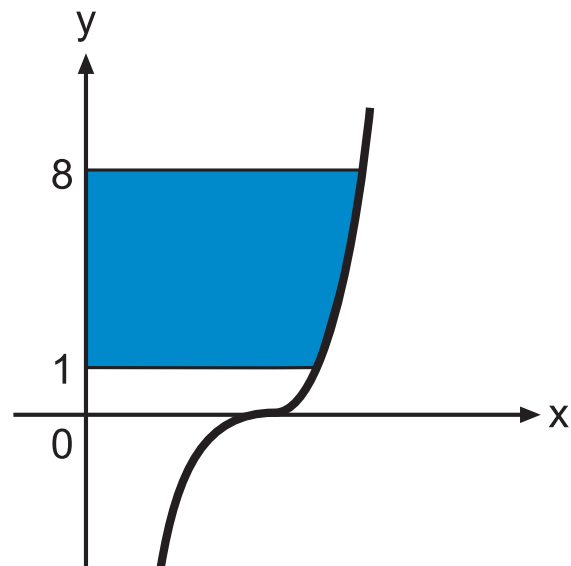
Therefore, area $= \displaystyle\int_{1}^{8} x\,dy$

$$= \int_{1}^{8} \left(y^{1/3} + 4\right) dy$$

$$= \left[\tfrac{3}{4}y^{4/3} + 4y\right]_{1}^{8}$$

$$= 44 - 4\tfrac{3}{4}$$

$$= 39\tfrac{1}{4}$$

Now try the questions in Exercise 10

Exercise 10

There is an on-line exercise at this point, which you might find helpful.

20 min

Q71: Find the area enclosed between the curve $y = x^2$ and the y-axis between $y = 0$ and $y = 4$ for $x > 0$

Q72: Make a sketch of the curve $y = (x + 2)^3$ and find the area enclosed between this curve and the y-axis between $y = 1$ and $y = 8$

Q73:

Find the shaded area in the diagram shown here.

The sketch represents the graph of $y = \frac{1}{x^2}$

Q74:

Find the area enclosed between the curve $y^2 = 9 - x$ and the y-axis, as shown in the diagram.
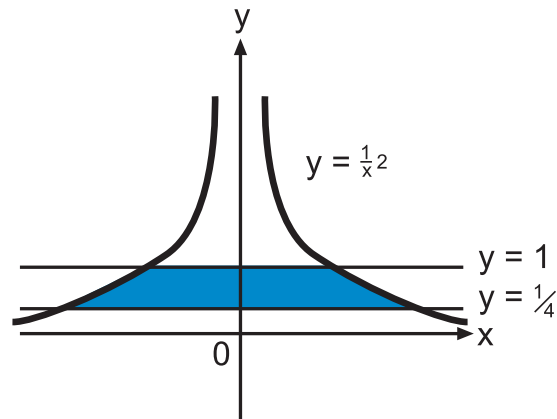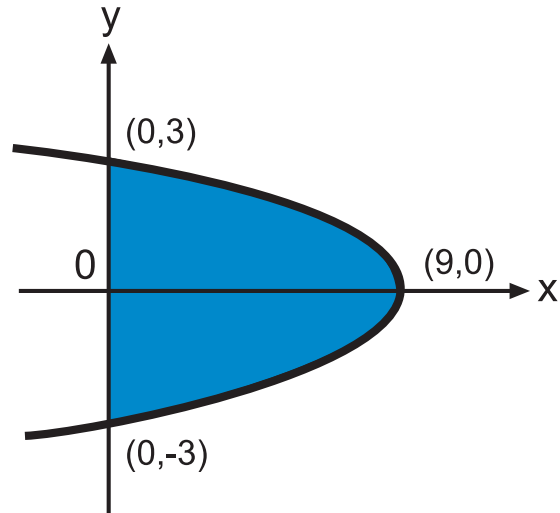
3.9   Volumes of Revolution

Learning Objective

Understand how to calculate the volume of a solid of revolution.

The shaded area enclosed by the curve $y = f(x)$ and the x-axis between $x = a$ and $x = b$ is shown in the diagram

Suppose this area is rotated $360°$ about the x-axis, then the solid that is formed is known as a solid of revolution.

We wish to calculate its volume, V
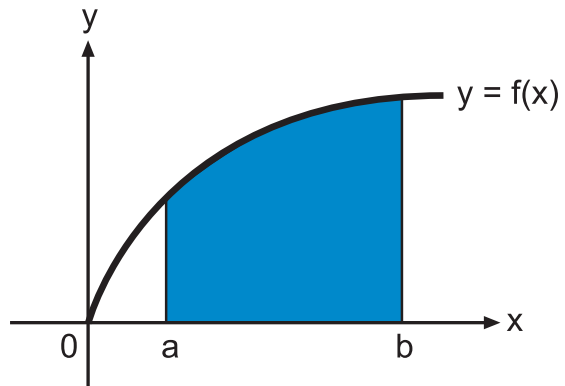
Now, consider the strip with thickness δx and height y. When this is rotated about the x-axis it produces a disc of radius y and thickness δx. Let δV be the volume of this disc, then $\delta V \approx \pi y^2 \delta x$

(Compare with the volume of a cylinder = $\pi r^2 h$)

Now as $\delta x \to 0$

then $V = \lim\limits_{\delta x \to 0} \sum\limits_{x=a}^{x=b} \pi y^2 dx$
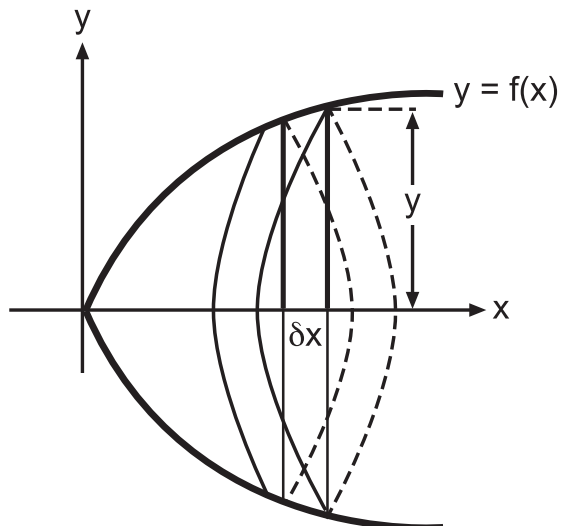
$\quad = \int\limits_{a}^{b} \pi y^2 dx$

> Therefore the volume of a solid of revolution is given by:
>
> $$V = \int_a^b \pi y^2 dx \quad \text{where} \quad y = f(x)$$

Examples

1.

The shaded area shown in the diagram opposite is the area between the curve y = 2x + x² and the x-axis from x = 2 to x = 3

Calculate the volume of the solid of revolution formed when this area is given a full turn about the x-axis.
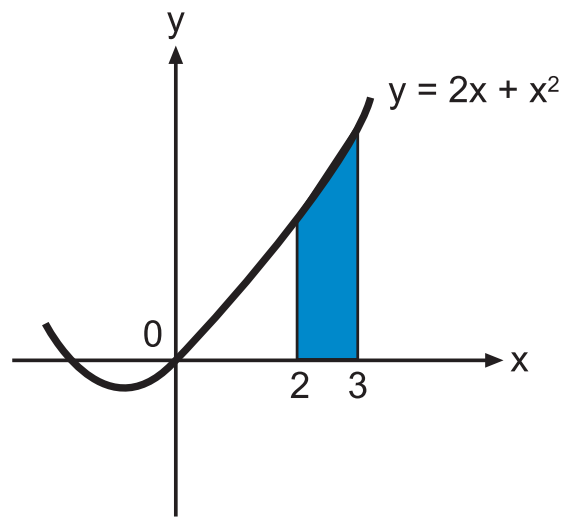
$y = 2x + x^2$

Answer

The volume is given by

$$V = \int_a^b \pi y^2 dx \quad \text{where} \quad y = 2x + x^2$$

$$= \int_2^3 \pi (2x + x^2)^2 dx$$

$$= \pi \int_2^3 \left(4x^2 + 4x^3 + x^4\right) dx$$

$$= \pi \left[\tfrac{4}{3}x^3 + x^4 + \tfrac{1}{5}x^5\right]_2^3$$

$$= \pi \left[(36 + 81 + 48\tfrac{3}{5}) - (10\tfrac{2}{3} + 16 + 6\tfrac{2}{5})\right]$$

$$= 132\tfrac{8}{15}\pi \quad \text{units}^3$$

2.

Another interesting example to consider is this one:

Calculate the volume of the solid formed when the semi-circle with equation x² + y² = r², y ≥ 0 is given a full turn about the x-axis.

From our formula we have

$$V = \int_{-r}^{r} \pi y^2 dx$$

$$= \int_{-r}^{r} \pi(r^2 - x^2)dx$$

$$= \int_{-r}^{r} (\pi r^2 - \pi x^2)dx$$

$$= \left[\pi r^2 x - \frac{1}{3}\pi x^3 \right]_{-r}^{r}$$

$$= \pi r^3 - \frac{1}{3}\pi r^3 + \pi r^3 - \frac{1}{3}\pi r^3$$

$$= \frac{4}{3}\pi r^3$$
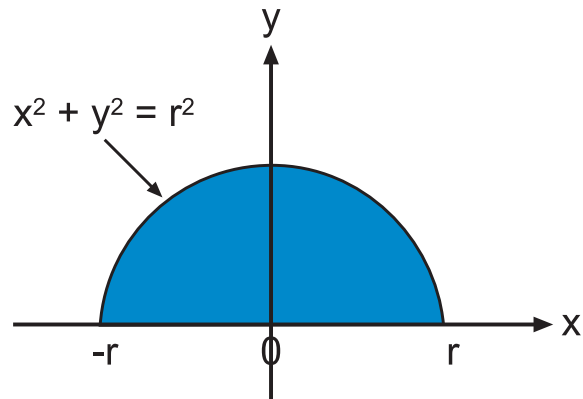
which, as you might expect, gives the formula for the volume of a sphere.

$x^2 + y^2 = r^2$

One of the first mathematicians to perfect a method for computing the volume of a sphere was **Archimedes of Syracuse**. You can read more about this mathematician in the Extended information chapter.

Now try the questions in Exercises 11 and 12

Exercise 11

There is an on-line exercise at this point, which you might find helpful.

15 min

Find the volumes of the solids of revolution formed when each of the following areas is given one complete turn about the x-axis. It might help to make a sketch. You should give your answers in terms of π.

Q75: The area between the curve $y = x^2 + 4x$ and the x-axis from $x = 0$ to $x = 2$

Q76: The area enclosed by the x-axis and $y = x^2 + 4x$ from $x = -4$ to $x = 0$

Q77: The area enclosed by the x-axis and $y = \sin x$ a from $x = 0$ to $x = \pi$

Q78:

Part of the graph for $y = 2x - \frac{8}{x}$ is shown here.

The shaded area is between x = 2 and x = 4

Calculate the volume of the solid formed when this shaded area is given a full turn about the x-axis.

Exercise 12

Q79:

15 min
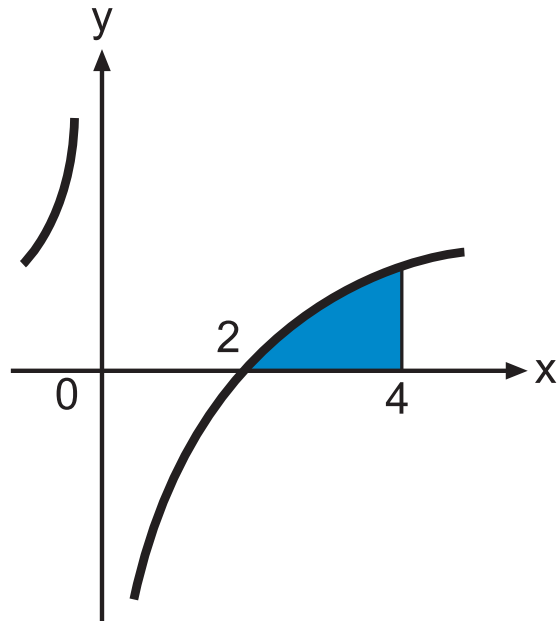
You now know that the formula for the volume of a solid formed by rotating an area one full turn about the **x-axis** is given by $V = \int_a^b \pi y^2 dx$ where $y = f(x)$

Write down your conjecture for the formula for the volume of the solid which is formed by rotating the area shown here one full turn about the **y-axis**.

Check your answer before trying the next two questions.
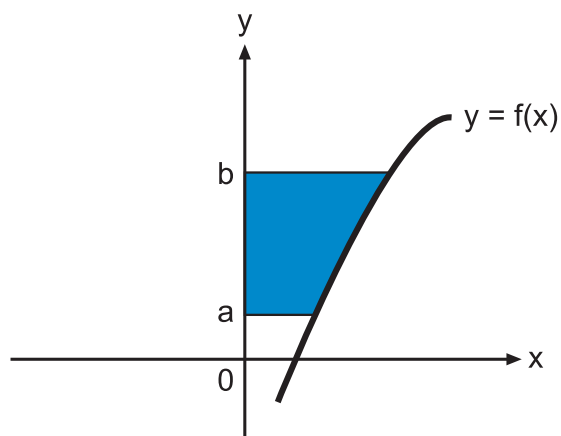
An online assessment is provided to help you review this topic.

Q80:

Find the volume of the solid formed when the shaded area shown here is given a full turn about the y-axis.
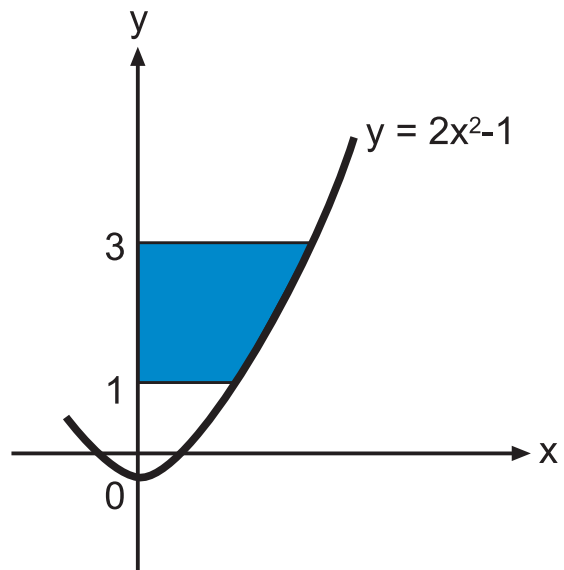
Q81:

Find the volume of the solid formed when the shaded area shown here is given a full turn about the y-axis.
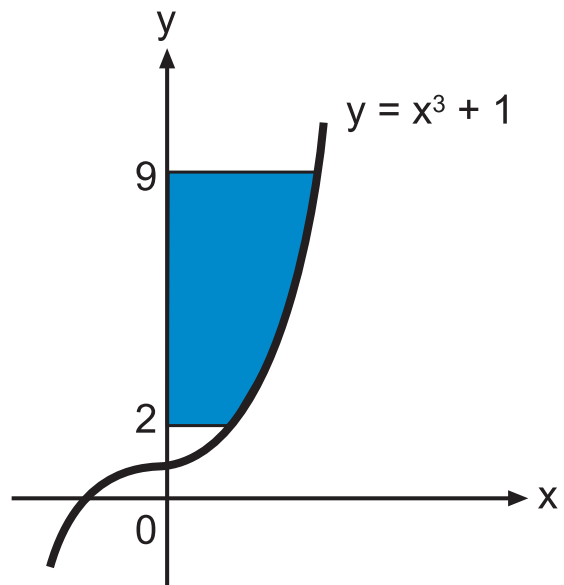
3.10   Rates of Change

Learning Objective

Establish the connection between acceleration, velocity and displacement

You should recall that *displacement* (s), *velocity* (v) and *acceleration* (a) are linked together by differentiation with respect to time.

In reverse order, *acceleration, velocity* and *displacement* are linked together by integration.

So from $v = \dfrac{ds}{dt}$ it follows that $s = \displaystyle\int v\,dt$

also from $a = \dfrac{dv}{dt}$ it follows that $v = \displaystyle\int a\,dt$

To summarise:
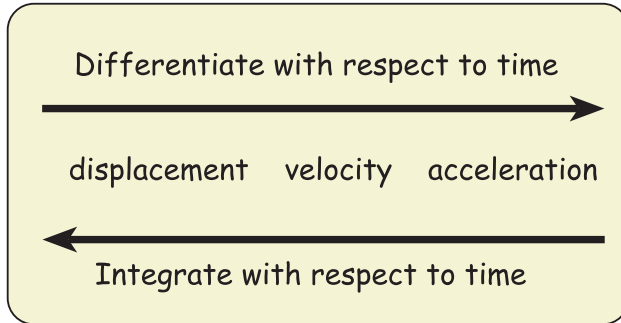
Example A particle moves in a straight line with a constant acceleration a m/s^2. It starts from the origin at time t = 0 and has initial velocity u m/s.

a) Obtain an expression for velocity at time t

b) Obtain an expression for the displacement of the particle at time t

Answer

a) Since $v = \int a\,dt$ integrating with respect to t gives v = at + C (1)

   We also know that when t = 0, v = u m/s

   Substituting this into equation (1) gives u = a × 0 + C and thus C = u

   Therefore the expression for velocity at time t is v = u + at

b) Since $s = \int v\,dt = \int (u + at)\,dt$ integrating with respect to t gives $s = ut + \frac{1}{2}at^2 + C$ (2)

   Since the particle starts from the origin when t = 0 then s = 0 also.

   Substituting these values into equation (2) gives C = 0

   Thus the expression for displacement of the particle at time t is

   s = ut + $^1/_2$ at^2

Physics scholars: perhaps you recognise these formulae!

Now try the Questions in Exercise 13

Exercise 13

There is an on-line exercise at this point, which you might find helpful.

20 min

Q82: The velocity of a car after t seconds is given by the formula v (t) = 20 + 5t m/s

a) Find an expression for s (t), the distance travelled in t seconds.

b) Calculate the distance travelled in the first 4 seconds.

Q83: A rocket is launched vertically from rest at ground level. The acceleration is given by the formula a (t) = 200 - 2t m/s^2 for $0 \leq t \leq 100$

    a) Find an expression for the velocity v (t) in terms of t

    b) Find an expression for the height of the rocket h (t) after t seconds

    c) Find the height of the rocket after 100 seconds

Q84:

The velocity/time graph of a particle is shown. At t = 0 the particle is a rest.

$v = t^3 - 4t^2 + 3t$

    a) Between t = 0 and t = 1 the particle moves to the right. Describe how its speed changes during this time.

    b) Describe its direction and speed between t = 1 and t = 3

    c) Calculate how far away the particle is from the origin at t = 3

    d) Calculate the total distance travelled between t = 0 and t = 3

    e) Calculate the acceleration of the particle at t = 3
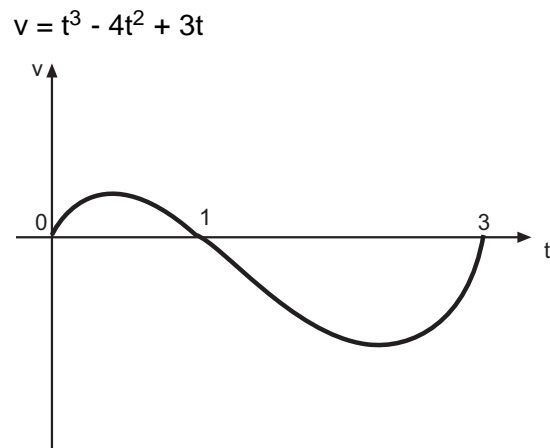
3.11   Summary

> **Learning Objective**
>
> Consolidation of new content in this topic

    1.

        $\int \sec^2 x dx = \tan x + C$

    2.

        $\int e^x dx = e^x + C$

    3.

        $\int \frac{1}{x} dx = \ln |x| + C$

    4.

        In many cases an integration is made simpler by using a substitution.
        In simple cases you will be expected to decide on the substitution to
        be used, whereas for more complex problems the substitution will be
        given.

5.

The area under a curve.

The area under the curve y = f (x) between the x-axis and the lines x = a and x = b, as shown here, is given by the integral

$$\int_a^b f(x)dx$$
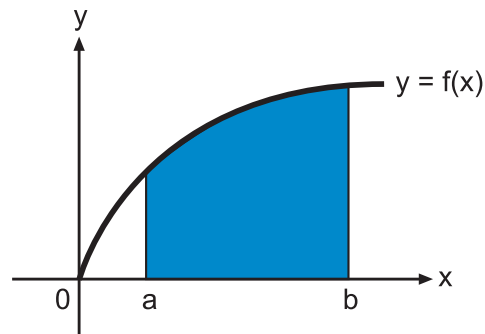
6.

Volumes of revolution.

When the shaded area shown above is given a full turn about the x-axis, the solid that is formed has a volume given by

$$V = \int_a^b \pi y^2 dx \quad \text{where} \quad y = f(x)$$

Note that a similar formula is obtained for a rotation of $360°$ about the y-axis.

3.12 Extended information

Learning Objective

To encourage an interest in related topics

There are links on-line which give a variety of web sites related to this topic.

Archimedes of Syracuse

Archimedes was born in 287 BC and died in 212 BC in Syracuse in Sciliy.

He achieved fame in his own time through his mechanical inventions. Many of these were war machines, which proved to be particularly effective in the defence of Syracuse when attacked by the Roman general Marcellus.

On a visit to Egypt he also invented a type of pump known as Archimedes' screw, which is still used today in many parts of the world.

Another of his inventions, which brought him great fame, was the compound pulley. It was written that using this small device he was able to move the great weight of 'a ship of burden out of the king's arsenal, which could not be drawn out of the dock without

great labour and many men' and 'holding the head of the pulley in his hand and drawing the cords by degrees, he drew the ship in a straight line, as smoothly and evenly as if she had been in the sea.'

He is noted as having stated: 'give me a place to stand and I will move the earth.' This was in the context of his work on levers.

give me a place to stand and I will move the earth.
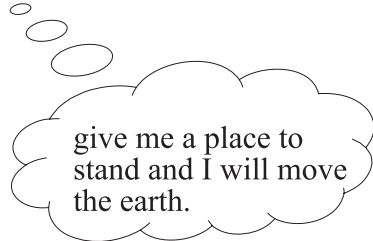
Although he achieved fame through his mechanical inventions, Archimedes believed that pure mathematics was the only worthy pursuit. He is considered by most historians to be one of the greatest mathematicians of all time. He demonstrated that he could approximate square roots accurately. He invented a system of expressing large numbers, and also perfected an integration method that allowed him to calculate the areas, volumes and surface areas of many solids. He found the volume and surface area of a sphere. Archimedes considered his most significant accomplishments were those concerning a cylinder circumscribing a sphere, and he gave instructions that his tombstone should have displayed on it a diagram consisting of a sphere with a circumscribing cylinder.

It is also claimed that his work on integration 'gave birth to the calculus of the infinite conceived and brought to perfection by Kepler, Cavalieri, Fermat, Leibniz and Newton'.

In mechanics, Archimedes discovered theorems for the centre of gravity of plane figures and solids. His most famous theorem gives the weight of a solid immersed in a liquid, called Archimedes' principle. This discovery is said to have inspired his famous shout 'Eureka!' ('I have found it!').

Archimedes was killed in 212 BC by a Roman soldier during the capture of Syracuse by the Romans in the Second Punic War. Cicero, a Roman statesman, while serving in Sicily, had Archimedes' tombstone restored. It has been claimed that 'the Romans had so little interest in pure mathematics that this action by Cicero was probably the greatest single contribution of any Roman to the history of mathematics'.

3.13   Review exercise

Review exercise in integration

There are randomised questions on-line which you can use for further practice.

10 min

These questions are intended to test basic competency in this topic.

Q85: Find $\int \frac{2x}{x^2 + 4} dx$

Q86: Find $\int \exp(4x) dx$

Q87: Find $\int \cos^4 x \sin x \, dx$ given the substitution $u = \cos x$

3.14 Advanced review exercise

Advanced review exercise in integration

There is an on-line exercise at this point, which you might find helpful.

These questions are intended to reflect the type of question that might be posed in the final exam.

30 min

Q88: Evaluate $\int_0^{\pi/3} \sin x \sec^2 x \, dx$ using the substitution $u = \cos x$

Q89: Evaluate $\int_e^{e^4} \frac{dx}{x(\ln x)^2}$ using the substitution $u = \ln x$

Q90: Evaluate $\int_0^3 \frac{x - 2}{\sqrt{9 - x^2}} dx$ using the substitution $x = 3 \sin t$

3.15 Set review exercise

Set review exercise in integration

An on-line assessment is available at this point, which you will need to attempt to have these answer marked. These questions are not randomised on-line. The questions on the web may be posed in a different manner but you should have the required answers in your working.

10 min

Q91: Find $\int \frac{3x^2}{x^3 + 5} dx$

Q92: Find $\int e^{7x} dx$

Q93: Find $\int \cos^3(x) \sin(x) \, dx$ using the substitution $u = \cos x$

Topic 4

Properties of Functions

Contents

Prerequisite knowledge

A sound knowledge of the following techniques is required for this unit.

- *Ability to perform simple differentiation.*

- *Understanding of coordinates in graph work.*

- *Understanding of set notation.*

Learning Objectives

- *Use properties of functions.*

Minimum Performance Criteria:

- *Find the vertical asymptote(s) of a rational function.*

- *Find the non vertical asymptote of a rational function.*

- *Sketch the graph of a rational function including appropriate analysis of stationary points.*

4.1   Revision exercise

This exercise should help identify any areas of weakness in techniques which are required for the study of this unit.   Some revision may be necessary if any of the questions seem difficult.

Revision exercise

There is a similar exercise on the web if you would like to try it.

10 min

Q1: Differentiate $3x^2$ - 4

Q2: Differentiate $\frac{3}{2x^2}$

Q3: Differentiate $\frac{2x}{(x + 3)^2}$

4.2   Introduction

Why study functions?

Functions provide a way of describing mathematically the relationship between two quantities.

The key idea is that the values of two variables are related. For example, the price of a bag of potatoes depends on how much they weigh.

Another example is the amount of tax paid on a product depends on its cost.

This topic will explore the rules defining functions and how to sketch the graphs of functions. Graphs are a useful way to gain information about a function.

Graphs of simple functions such as f (x) = x^2 are easily sketched, but there are also many relationships between functions that can be used to sketch other slightly more complex graphs.

These techniques will be developed and explored in order to identify some useful ways of dealing with the graphs of rational functions.

The emphasis will be on sketching the graphs.

4.3   Function definitions

Learning Objective

Use the basic concepts of a function to determine aspects of the related graph

The standard number sets shown are used in this section.

Standard number sets

The standard number sets are:

- $\mathbb{N}$ = {1, 2, 3, 4, 5, ...} the set of natural numbers.

- $\mathbb{W}$ = {0, 1, 2, 3, 4, 5, ...} the set of whole numbers.

- $\mathbb{Z}$ = {..., -3, -2, -1, 0, 1, 2, 3, ...} the set of integers.

- $\mathbb{Q}$ = the set of all numbers which can be written as fractions, called the set of rational numbers.

- $\mathbb{R}$ = the set of rational and irrational numbers, called the set of real numbers.

These sets and their subsets (such as $\mathbb{Z}^+$ = {1, 2, 3, ...} or $\mathbb{R}^+$ = { $x \in \mathbb{R}$, $x > 0$ }) are the only standard number sets used in this unit.

Function f

A function f from set A to set B is a rule which assigns to each element in A exactly one element in B. This is often written as f : A $\rightarrow$ B.

Domain

For a function f : A $\rightarrow$ B, A is called the domain of the function f.

In this unit unless otherwise stated, the domain of a function will be the largest possible set of x values for which the rule defining f (x) makes sense.

For example if f (x) = $\frac{1}{x}$ then the domain is {x $\in \mathbb{R}$: $x \neq 0$}

Codomain

For a function f : A $\rightarrow$ B, B is called the codomain of the function f.

To define a function fully it is necessary to specify:

- The **domain**.

- The **codomain**.

- The rule.

A function can be defined using different notation.

Other function notations may appear so it is important to understand the concepts. The terminology used will then be easier to understand.

Here is a selection of ways of doing this:

- Let f be the function defined by f (x) = x^2 where x $\in$ [-2, 2]

- f : [-2, 2]$\to$ $\mathbb{R}$

  f : x $\to$ x^2

- f is defined by f : x $\to$ x^2 where x $\in$ [-2, 2]

- f is defined on the set T where T = {x: x $\in$ [-2, 2]} by f : x $\to$ x^2

- f is defined by f (x) = x^2 with domain { x : -2 $\leq$ x $\leq$ 2}

The above function is well known. A different type of function (with a different notation) is

$$f(x) = \begin{cases} 2x : x \geqslant 2 \\ x^2 : x > 2 \end{cases}$$

When dealing with functions it is common practice to use the letter defining the function such as f instead of f (x).

The graph of a function f (x) is usually drawn as the curve y = f (x).
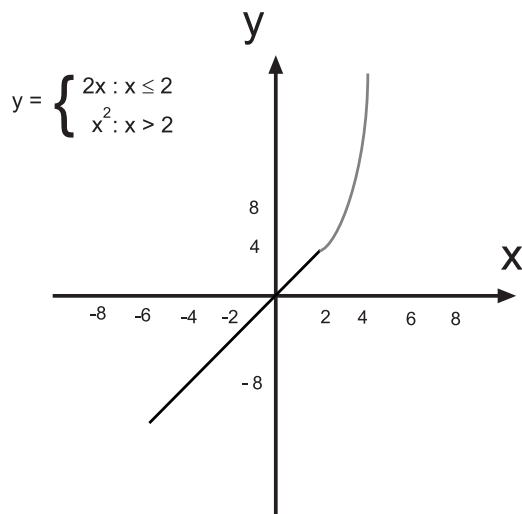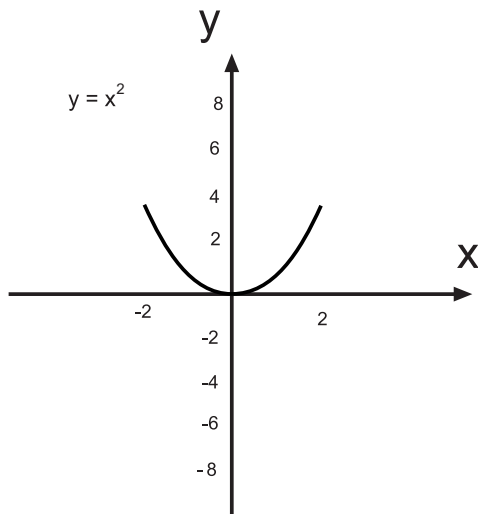
> **image set or range**
>
> For a function f : A → B, the set C of elements in B which are images of the elements in A under the function f is called the image set or range of the function f. C is always contained in or equal to B. This is written C ⊆ B

.
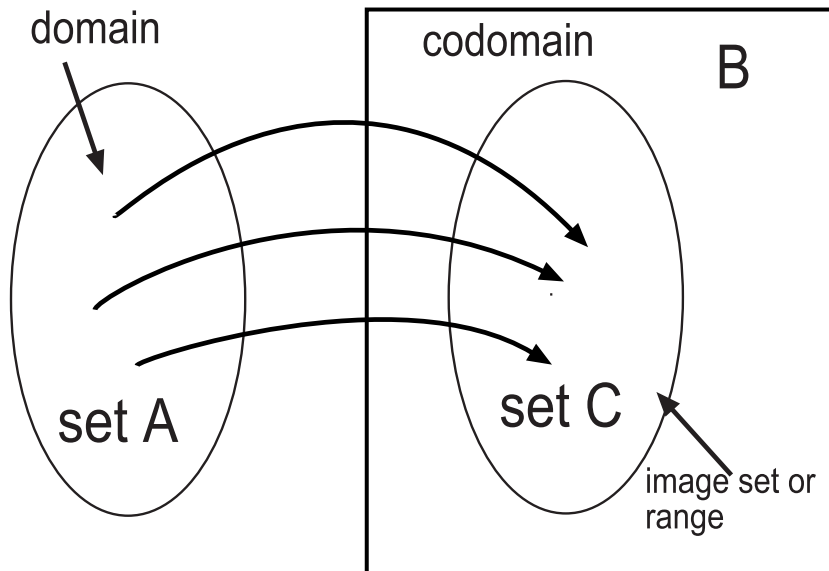
Examples

1. f (x) = x + 7 with a domain of {1, 2, 3}

This function has a **codomain** of ℤ and a **range** of (8, 9, 10}

This shows that the range ⊆ codomain.

2. As shown earlier the function f (x) = x^2 has the graph of y = x^2

It had a domain of [-2, 2] and a codomain of the real numbers. The rule is that every value of x maps to x^2

Note that the range is actually [0, 4]

This is all possible values of y for x ∈ [-2, 2]

Care must be taken to cover all situations. Here, if the end points of the domain are used then f (x) = 4 in both cases and the fact that 0^2 = 0 could be missed.

Notice that the range [0, 4] lies inside the codomain of ℝ as required.

(Finding the range of a function is not always easy.)

Examples

1. Determine the largest possible domain for the following rule

k (z) = $\sqrt{\frac{1}{z}}$

Answer:

$\sqrt{\frac{1}{z}}$ is undefined when z = 0 since $^1/_0$ is not defined and also when z is negative.

This means that z must be greater than zero.

The largest possible domain is therefore $\mathbb{R}^+$ (the set of all positive real numbers).

2. Using the second definition shown earlier

f : [0, π] $\to \mathbb{R}$

f : x $\to$ sin x

This means that the function f has domain of [0, π], codomain of the real numbers and maps every value of x to the corresponding value sin x.

Note again that only part of this codomain of $\mathbb{R}$ will actually be the range. This range however will be totally included in $\mathbb{R}$.

3. Write the function f defined by

f : [0, π] $\to \mathbb{R}$

f : x $\to$ sin x

using two other methods of notation for a function.

Answer:

f is defined by f : x $\to$ sin x where x $\in$ [0, π] or

f is defined by f (x) = sin x with domain { x : 0 $\leq$ x $\leq \pi$}

4. What is the range of the function f which maps x to sin x where x has a domain of [0, π] and codomain of $\mathbb{R}$?

Answer:

f (x) = sin x and the domain is [0, π]

Draw the graph and note that f (0) = 0 and f (90) = 1

The range is therefore [0, 1]

The following points need to be considered when determining the range of the function.

SHAPE: Think about the shape of a sin graph and its maximum and minimum values. It may be that this is enough to determine the range.

GRAPH VALUES: Consider the values which can be given to x.

For example 0, $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$ and so on. Work out the values of sin x. Sketch the graph if it helps.

$y = \sin x$
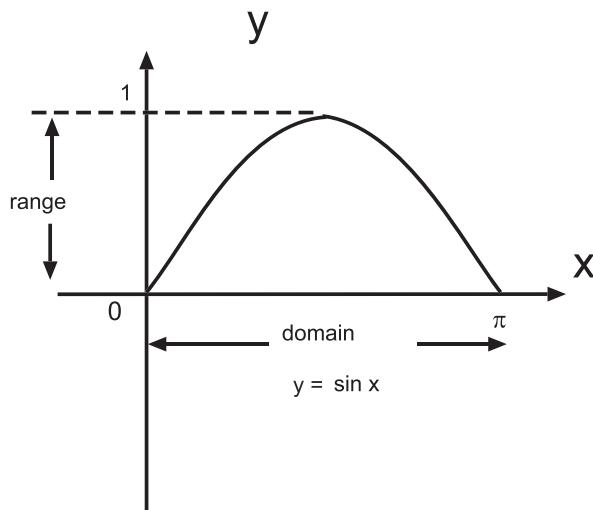
Here the range is [0, 1]

This can also be written as $0 \le f(x) \le 1$ or $\{f(x) \in \mathbb{R} : 0 \le f(x) \le 1\}$

Domain and range exercise

There is a web exercise if you prefer it.

5 min

Q4: What is the domain for the function $f(x) = \sqrt{x - 2}$?

Q5: What is the domain for the function $g(y) = \frac{1}{y}$?

Q6: Find the domain for the function $h(z) = (3 - \sqrt{z})^2$?

Q7: Work out the range of the function f with domain $\mathbb{N}$ (the natural numbers) and defined by $f(x) = \frac{1}{x}$

Q8: The function g is defined by $g(y) = y - 4$ with domain $\{y \in \mathbb{Z} : y > 4\}$. Work out the range for it.

Q9: Find the range of the function $h(w) = w^2 - 3$ with domain $\{w \in \mathbb{Z} : w > 1\}$

4.4   One-to-one and onto functions

Learning Objective

Display graphically the concept of a one-to-one or an onto function

In the definition of a function the elements in set A must map to one and only one element of set B.

One-to-one

A function $f : A \to B$ is a one-to-one function if whenever $f(s) = f(t)$

then s = t where $s \in A$ and $t \in A$.

The function is said to be in one-to-one correspondence.

This means that each value of f (x) in the range is produced by one and only one value of x in the domain.

Consider the function f (x) = $\frac{x}{2}$
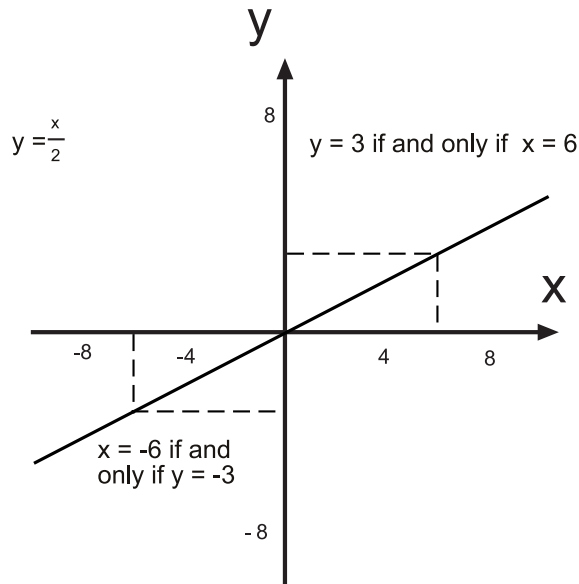
Here there is only one value in the domain, x = 6, which produces the value y = 3 in the codomain.

At any point on the graph of a one-to-one function this property will exist.

A one-to-one function can be identified from its graph.

One-to-one demonstration

Take a horizontal line and move it up and down over the graph.

If at any point the line crosses the graph more than once then the function is not one-to-one.

5 min
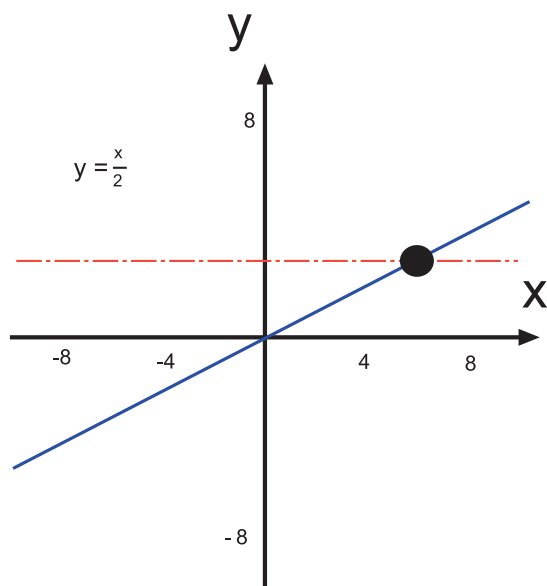
The function f (x) = $\frac{x}{2}$ is a one-to-one function.

There is also a web demonstration of a one-to-one function if you wish to view it.

It is possible, however, for more than one element of set A to map to the same element in set B.

Many-to-one

A function which maps more than one element in the domain to the same element in the range or image set is called a many-to-one or a many-one function.

The function is said be in many-to-one correspondence.

It is also common to say that such a function is not one-to-one.

Note that f (3) = 3^2 = 9 and f (-3) = $(-3)^2$ = 9. So there are two different elements, 3 and -3, in the domain (set A) which map to 9 in the codomain (set B).

The function is not one-to-one.

$$y = x^2$$
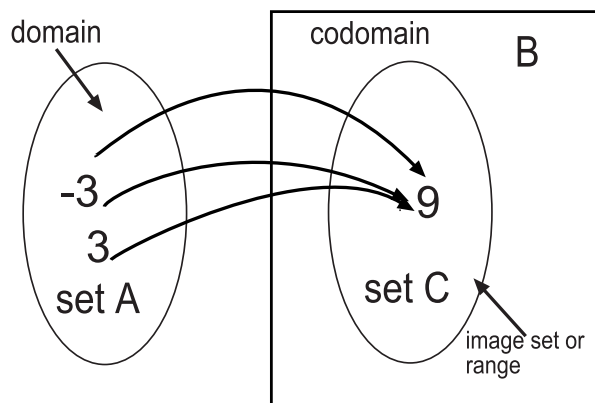
When the range is equal to the codomain the function has a special name.

Onto

An onto function is one in which the range is equal to the codomain.

The method of proof for these properties is shown in the section headed Proofs.

One-to-one and onto function exercise

There is a web version of this exercise if you prefer it.

5 min

Q10: Is the function h (z) = z^2 + 1 where z $\in$ $\mathbb{R}$,
codomain of { h (z) $\in$ $\mathbb{R}$, h (z) $\geq$ 1} an onto function?

Q11: Is the function f (x) = -x where x $\in$ $\mathbb{R}$ one-to-one?

Q12: Is the function g (y) = - y^2 where y $\in$ $\mathbb{Z}$ one-to-one?

Q13: Is the function k (w) = 4w - w^2 where w $\in$ [0, 4] one-to-one?

Q14: Is the function h (z) = 4z - z^2 where z $\in$ $\mathbb{Z}$: -2 $\leq$ z $\leq$ 2 one-to-one?

Q15: Is the function f (x) = -x where x $\in$ $\mathbb{R}$, codomain of $\mathbb{R}$ an onto function?

Q16: Is the function g (y) = - y^2 where { y $\in$ $\mathbb{Z}$: -2 $\leq$ y $\leq$ 2}, codomain of [-4, 4] an onto function?

4.5 Inverse functions

Learning Objective

Sketch the inverse of a function

Consider the function f (x) = y where x $\in$ A and y $\in$ B

This function maps the elements of A to elements of B

Is it possible to find a function that reverses this?

Such a function will map elements of B back to elements of A

If this function exists it will be the inverse function of f and is denoted f $^{-1}$

One condition of a function is that it maps an element in the domain (set A) to only one element in the range (set B).

For an inverse of this function to exist, each element of the set B will have to map back to the element in set A from which it came.

This is precisely what a one-to-one function does.

However this only occurs when the image set of f is the whole of the codomain B.

Otherwise the range of the inverse function is not contained in the domain of the original function.

This is precisely the definition of an onto function.

Inverse function

Suppose that f is a one-to-one and onto function. For each y $\in$ B (codomain) there is exactly one element x $\in$ A (domain) such that f (x) = y

The inverse function is denoted f^{-1} (y) = x

This means that each element in the range of the function f is mapped back to the element from which it came.
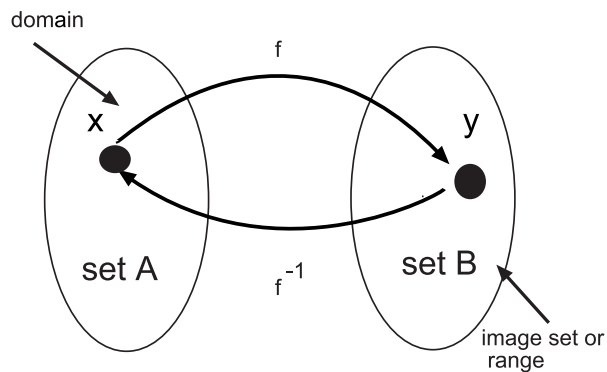
The **domain** of the inverse function is the range of the original function.

The **codomain** of the inverse function is the domain of the original function.

The relationship between functions and their inverses is clearly seen using graphs.

To find the rule for an inverse function interchange y and x in the original rule,

then rearrange to give a new equation for y = expression in x.

The effect of interchanging x and y in the equation is the same as interchanging the axes on the graph.

This is the same as reflecting the graph of a function in the line y = x

Example Sketch the inverse function f^{-1} where f (x) = 2x

Answer:

First sketch the function f which is the line y = 2x

Reflect this graph in the line y = x to give the inverse function f^{-1}

By interchanging x and y the equation y = 2x becomes x = 2y and solving for y gives the line y = $^1/_2$ x

The inverse function is f^{-1} (x) = $^1/_2$x
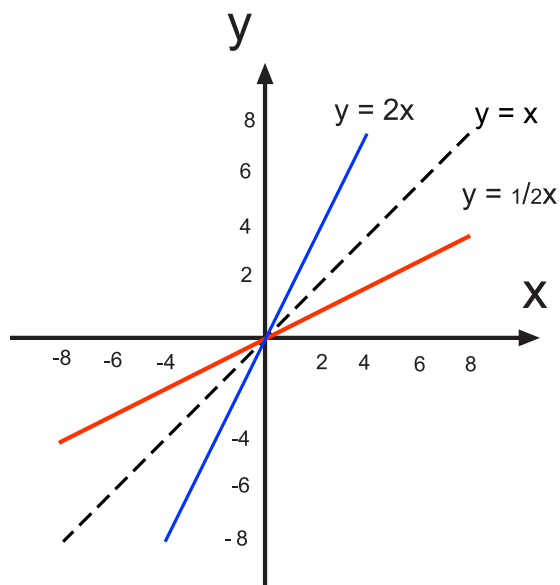
This shows that the element 2 has an image of 4 under f and the element 4 has an image of 2 under f⁻¹

When asked to find and sketch the inverse of a function it is important to check that the function in question is actually one-to-one and onto.

It could be that the codomain of the original function will have to be restricted in order to find an inverse which actually exists.

This is very important.

Example

Sketch the graph of the function $f : \mathbb{R}^+ \to [-3, \infty)$ where $f(x) = 4x^2 - 3$. Find and sketch its inverse if it exists.

Answer:

First of all check that the function is one-to-one.
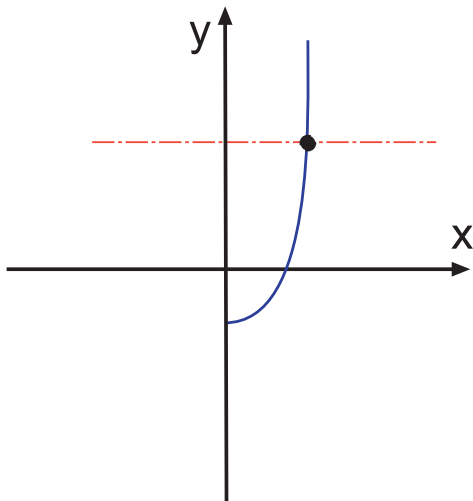
Check now whether the function is onto.

The definition of onto means that the range equals the codomain. Here the **codomain** is $[-3, \infty)$ and the range is the same. The function is onto.

An inverse exists.

Now draw the graphs and compare.
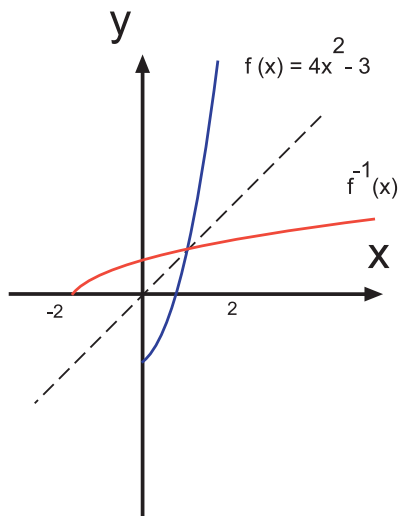
Check the answers in a graphics calculator. If a graphics calculator is not available, an on-line calculator can be used at the following web site:

http://www.univie.ac.at/future.media/moe/fplotter/fplotter.html

This on-line tool is useful for the other exercises which follow if there is a shortage of graphic calculators.

Q17: Use the graph of $y = \log_{10} x$ for $x \in \mathbb{R}^+$ to sketch the graph of $y = 10^x$

Note that 10^x is the inverse of the graph $y = \log_{10} x$

Q18: Using the graph of $y = 2^x$ sketch the graph of $y = \log_2 x$

Q19: Find and sketch the inverse of the function f (x) = 8 - 2x where $x \in \mathbb{R}$.

A graphics calculator can be used to check your understanding of the concept.

Q20: Find and sketch the inverse of the function $g(x) = \frac{1}{2 - x}$ where $x \in \mathbb{R}: x \neq 2$

Q21: Find and sketch the inverse of the function h (x) = 4x - 6 where $x \in \mathbb{R}$

Now consider the graphs of trig functions.

Is it possible to define any type of inverse function for sin, cos or tan functions?

The graph of the function f : $\mathbb{R} \to [-1,1]$ where f (x) = sin x is a very familiar one.

It is plain to see that it is not a one-to-one function.

Check this on a graphic calculator.

For every value of f (x) there are various values of x for which sin x = f (x)

For example sin x = 0 gives x = 0, π, 2π, ...

The graph however is onto.

The range of the function is [-1,1] which is equal to the codomain given.

By changing the minimum and maximum values for x on the calculator is it possible to find a domain for x such that sin x is a one-to-one function?

The answer is that there are many different restrictions which will produce a suitable domain. One of these is $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

So if the domain is restricted to give a one-to-one, onto sine function then the inverse sine function exists. This is denoted sin $^{-1}$ x and should not be confused with $\frac{1}{\sin x}$

y = sin $^{-1}$ x means 'y is the angle whose sin is the value x'. The functions cos and tan have similarly defined inverses.

Notice that the graph of y = tan^{-1} x has **horizontal asymptotes** and these are related to the **vertical asymptotes** of tan x. (Remember that an inverse is a reflection in the line y = x which is the same as interchanging the axes.)

Q22: Activity to explore trig. inverse functions:

- Show that the sine function when restricted to a domain of $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and codomain of [-1,1] is both one-to-one and onto.
- By exploring the cosine function, with or without a graphic calculator, suggest a possible restricted domain on cos x which will make the function a one-to-one function.
- Explore the graph of y = tan x and suggest a possible restricted domain to make this a one-to-one function.
- Use a graphics calculator to graph the inverse functions of sin, cos and tan. Use the zoom box to gain an understanding of the reflection in y = x
- What is the domain of y = sin $^{-1}$ x if sin x has a domain of $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$?
- Sketch the graph of y = tan $^{-1}$ x from the graph of tan x where $x \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ by using reflection in the line y = x
- Sketch the graph of y = cos $^{-1}$ x from the graph of cos x where $x \in [0, \pi]$ by using reflection in the line y = x

The inverses of sine, cosine and tangent functions

There are demonstrations of sin, cos and tan with their inverses on the web.

10 min

Q23: With the aid of a graphics calculator explore the inverses of functions of the type

a sin x, cos bx and tan cx over the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Define a general rule for the domain of the function in each case which will give an inverse.

Q24: Complete the blanks in the following table giving exact values throughout. Give angles in radians.

x	sin^{-1} x	tan^{-1} x	cos^{-1} x
$\frac{\sqrt{3}}{2}$		********	
		$\frac{\pi}{4}$	
	$\frac{\pi}{4}$		
	********	$\frac{\pi}{3}$	********
	$\frac{\pi}{6}$	********	

Q25: Complete the blanks in the following table giving exact values throughout. Give angles in degrees.

x	$\sin^{-1} x$	$\tan^{-1} x$	$\cos^{-1} x$
		********	$30°$
		$45°$	
	$30°$	********	
$\sqrt{2}$		********	
		$60°$	

Q26: The fact that $\ln x$ is the inverse of e^x has already been stated and used in this unit. The graph of each clearly shows the reflection in $y = x$

Given the fact that $\ln x$ crosses the x-axis at the point (1, 0), determine where the graph of e^x will cross the y-axis.

Q27: Explore the functions of the form $\frac{1}{a} \ln x$ and e^{ax}. Determine the relationship between these two functions.

4.6   Odd and even functions

Learning Objective

Recognise the symmetry and follow the proofs of the properties of a function being odd or even

The symmetry of some graphs can make sketching easier.

Take the curve $y = x^2$. This graph is symmetrical about the line $x = 0$

A simple symmetry involves reflection in the primary axis. That is:

- $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis.

- $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.

Reflections in the x-axis and in the y-axis

There are some examples of these reflections on the web.

10 min

There is a further relationship between the function $f(-x)$ and either $-f(x)$ or $f(x)$ which determines whether a function is odd, even or neither.

Odd function

A function is odd if $f(-x) = -f(x)$ for every value of x within the domain of the function.

The graph is symmetrical under $180°$ rotation about the origin.

It is said to have rotational symmetry of order 2

Even function

A function is even if f (-x) = f (x) for every value of x within the domain of the function.

The graph is symmetrical under reflection in the y-axis.
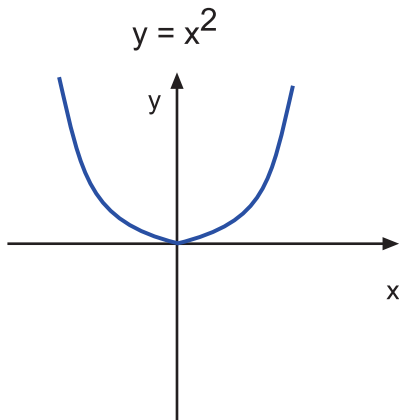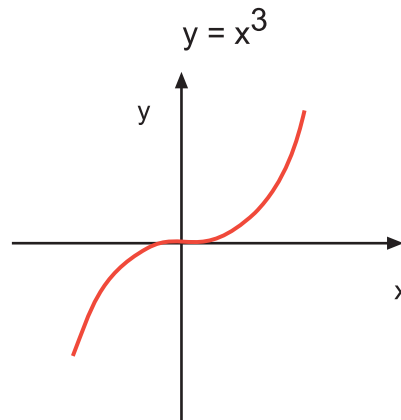
Activity

Experiment with your graphics calculator and try to find some odd or even functions.

Hint : use the examples as a basis.

This property of being odd or even certainly aids the sketching of such a graph but beware, many graphs are neither.

A graph on its own however does not formally prove that a function is odd or even. Certain properties have to be established by algebraic means.

The following examples will demonstrate the form of such proofs.

Examples

1. Prove that the function $f(x) = x^2 \cos x$ is even.

$f(x) = x^2 \cos x$

$f(-x) = (-x)^2 \cos(-x) = x^2 \cos x = f(x)$

Since $f(x) = f(-x)$ the function is even.

2. Prove that the function $g(x) = \frac{x}{x^2 + 1}$ is odd.

$g(x) = \frac{x}{x^2 + 1}$

$g(-x) = \frac{-x}{(-x)^2 + 1} = \frac{-x}{x^2 + 1} = -g(x)$

Since $g(-x) = -g(x)$ the function is odd.

Note that if a function is a product or quotient of two other functions the following rules apply:

- even $\times$ even = even

- odd $\times$ odd = even

- odd $\times$ even = odd

So g (x) = $\frac{x}{x^2 + 1}$ = x $\cdot$ $\frac{1}{x^2 + 1}$ = odd $\times$ even = odd (as already proved).

Q28:   Prove whether the following graphs are odd, even or neither. If the graph is odd or even, sketch it using the property found.

1. f (x) = cos x
2. g (x) = $\frac{1}{2x - 1}$
3. k (x) = $\frac{1}{x}$

4.7   Critical and stationary points

Learning Objective

Identify any critical and stationary points

Critical point

A critical point is any point on a curve where the slope of the tangent to the curve is zero (parallel to the x-axis) or where the slope of the tangent to the curve is undefined (parallel to the y-axis).

Stationary point

A stationary point is any point on a curve where the slope of the tangent to the curve is zero (parallel to the x-axis).

The term stationary point can refer to:

- A maximum turning point.

- A minimum turning point.

- A horizontal point of inflection.

Local maximum and minimum turning points.

A local maximum point occurs when a function has a greater value at that point than at any points close to it. It is not necessarily the greatest value of the function.

There can be more than one local maximum turning point.

A local minimum point occurs when a function has a lesser value at that point than at any points close to it. It is not necessarily the least value of the function.

There can be more than one local minimum turning point.

Global maximum and minimum turning points.

A global maximum point occurs when f is defined over a domain A and the value of the function at this point is greater than or equal to that at any other point within the domain.

A global minimum point occurs when f is defined over a domain A and the value of the function at this point is less than or equal to that at any other point within the domain.

Recall that a maximum turning point occurs when the slope of the tangent has a decreasing positive value as it approaches the stationary point from the left and changes to an increasing negative value as it continues past the point.

A minimum turning point occurs when the slope of the tangent has a decreasing negative value as it approaches the stationary point from the left and changes to an increasing positive value as it continues past the point.

A horizontal point of inflection occurs when the value of the slope of the tangent remains either positive or negative on both sides of the stationary point but is zero at the particular point.

Q29: Identify any maximum turning points, minimum turning points or points of inflection on the following curves.

Use X for the maximum points, N for the minimum points and P for the points of inflection.
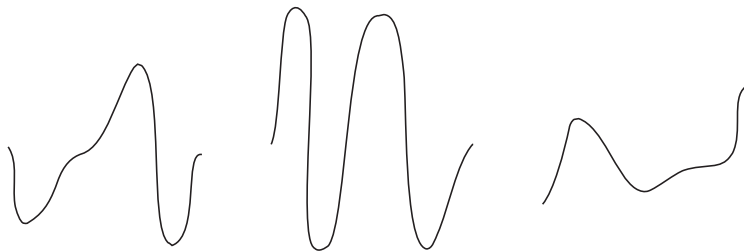
Example Identify the turning points on the following graph and indicate if they are local or global maxima or minima.
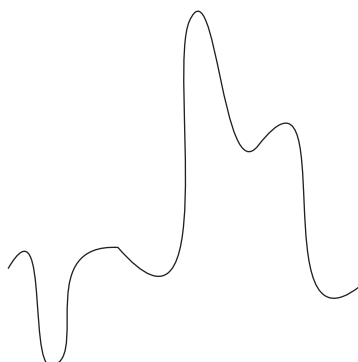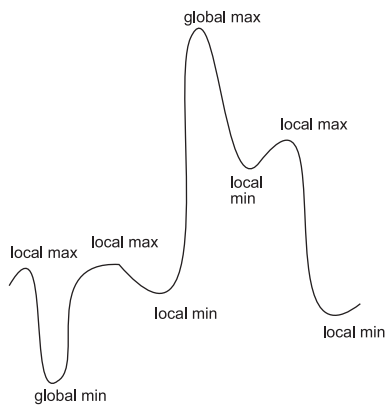
Local and global maxima and minima exercise

5 min

There is a web exercise if you wish to try it.

Q30: Identify the stationary points on the following graph and indicate if any of the maximum or minimum turning points are local or global maxima or minima.
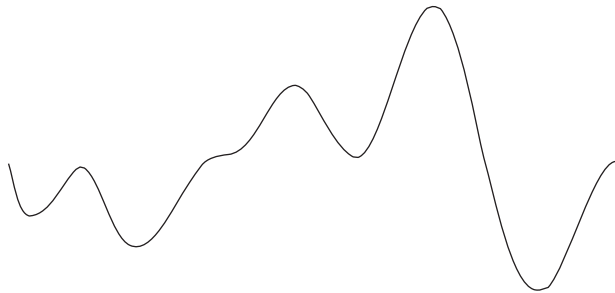
4.8   Derivative tests

Learning Objective

Use the first and second derivative tests to determine the nature of a turning point

Finding stationary points can help with the sketching of functions.

At stationary points the slope of the tangent to the curve is zero.

The value of the derivative is the slope of the tangent to a curve and hence by solving the equation of this tangent equal to zero it is possible to identify the x coordinate of a stationary point.

By substitution in the function expression the y-coordinate can be found.

The nature of the turning point has yet to be determined and this can be done in two ways called the first and second derivative tests.

4.8.1   The first derivative test

The first derivative test works as follows:

- Calculate values for the first derivative of the function using values on either side of the x coordinate of the turning point. The signs of the values are what is really needed.

- Construct a table of signs from this data to show the nature of the turning point.

(Note: the variable allocated to the horizontal axis may be denoted other than x.)

Example Determine the nature of the turning point on the curve $y = 3x^2 - 12x + 4$

Answer:

Calculate the first derivative of the function $y = 3x^2 - 12x + 4$

$\frac{dy}{dx} = 6x - 12$

Solve the equation $\frac{dy}{dx} = 0$

This is when $6x - 12 = 0 \Rightarrow x = 2$

So the turning point is at $x = 2$

The y-coordinate of this turning point is $y = 3 \times 2^2 - 12 \times 2 + 4 = -8$

The turning point is at (2, -8)

Now determine the nature of the point (2, -8) using the **first derivative test**.

At this turning point, $x = 2$

look at the value of the derivative on either side of this.

In the table the notation 2^- is used for just below $x = 2$ and 2^+ for just above $x = 2$

Do not confuse these with - 2 and + 2

	2^-	2	2^+
signs of $\frac{dy}{dx}$	-	0	+
slope	↘	→	↗

This clearly shows that the turning point is a minimum.

(If the arrows show as ↗ → ↘ then the turning point will be a maximum.)

First derivative test

The first derivative test is a means of determining the nature of the turning point by finding the signs of the derivative to the left and to the right of the turning point.

First derivative test exercise

There is an alternative exercise on the web that you can try if you prefer it.

10 min

Q31: Use the first derivative test to determine the nature of the turning points of the function $y = \frac{1}{3} x^3 - 4x$

Q32: Use the first derivative test to determine the nature of the turning point of the function $y = 4x^2 + 8x - 3$

Q33: Use the first derivative test to determine the nature of turning point of the function $y = 3x^2$

4.8.2   The second derivative test

The second derivative of a function gives an alternative way of determining the nature of a turning point.

The second derivative test works as follows:

- Once the turning point has been identified, the second derivative f" of the function is calculated.

- The x value of the stationary point is substituted into the expression for f"

If the value of f" is negative (- ve) the turning point will be a maximum.

If the value of f" is positive (+ ve) the turning point will be a minimum.

If the value of f" is zero the nature is indeterminate (using this method) and the first derivative test has to be used instead.

Example Find the coordinates of any turning points on the graph of $y = x^3 + 3x^2 - 4$
Determine the nature of them.

Answer:

$y = x^3 + 3x^2 - 4$

$\frac{dy}{dx} = 3x^2 + 6x$

At turning points $\frac{dy}{dx} = 0 \Rightarrow x = 0$ or $x = -2$

This means that there are two turning points.

The y coordinate when $x = 0$ is found by substituting the value $x = 0$ into the original equation.

So the turning point is (0, -4)

Similarly when $x = -2$ the turning point is (-2, 0)

Now using the **second derivative test** $\frac{d^2y}{dx^2} = 6x + 6$

At $x = 0$, $\frac{d^2y}{dx^2} = 6$ which is positive and this turning point is a minimum.

At $x = -2$, $\frac{d^2y}{dx^2} = -6$ which is negative and this turning point is a maximum.

The answer is therefore:

There are two turning points; a maximum at (-2, 0) and a minimum at (0, -4)

Second derivative test

The second derivative test of a function is a means of determining the nature of a turning point. A positive second derivative gives a minimum turning point and a negative second derivative gives a maximum turning point.

The second derivative test is very useful but does not always work because:

- The second derivative may not exist.

- If it does exist and is 0, the nature of the point is still undecided
  (i.e. when f" is zero).

In these circumstances it is necessary to revert to the first derivative test and complete a table of signs for f' for x in the close neighbourhood of the turning point.

The following exercise will give practice in finding turning points and using the derivative tests to determine their nature.

First of all look at the example.

Example For the curve $y = 2x^3 - 3x^2$ find the coordinates of the turning points and determine their nature.

Answer:

$y = 2x^3 - 3x^2$

$\frac{dy}{dx} = 6x^2 - 6x$

$\frac{d^2y}{dx^2} = 12x - 6$

Let

$\frac{dy}{dx} = 6x^2 - 6x = 0 \Rightarrow x = 0 \text{ or } x = 1$

When $x = 0$, $\frac{d^2y}{dx^2} = 12x - 6 = -6$ which is negative.

This indicates a maximum turning point at (0, 0).

When $x = 1$, $\frac{d^2y}{dx^2} = 12x - 6 = 6$ which is positive.

This indicates a minimum turning point at (1, -1).

Turning points exercise

There is a different exercise on the web if you prefer to try it.

15 min

Q34: The turning points of the function $y = \frac{1}{3} x^3 - 4x$ are $(2, \frac{-16}{3})$ and $(-2, \frac{16}{3})$

Use the second derivative test to determine their nature.

Q35: The turning point of the function $y = 4x^2 + 8x - 3$ is (-1, -7)

Use the second derivative test to determine its nature.

Q36: The turning point of the function $y = 3x^2$ is (0, 0)

Find the nature of it using the second derivative test.

Q37: For the curve $y = 4x^3 - 3x^2$ find any turning points and determine their nature. Sketch the graph.

Q38: For the curve $y = 3x^2 - 2x$ find any turning points. Determine their nature. Sketch the graph.

Q39: Find any turning points, determine their nature and sketch the graph for the curve
y = 3 sin x for x $\in$ [-2π, 2π]

4.9   Concavity

> **Learning Objective**
>
> Recognise the terms related to concavity and the shape displayed by these graphs

The second derivative of a function can provide information about the shape of the graph
of the function.

Concave downward

The graph of y = f (x) is concave downward in an interval if f (x) is a function which has
a second derivative $\frac{d^2y}{dx^2} < 0$ for all x in the open interval.

Concave downward can be pictured as an 'n' shaped graph ($\cap$).

Concave upward

The graph of y = f (x) is concave upward in an interval if f (x) is a function which has a
second derivative $\frac{d^2y}{dx^2} > 0$ for all x in the open interval.

Concave upward can be pictured as a 'u' shaped graph ($\cup$). Think of 'u' for UP.

Thus:

- concave downwards $\Leftrightarrow \frac{d^2y}{dx^2} < 0$

- concave upwards $\Leftrightarrow \frac{d^2y}{dx^2} > 0$

The sign $\Leftrightarrow$ means 'if and only if'. That is, the condition works both ways.

For example, if it is known that the second derivative is positive then the curve is concave
upwards.

If, however, the curve is concave downwards then the second derivative is negative.

When the concavity of a curve changes, a point of inflexion occurs. In fact this is a
sufficient and necessary condition for a point of inflexion.

Example Determine the concavity of the following functions:

1

2

3

$y = -x^2$

$y = x^2$

$y = x^3 - 3x + 4$
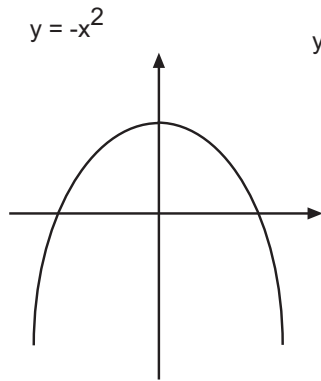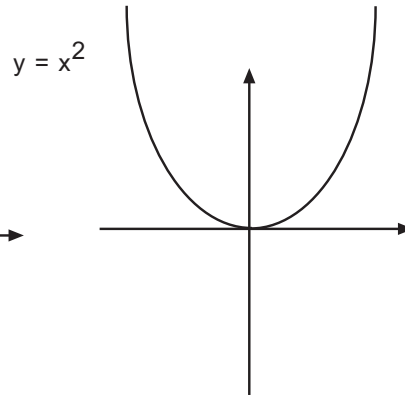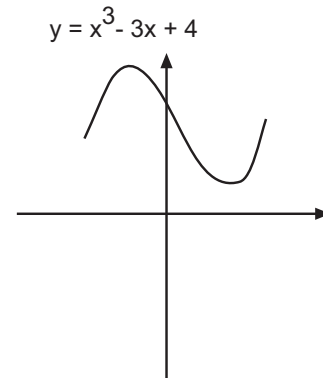
Answer:

1. This is concave downwards. Check: $\frac{d^2y}{dx^2}$ = -2 which is negative. It is an 'n' shape.

2. This is concave upwards. Check: $\frac{d^2y}{dx^2}$ = 2 which is positive. It is a 'u' shape.

3. This is concave downwards leading to concave upwards.

   Although the maximum and minimum turning points are easily identified, the point of inflexion when the concavity changes is not identified by the methods shown so far in this unit. (It is not a horizontal point of inflexion)

Q40: Identify the following curves using the following key:

a = concave upwards

b = concave downwards

c = concave upwards to concave downwards through a point of inflexion (non horizontal)

d = concave downwards to concave upwards through a point of inflexion

---

Non horizontal point of inflexion

A non horizontal point of inflexion occurs when $\frac{dy}{dx} \neq 0$ but $\frac{d^2y}{dx^2} = 0$

It is also worth noting that any point of inflexion occurs at a point P on a curve when the tangent to the curve crosses the curve **at that point**.

This is a useful visual check to note the existence of points of inflexion.

Example Find the points of inflexion on the curve $y = 2x^3 - 3x^2$

$\frac{dy}{dx} = 6x^2 - 6x$

$\frac{d^2y}{dx^2} = 12x - 6 = 0 \implies x = {}^1/_2$

When $x = {}^1/_2$, $\frac{dy}{dx} = 6x^2 - 6x \neq 0$

At x = $^1/_2$, $\frac{dy}{dx} \neq 0$ but $\frac{d^2y}{dx^2} = 0$

There is a non horizontal point of inflexion at $\left(\frac{1}{2}, -\frac{1}{2}\right)$

Concavity exercise

10 min

There is a short web exercise on concavity if you wish to try it as well.

Q41: For the curve y = 4x^3 - 3x^2 locate any points of inflexion.

Q42: For the curve y = 3x^2 - 2x locate any points of inflexion.

Q43: For the curve y = 3 sin x for x $\in$ [-2π, 2π] locate any points of inflexion.

4.10   Continuity and asymptotic behaviour

Learning Objective

Identify and sketch the asymptotes of a function

A function f (x) is said to be continuous if there is no break in the curve of the function for all values of x in the domain of the function.

Formally this is stated as:

Continuous function

A continuous function f (x) is a function where at every point P on the domain

$\lim\limits_{x \to P} f(x) = f(P)$

This means that as any point on the curve is approached (from either side), the value of the function tends closer and closer to the value of the function at the point itself.

If a graph can be drawn without lifting the pencil from the paper, the function of the graph is continuous.

It follows from this that a function f (x) is said to be discontinuous if there is a break in the curve at any value of x within its domain.

Again this can be stated formally.

Discontinuous function

A function f (x) is discontinuous at a point P if f (x) is not defined at P or if

$\lim\limits_{x \to P} f(x) \neq f(P)$

Example Examine the following functions

f (x) = x^2 and

$g(x) = \begin{cases} x & : x \leqslant 1 \\ 3 & : x > 1 \end{cases}$

The function f (x) = x² is continuous.

The function $g(x) = \begin{cases} x & : x \leqslant 1 \\ 3 & : x > 1 \end{cases}$ is discontinuous.

$y = x^2$
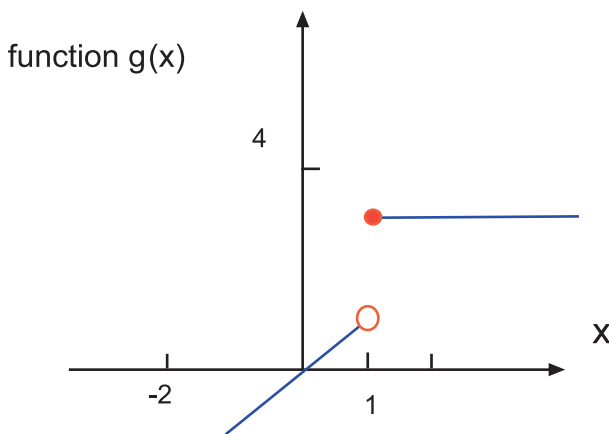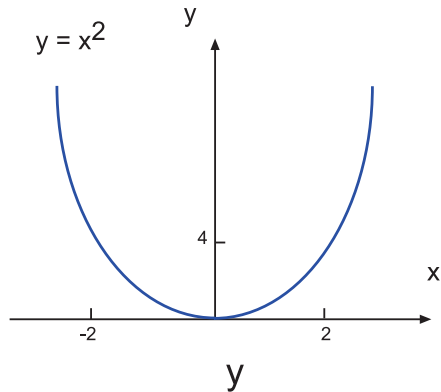
function g(x)

This graph for the function g has a distinct 'jump' in it after x = 1

The graph of y = tan x however shows a different type of situation.

For example, at x = 90° the value of tan x is undefined.

When the point x = 90° is approached from the left the value of tan x approaches ∞ as x approaches 90°. Check this by taking values of tan x for x between 89° and 89.9° on a calculator. The answers are larger and larger positive values.

When the point is approached from the right the value of tan x approaches -∞ as x approaches 90°. Again check this by taking values of tan x for x from 91° down to 90.1°. The answers in this case are larger and larger negative values.

In these circumstances the line x = 90 is called an asymptote.

Asymptotes are also present when sketching functions other than tan x.

Asymptotes of rational functions

For rational functions, a vertical asymptote is a vertical line with equation of the form x = k at which the function in question is undefined.

The function values either increase rapidly towards + ∞ or decrease rapidly towards - ∞ as x gets closer and closer to the value k

For rational functions, a horizontal asymptote is a horizontal line with equation of the form y = m for which the function value gets closer and closer to the value m as x tends towards + ∞ and / or - ∞

A horizontal asymptote occurs when the degree of the numerator is less than or equal to the degree of the denominator.

For rational functions, a slant or oblique asymptote is a line, neither horizontal nor vertical, with equation of the form y = ax + b for which the function value gets closer and closer to the line y = ax + b as x tends towards +∞ and / or towards - ∞

A slant asymptote occurs when the degee of the numerator is exactly one greater than the degree of the denominator.

For a vertical asymptote, values of x on either side of the asymptote can be chosen and the function evaluated to see if the function is greatly increasing (towards + ∞) or greatly decreasing (towards - ∞). This will give enough information to show the direction of the graph near this asymptote.

For a horizontal asymptote, check what happens to the function value as x $\rightarrow$ - ∞ and as x $\rightarrow$ + ∞. The values obtained for the function indicate whether the graph will continue above the asymptote, or below it. Again this gives enough information to show the direction of the graph near the asymptote.

For example:

If there is an asymptote at x = 0, the behaviour of the graph is determined in two stages.

1. Take several values immediately below 0 such as -2, -2, -0.5 and calculate the value of the function.

   These values will either become:

   - Increasingly large negative values
     (this is written: as x $\rightarrow$ 0⁻ f (x) $\rightarrow$ - ∞).
   - Increasingly large positive values
     (this is written: as x $\rightarrow$ 0⁻ f (x) $\rightarrow$ + ∞).

2. Take several values immediately above 0 such as 2, 1, 0.5 and calculate the value of the function.

   These values will either become:

   - Increasingly large negative values
     (this is written: as x $\rightarrow$ 0⁺ f (x) $\rightarrow$ - ∞).
   - Increasingly large positive values
     (this is written: as x $\rightarrow$ 0⁺ f (x) $\rightarrow$ + ∞).

Example What are the asymptotes for the graph $y = \frac{1}{x}$?

Answer:

y is undefined if $x = 0$ since $\frac{1}{0}$ is not possible.

There is a vertical asymptote at $x = 0$

As $x \to 0^+$, $y \to + \infty$ ($\frac{1}{x}$ is positive).

As $x \to 0^-$, $y \to - \infty$ ($\frac{1}{x}$ is negative).

$y = 0 \Rightarrow 0 = \frac{1}{x} \Rightarrow 0 = 1$ which is impossible.

There is a horizontal asymptote at $y = 0$

As $x \to + \infty$, $y \to 0$ from above ($\frac{1}{x}$ is positive).

As $x \to - \infty$, $y \to 0$ from below ($\frac{1}{x}$ is negative).

This is enough information to sketch this graph to show the asymptotes. Note the dashed axes indicate that these are asymptotes. All asymptotes should be shown with dashed lines.
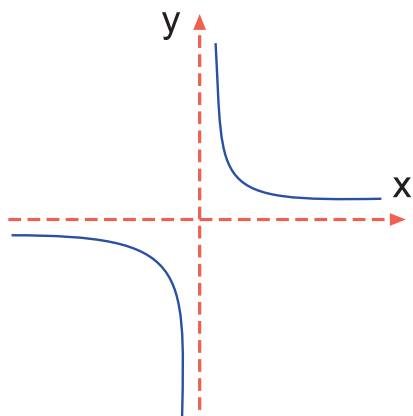
ACTIVITY

Examine the graphs of $y = \frac{1}{x^2}$ and $y = \frac{1}{x+1}$. Note the equations of the asymptotes.

Calculator activity

There are examples of functions with asymptotes on the web.

Using a graphics calculator examine the graphs of the following functions and note where the asymptotes occur:

- ln x

- tan (2x)

- e^x

- 1/(x - n) for several values of n

10 min

4.11   Sketching of rational functions

Learning Objective

Sketch a range of rational functions identifying their important features

Before considering a formal strategy for sketching graphs, the following example shows some of the techniques already explained.

Example Sketch the graph of f (x) = x - $\frac{4}{x}$

Answer :

Let y = f (x)

y = x - $\frac{4}{x}$

Symmetry	This function is odd since f (-x) = -f (x) So it has rotational symmetry of order 2
x-axis crossing	It cuts the x-axis if y = 0 $\Rightarrow$x - $\frac{4}{x}$ = 0 $\Rightarrow$x^2 = 4 and so x =$\pm$ 2 It crosses the x-axis at the points (-2, 0) and (2, 0)
y-axis crossing	It cuts the y-axis if x = 0 Here y is not defined if x = 0 since $\frac{4}{0}$ is not defined. It does not cut the y-axis.
Turning points	$\frac{dy}{dx}$ = 1 + $\frac{4}{x^2}$ $\frac{dy}{dx} \neq$ 0 at any point. It is always positive. There are no turning points. Since $\frac{d^2y}{dx^2} \neq$ 0 at any point there can be no points of inflection.
x $\rightarrow \pm\infty$	As x $\rightarrow$ -∞, y $\rightarrow$ x because $\frac{4}{x}$ becomes increasingly smaller. As x $\rightarrow$ + ∞, y $\rightarrow$ x There is an asymptote at y = x Note that this is a slant asymptote.

discontinuity

-y undefined

y is undefined where x = 0

There is a vertical asymptote at x = 0

As x → 0 $^+$, y → -∞

The small + sign indicates that x is approached from the right.

As x → 0 $^-$, y → ∞

The small - sign indicates that x is approached from the left.

Sketch the graph. Recall the symmetry to check if the sketch seems reasonable.

(x -4)/x
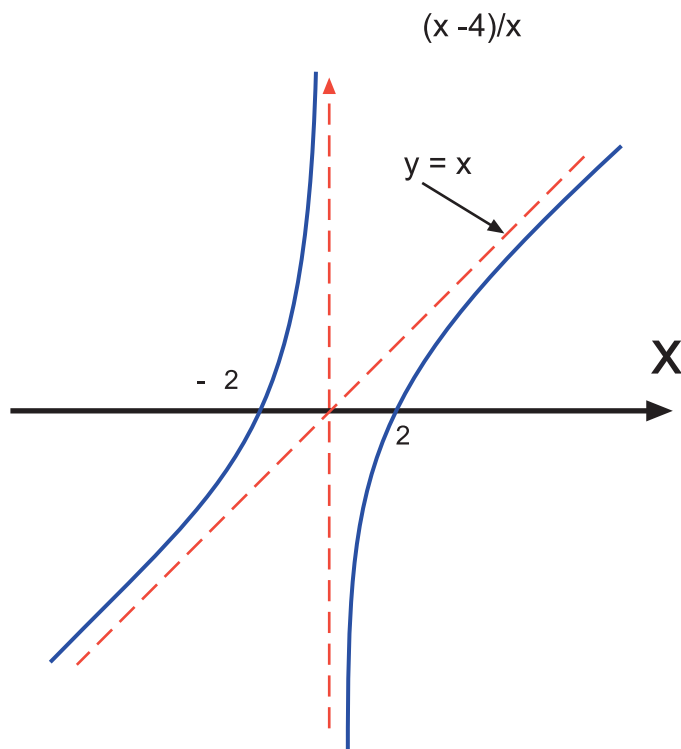
Here the dashed lines indicate asymptotes.

It is possible to identify the relevant features of a graph and develop a strategy for sketching curves by investigating functions in this manner.

Strategy for sketching curves

The **strategy for sketching curves** follows the following steps:

- Identify whether the function is odd or even.

- Identify crossing of x-axis.

- Identify crossing of y-axis.

- Look for turning points and points of inflection and their nature.

- Consider the behaviour of the curve as x → ± ∞.

- Check the behaviour where y is undefined i.e. is there a discontinuity?

- Check, if necessary, the behaviour of x or y further as the asymptotes are approached.

With the information obtained from this topic it is possible to sketch a wide variety of functions, including rational functions.

There are five general types of rational function to consider in this unit. Specific examples of each type including a sketch are shown in turn.

Type 1	$\dfrac{\text{constant}}{\text{linear}}$	$\dfrac{c}{ex+f}$	$\dfrac{1}{x+1}$
Type 2	$\dfrac{\text{linear}}{\text{linear}}$	$\dfrac{bx+c}{ex+f}$	$\dfrac{2x+3}{4x-1}$
Type 3	$\dfrac{\text{constant/linear}}{\text{quadratic}}$	$\dfrac{c \text{ or } (bx+c)}{dx^2+ex+f}$	$\dfrac{5 \text{ or } (2x-3)}{x^2+3x-4}$
Type 4	$\dfrac{\text{quadratic}}{\text{quadratic}}$	$\dfrac{ax^2+bx+c}{dx^2+ex+f}$	$\dfrac{3x^2-4x+2}{x^2-2x+1}$
Type 5	$\dfrac{\text{quadratic}}{\text{linear}}$	$\dfrac{ax^2+bx+c}{ex+f}$	$\dfrac{x^2+x-6}{x-1}$

The strategy for sketching these types of rational functions is given in the following examples. The strategy is basically the same for each type.

Make sure that any common factors on the top and bottom are first cancelled before classifying rational functions into the above types.

For example $\frac{x^2-3x+2}{x^2-4x+3} = \frac{(x-1)(x-2)}{(x-1)(x-3)} = \frac{x-2}{x-3}$.

This is a type 2 rational function and not as it might first seem, a type 4

After each example, simplified techniques are explained which help to reduce the effort when a sketch rather than a detailed graph is required.

4.12   Type 1 rational function: constant over linear

Example Sketch the curve of $y = \frac{1}{x+1}$

Answer: $y = \frac{1}{x+1}$

Symmetry:	This function is neither odd nor even.
	(f (-x) $\neq$ f (x) or -f (x))
x-axis crossing	$y = 0 \Rightarrow 0 = 1$ which is impossible.
	The curve does not cross the x-axis.
y-axis crossing	When $x = 0$, $y = 1$

Turning points $\qquad \dfrac{dy}{dx} = \dfrac{-1}{(x+1)^2}$

$\dfrac{dy}{dx} \neq 0$ at any point so there are no turning points

or horizontal points of inflection.

$\dfrac{d^2y}{dx^2} \neq 0$ at any point so there are no points of inflection.

x → ±∞ $\qquad$ As x → ± ∞, y → 0

There is a horizontal asymptote at y = 0

discontinuity $\qquad$ y is undefined at x = -1

-y undefined $\qquad$ There is a vertical asymptote at x = -1

As x → -1$^+$, y → + ∞

As x → -1$^-$, y → - ∞
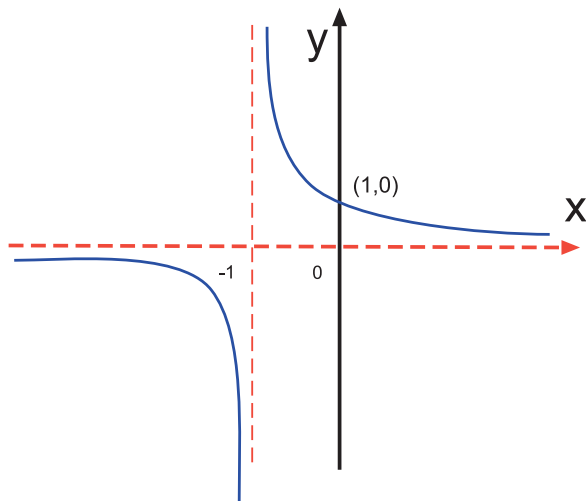
The horizontal axis is drawn with a dashed line to show that it is an asymptote.

Type 1 exercise

There are alternative questions on the web for you to try if you wish.

15 min

Q44: Sketch the graph of $y = \dfrac{16}{x^2}$

Q45: Sketch the graph of $y = \dfrac{3}{x}$

Q46: Sketch the graph of $y = \dfrac{4}{3x - 2}$
Include the asymptotes.

4.12.1   Type 1: Shortcuts

The equation takes the form $\frac{c}{ex + f}$

The vertical asymptote has the equation $x = \frac{-f}{e}$

All graphs of this type have a horizontal asymptote at $y = 0$

The graph crosses the y-axis at $y = \frac{c}{f}$ if $f \neq 0$

Apply these shortcuts to the previous graph of $\frac{1}{x + 1}$

$a = 0, b = 0, c = 1, d = 0, e = 1, f = 1$

The vertical asymptote is $x = \frac{-f}{e} = \frac{-1}{1} = -1$

The horizontal asymptote is at $y = 0$ as it is type 1

The graph crosses the y-axis at $y = \frac{1}{1} = 1$

The results match those found by the formal techniques but of course the turning points may still be needed.

Example What are the asymptotes for the function $y = \frac{2}{x + 5}$ and where does it cross the y-axis?

Answer :

$c = 2, e = 1, f = 5$

The vertical asymptote is at $x = \frac{-f}{e} = \frac{-5}{1} = -5$

The horizontal asymptote is at $y = 0$

The graph crosses the y-axis at $\frac{c}{f} = \frac{2}{5}$

4.13   Type 2 rational function: linear over linear

Example Sketch the graph of $y = \frac{2x + 3}{4x - 1}$

Answer: $y = \frac{2x + 3}{4x - 1}$

Symmetry	The function is neither odd nor even.
x-axis crossing	$y = 0 \Rightarrow 2x + 3 = 0 \Rightarrow x = -^3/_2$
	The graph crosses the x-axis at $x = -^3/_2$
y-axis crossing	$x = 0 \Rightarrow y = -3$
	The graph crosses the y-axis at $y = -3$

Turning points $\qquad \dfrac{dy}{dx} = \dfrac{-14}{(4x-1)^2}$

$\dfrac{dy}{dx} \neq 0$ at any point so there are no turning points or horizontal points of inflection.

$\dfrac{d^2y}{dx^2} \neq 0$ at any point

there are no points of inflection.

As $x \to \pm \infty$ To examine this the function has to be rearranged.

$\dfrac{2x+3}{4x-1} = \dfrac{2+\frac{3}{x}}{4-\frac{1}{x}}$

With any function of this type where the power of x is the same in the numerator and denominator divide the top and bottom by this power of x.

As $x \to \pm \infty$, $y \to {}^2/_4 = {}^1/_2$

There is a horizontal asymptote at $y = {}^1/_2$

discontinuity   y is undefined when $x = {}^1/_4$

- y undefined   As $x \to {}^1/_4{}^+$, $y \to +\infty$

As $x \to {}^1/_4{}^-$, $y \to -\infty$

There is a vertical asymptote at
$x = {}^1/_4$
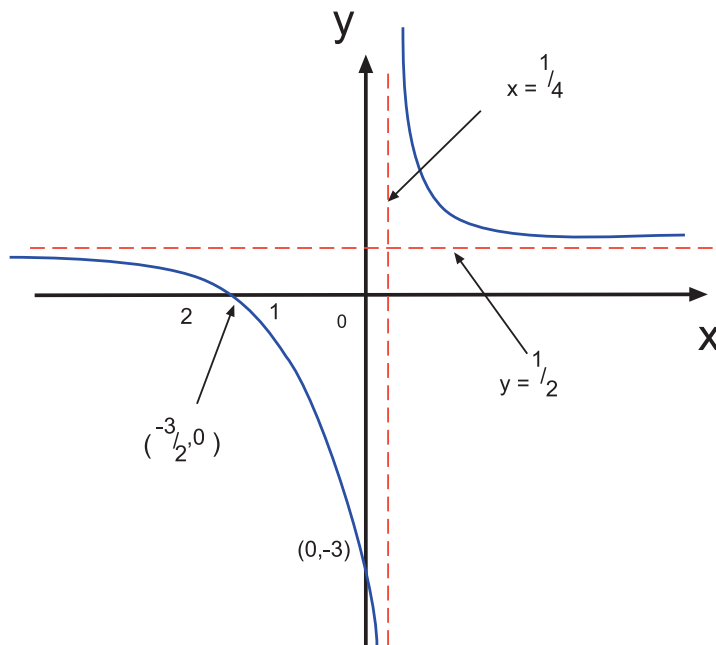
Type 2 exercise

There is an alternative exercise on the web for you to try.

15 min

Q47: Sketch the graph of $y = \dfrac{3x-4}{x+3}$

4.13.1   Type 2: Shortcuts

This takes the form $\frac{bx + c}{ex + f}$

The vertical asymptote for a graph of this type will have the equation $x = \frac{-f}{e}$

The horizontal asymptote will have the equation $y = \frac{b}{e}$

The graph will cross the y-axis at the point $(0, \frac{c}{f})$ if $f \neq 0$

The graph will cross the x-axis at the point $(\frac{-c}{b} , 0)$

Apply these shortcuts to the previous graph of $\frac{2x + 3}{4x - 1}$

$a = 0, b = 2, c = 3, d = 0, e = 4, f = -1$

The vertical asymptote is $x = \frac{-f}{e} = \frac{1}{4}$

The horizontal asymptote is at $y = \frac{b}{e} = \frac{1}{2}$

The graph crosses the y-axis at $y = \frac{c}{f} = \frac{3}{-1} = -3$

The graph crosses the x-axis at $x = \frac{-c}{b} = \frac{-3}{2}$

The results match those found by the formal techniques but of course the turning points may still be needed.

Example   Find the asymptotes and the points where the graph of $y = \frac{2x - 3}{3x + 4}$ crosses the axes.

Answer :

$b = 2, c = -3, e = 3, f = 4$

The vertical asymptote has equation $x = \frac{-f}{e} = \frac{-4}{3}$

The horizontal asymptote has the equation $y = \frac{b}{e} = \frac{2}{3}$

The graph crosses the y-axis at the point $(0, \frac{c}{f}) = (0, \frac{-3}{4})$

The graph crosses the x-axis at the point $(\frac{-c}{b}, 0) = (\frac{3}{2}, 0)$

4.14   Type 3 rational function: constant or linear over quadratic

Example   Sketch the graph of $y = \frac{5}{x^2 + 3x - 4}$

Answer: $y = \frac{5}{x^2 + 3x - 4}$

Symmetry	The function is neither odd nor even.
x-axis crossing	$y = 0 \Rightarrow 5 = 0$ which is impossible.
	The curve does not cross the x-axis.

y-axis crossing	when x = 0, y = $-5/4$. The graph crosses the y-axis at y = $-5/4$
Turning points	$$\frac{dy}{dx} = \frac{-10x - 15}{(x^2 + 3x - 4)^2}$$ $\frac{dy}{dx} = 0$ at turning points $\Rightarrow -10x - 15 = 0 \Rightarrow x = -3/2$ Use either the second derivative test or examine the tangent values as $x \to -3/2$ to give the nature of the turning point. Here the signs show that there is a maximum turning point at $\left(-\frac{3}{2}, -\frac{4}{5} \right)$
As $x \to \pm \infty$	As $x \to \pm \infty$, $y \to 0$ since $x^2 + 3x - 4$ becomes increasingly large There is a horizontal asymptote at y = 0
discontinuity - y undefined	y is undefined if $x^2 + 3x - 4 = 0$ Here the denominator factorises. If it did not, the quadratic formula would have to be used to determine if there were any real roots. So $x^2 + 3x - 4 = (x + 4)(x - 1)$ y is undefined if x = -4 or x = 1 There are vertical asymptotes at each. As $x \to -4^+$, $y \to -\infty$ As $x \to -4^-$, $y \to +\infty$ As $x \to 1^+$, $y \to +\infty$ As $x \to 1^-$, $y \to -\infty$
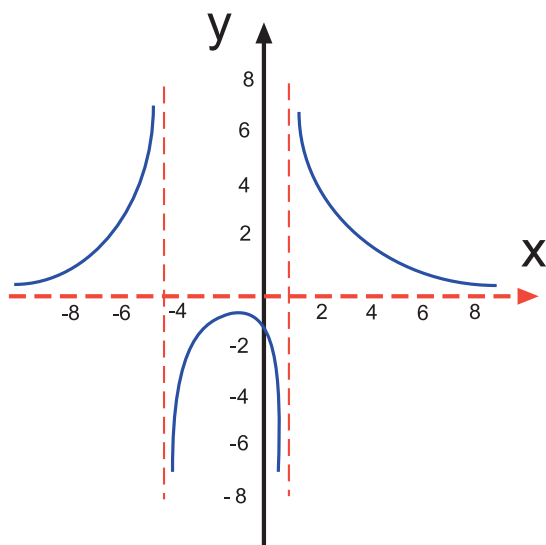
Type 3 exercise

There is another exercise on the web for you to try if you wish.

15 min

Q48: Sketch the graph of $y = \frac{x-4}{x^2-4}$

4.14.1　Type 3: Shortcuts

This takes the form $\frac{c}{dx^2 + ex + f}$ or $\frac{bx + c}{dx^2 + ex + f}$

The vertical asymptotes exist only if the equation $dx^2 + ex + f = 0$ has real solutions for x. This may not factorise, in which case the quadratic formula will have to be used.

Suppose that there are two roots, namely, $x = w$ and $x = z$ then these equations will be the equations of the vertical asymptotes.

If there are no solutions then there are no vertical asymptotes.

The horizontal asymptote will have the equation $y = 0$

The graph crosses the x-axis at the point $(\frac{-c}{b}, 0)$ **only when** the numerator is of the form $bx + c$. Otherwise there is no point at which the graph crosses the x-axis.

The graph crosses the y-axis at the point $(0, \frac{c}{f})$ if $f \neq 0$

Apply these shortcuts to the previous graph of $y = \frac{5}{x^2 + 3x - 4}$

$a = 0$, $b = 0$, $c = 5$, $d = 1$, $e = 3$, $f = -4$

The denominator factorises to give $(x + 4)(x - 1)$

The vertical asymptotes are $x = 1$ and $x = -4$

The horizontal asymptote is at $y = 0$

The graph crosses the y-axis at $y = \frac{c}{f} = -\frac{5}{4}$

The graph does not cross the x-axis since $b = 0$

The results match those found by the formal techniques but of course the turning points may still be needed.

Examples

1. What are the asymptotes for the function $f(x) = \frac{2x-3}{x^2 + 3x - 4}$ and where does it cross the axes?

Answer :

$b = 2$, $c = -3$, $d = 1$, $e = 3$, $f = -4$

If there are vertical asymptotes then the quadratic on the denominator will have real roots.

This quadratic factorises to $(x + 1)(x - 4)$ giving the solutions when equated to zero of $x = -1$ and $x = 4$

There are two vertical asymptotes with equations $x = -1$ and $x = 4$

The horizontal asymptote has the equation $y = 0$

The graph crosses the x-axis at the point $(\frac{-c}{b}, 0) = (\frac{3}{2}, 0)$

The graph crosses the y-axis at the point $(0, \frac{c}{f}) = (0, \frac{-3}{-4}) = (0, \frac{3}{4})$

2. What are the asymptotes for the function $f(x) = \frac{-4}{x^2 - 2x + 5}$ and where does it cross the axes?

Answer :

c = -4, d = 1, e = -2, f = 5

x^2 - 2x + 5 has no real roots so there are no vertical asymptotes.

There is a horizontal asymptote at y = 0

The graph does not cross the x-axis since the numerator is not of the form bx + c

The graph crosses the y-axis at the point $(0, \frac{c}{f}) = (0, \frac{-4}{5})$

4.15 Type 4 rational function: quadratic over quadratic

Example Sketch the graph of $y = \frac{3x^2 - 4x + 2}{x^2 - 2x + 1}$

Answer: $y = \frac{3x^2 - 4x + 2}{x^2 - 2x + 1}$

Symmetry	The function is neither odd nor even.
x-axis crossing	$y = 0 \Rightarrow 3x^2 - 4x + 2 = 0$
	But this function has no real roots.
	(b^2 - 4ac is negative) and so the curve does not cross the x-axis.
y-axis crossing	$x = 0 \Rightarrow y = 2$
	The curve crosses the y-axis at y = 2

Turning points

$$\frac{dy}{dx} = \frac{-2x^2 + 2x}{(x^2 - 2x + 1)^2} = \frac{-2x(x-1)}{((x-1)^2)^2} = \frac{-2x}{(x-1)^3}$$

At turning points $\frac{dy}{dx} = 0 \Rightarrow -2x = 0 \Rightarrow x = 0$

At x = 0 the signs or second derivative test show that the turning point at (0, 2) is a minimum.

As x → ±∞

To examine this, as with type 2, divide through by the highest power of x.

Here it is x^2

So $\frac{3x^2 - 4x + 2}{x^2 - 2x + 1} = \frac{3 - \frac{4}{x} + \frac{2}{x^2}}{1 - \frac{2}{x} + \frac{1}{x^2}}$

As x → ± ∞, y → 3

There is a horizontal asymptote at y = 3

discontinuity	y is undefined if $x^2 - 2x + 1 = 0$.
- y undefined	This factorises to $(x - 1)^2 = 0$
	y is undefined if $x = 1$

There is a vertical asymptote.

As $x \to 1^+$, $y \to + \infty$

As $x \to 1^-$, $y \to + \infty$
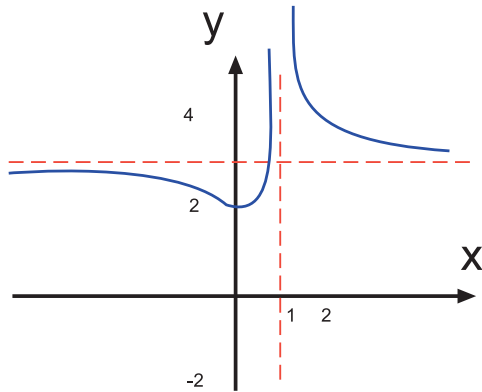
Type 4 exercise

There is an alternative exercise on the web for you to try if you prefer it.

15 min

Q49: Sketch the graph of $y = \frac{2x^2 - 4x - 1}{x^2 + 3x + 2}$

4.15.1   Type 4: Shortcuts

This takes the form $\frac{ax^2 + bx + c}{dx^2 + ex + f}$

Vertical asymptotes exist only if the equation $dx^2 + ex + f = 0$ has solutions for x.

As for type 3 if they exist then these solutions are the equations of the asymptotes.

The horizontal asymptote has equation $y = \frac{a}{d}$

The graph crosses the y-axis at the point $(0, \frac{c}{f})$ if $f \neq 0$

The graph crosses the x-axis at the real roots of the equation $ax^2 + bx + c = 0$ if there are any.

Apply these shortcuts to the previous graph of $y = \frac{3x^2 - 4x + 2}{x^2 - 2x + 1}$

$a = 3$, $b = -4$, $c = 2$, $d = 1$, $e = -2$, $f = 1$

The denominator factorises to give $(x - 1)^2$

The vertical asymptote is $x = 1$

The horizontal asymptote is at $y = \frac{a}{d} = \frac{3}{1}$

The graph crosses the y-axis at $y = \frac{c}{f} = \frac{2}{1} = 2$

The numerator does not have real roots and so the graph does not cross the x-axis.

The results match those found by the formal techniques but of course the turning points may still be needed.

Example Find the asymptotes for the function $f(x) = \frac{2x^2 + 3x - 2}{4x^2 + 13x + 3}$ and the points where it crosses the axes.

Answer :

$a = 2, b = 3, c = -2, d = 4, e = 13, f = 3$

The denominator factorises to give $(4x + 1)(x + 3)$. Solving this equal to zero gives the asymptote equations.

These vertical asymptote equations are $x = -3$ and $x = \frac{-1}{4}$

The horizontal asymptote has the equation $y = \frac{a}{d} = \frac{2}{4} = \frac{1}{2}$

The graph crosses the y-axis at the point $(0, \frac{c}{f}) = (0, \frac{-2}{3})$

The numerator factorises to give $(2x - 1)(x + 2)$

Solving this equal to zero gives $x = -2$ and $x = \frac{1}{2}$

The graph crosses the x-axis at the points $(-2, 0)$ and $(\frac{1}{2} , 0)$

4.16 Type 5 rational function: quadratic over linear

Example Sketch the graph of $y = \frac{x^2 + x + 2}{x - 1}$

Answer: $y = \frac{x^2 + x + 2}{x - 1}$

This function is an improper function as the numerator has degree higher than the denominator.

Divide through by the denominator to give $y = x + 2 + \frac{4}{x - 1}$

(See topic 1 on algebra: subsection on algebra long division.)

Symmetry	The function is neither odd nor even.
x-axis crossing	$y = 0 \Rightarrow x^2 + x + 2 = 0$
	$\Rightarrow$ there are no real roots.
	The graph does not cross the x-axis.
y-axis crossing	$x = 0 \Rightarrow y = -2$
	The curve crosses y at $y = -2$
Turning points	$\frac{dy}{dx} = 1 - \frac{4}{(x - 1)^2}$
	At turning points $\frac{dy}{dx} = 0 \Rightarrow 1 - \frac{4}{(x - 1)^2} = 0$
	$\Rightarrow x - 1 = \pm 2$: So $x = -1$ or $x = 3$

$$\frac{d^2y}{dx^2} = \frac{8}{(x-1)^3}$$

When x = -1 this is a negative $\Rightarrow$ maximum turning point at (-1,-1)

When x = 3 this is a positive $\Rightarrow$ minimum turning point at (3, 7)

As x $\rightarrow \pm \infty$ x $\rightarrow \pm \infty$, y $\rightarrow$ x + 2

$(\frac{4}{x-1} \rightarrow 0)$

There is a slant asymptote at y = x + 2

This is the quotient found by dividing the top by the bottom.

discontinuity   y is undefined if x - 1 = 0

-y undefined   There is a vertical asymptote at x = 1

As x $\rightarrow$ 1$^+$, y $\rightarrow$ + ∞

As x $\rightarrow$ 1$^-$, y $\rightarrow$ - ∞
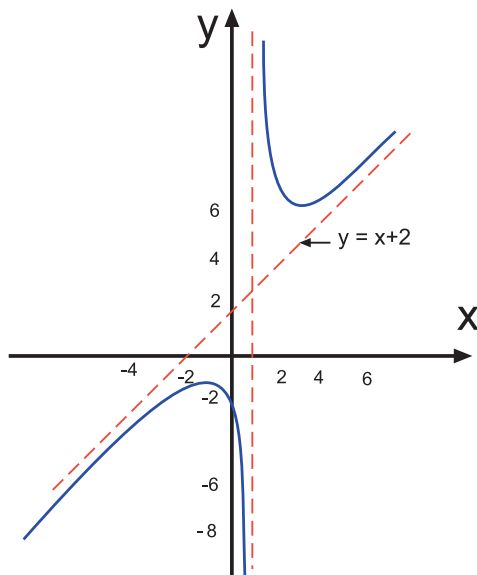
Type 5 exercise

There is an alternative exercise on the web for you to try if you prefer it.

15 min

Q50: Sketch the graph of y = $\frac{x^2 + 2x - 3}{x + 2}$

4.16.1   Type 5: Shortcuts

This takes the form $\frac{ax^2 + bx + c}{ex + f}$

The vertical asymptote has the equation x = $\frac{-f}{e}$

The slant asymptote is the quotient resulting from the division of the numerator by the denominator.

The graph crosses the y-axis at the point (0, $\frac{c}{f}$) if f $\neq$ 0

The graph crosses the x-axis at the real roots of the equation $ax^2 + bx + c = 0$ if there are any.

Otherwise it does not cross the axis.

Apply these shortcuts to the previous graph of $y = \frac{x^2 + x + 2}{x - 1}$

a = 1, b =1, c = 6, d = 0, e = 1, f = -1

The vertical asymptote has the equation $x = \frac{-f}{e} = \frac{1}{1} = 1$

The slant asymptote is at y = x + 2

(The quotient found upon dividing the numerator of $x^2 + x + 2$
by the denominator of x - 1)

The graph crosses the y-axis at $y = \frac{c}{f} = \frac{2}{-1} = -2$

The numerator does not have real roots and so the graph does not cross the x-axis.

The results match those found by the formal techniques but of course the turning points may still be needed.

Example Find the asymptotes for the function $y = \frac{x^2 + 1}{x}$ and where the graph crosses the axes.

Answer :

a = 1, b = 0, c = 1, e = 1, f = 0

A vertical asymptote does not exist since f = 0

The slant asymptote, found by dividing $x^2 + 1$ by x, is y = x

The graph does not cross the y-axis since f = 0 and the point $\frac{c}{f}$ is undefined.

The graph does not cross the x-axis since $x^2 + 1$ has no real roots.

4.17   Summary of shortcuts to sketching rational functions

Using the techniques and shortcuts mentioned, most of the rational functions of the types shown can be sketched easily.

Remember however that the turning points will still have to be calculated and shown on any questions which ask for the important features of a graph.

The following table shows a simplified picture of the various formulae used.

The types explained earlier in the unit can be identified by setting any of the coefficients (a, b, c, d, e or f) equal to zero if it is not present.

Remember that there will be instances when the formula is undefined.

In such instances it means that the graph does not possess the property shown.

general formula	$\frac{ax^2 + bx + c}{dx^2 + ex + f}$

	type 1 a, b, d = 0	type 2 a, d = 0	type 3 a = 0	type 4	type 5 d = 0
Horizontal asymptote	$y = 0$	$y = \frac{b}{e}$	$y = 0$	$y = \frac{a}{d}$	slant
Vertical asymptote	$x = \frac{-f}{e}$	$x = \frac{-f}{e}$	roots of denom.	roots of denom.	$x = \frac{-f}{e}$
crosses x-axis	no	$x = \frac{-c}{b}$	$x = \frac{-c}{b}$	roots of numer.	roots of numer.
crosses y-axis	$y = \frac{c}{f}$	$y = \frac{c}{f}$	$y = \frac{c}{f}$	$y = \frac{c}{f}$	$y = \frac{c}{f}$

Use the shortcut techniques to help with the sketching in the following questions.

Sketching exercise using shortcuts

20 min

There is an alternative exercise on the web for you to try.

Q51: Sketch the graph of $y = \frac{3x^2 - 4}{x + 1}$

Q52: Sketch the graph of $y = \frac{4x - 5}{x^2 - 1}$

Q53: Sketch the graph of $y = \frac{x^2 - 2x}{x^2 - 1}$

Q54: Sketch the graph of $y = \frac{2x^2 - 5x}{x^2 - 1}$

Q55: Sketch the graph of $y = \frac{2x^2}{x - 2}$

Q56: Sketch the graph of $y = \frac{x^2 - 3x + 3}{x - 1}$

4.18   Graphical relationships between functions

Learning Objective

Sketch a variety of functions using transformations

In previous sections simple relationships between functions have been used to aid with the sketching of the graph.

In this section several other types of relationships will be examined.

The following two relationships have already been encountered:

- $y = -f(x)$ is a reflection of $y = f(x)$ in the x-axis.

- $y = f(-x)$ is a reflection of $y = f(x)$ in the y-axis.

There are other relationships known as transformations. There are four to consider in this section.

If $y = f(x)$ then the following functions can be related to it:

1. $y = f(x) + k$

2. $y = k f(x)$

3. $y = f(x + k)$

4. $y = f(k x)$

where k is a constant.

Note that more than one transformation may occur in a function.

For example the graph of $y = -2x^3$ is a combination of a type 2 transformation and a reflection in the x-axis of the function $y = x^3$

Each of the transformation types shown is explained in the remainder of this section.

4.18.1   Type 1: $y = f(x) + k$

Example Sketch the graph of $y = 2x + 3$ for $x \in [-2, 2]$

Answer :

First of all complete a table of values with x from -2 to 2 as column headings and calculate values for the rows $y = 2x$ and $y = 2x + 3$

values of x	-2	-1	0	1	2
y = 2x	-4	-2	0	2	4
y = 2x + 3	-1	1	3	5	7

Plotting the points for $y = 2x$ and those for $y = 2x + 3$ gives:
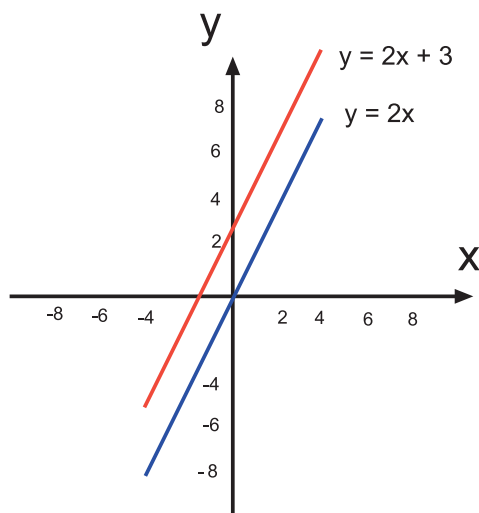
The effect of adding 3 to $y = 2x$ is clearly seen. The graph of $y = 2x + 3$ has moved three units up the y-axis.

Graphical relationships demonstration

There are demonstrations of each type of relationship on the web.

5 min

y = f (x) + k calculator investigation

5 min

Either use a graphics calculator or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

a) f (x) = 3x and g (x) = 3x - 4

b) f (x) = sin (x) and g (x) = sin (x) + 5

c) f (x) = x^2 and g (x) = x^2 + 1

d) f (x) = -2x and g (x) = -2x + 4

e) f (x) = cos x and g (x) = cos (x) - 3

The activities should clearly demonstrate the following rule:

Rule for graphing y = f (x) + k

To obtain the graph of y = f (x) + k take the graph of y = f (x).

- For k > 0 slide the graph UP the y axis by k units.

- For k < 0 slide the graph DOWN the y axis by k units.

This type of transformation is known as a vertical translation.

Vertical translation exercise

10 min

Q57: What is the relationship between y = 6x and y = 6x - 3?

Q58: What is the relationship between y = 2x and y = 2x + 8?

Q59: Describe the relationships between the two graphs y = 3x - 6 and y = 3x - 9

Q60: Using a graphics calculator explore and describe the relationships between the two graphs y = x^2 + 3x and y = x^2 + 3x + 2

Q61: Using a graphics calculator explore and describe the relationships between the two graphs y = ln x and y = ln x + 4

Q62: Using a graphics calculator explore and describe the relationships between the two graphs y = cos (2x) and y = cos (2x) - 3

4.18.2   Type 2: y = k f (x)

Example Sketch the graph of y = 4 sin x for x ∈ [-180°, 360°]

Answer:

Complete a table of values with x = -180° to 360° as column headings.

Calculate the values for the rows y = sin x and y = 4 sin x and plot the points for the two graphs.

values of x	-180°	-90°	0	90°	180°	270°	360°
y = sin x	0	-1	0	1	0	-1	0
y = 4 sin x	0	-4	0	4	0	-4	0

The relationship between the two graphs can be seen on the following diagram. This time the original graph is stretched vertically.
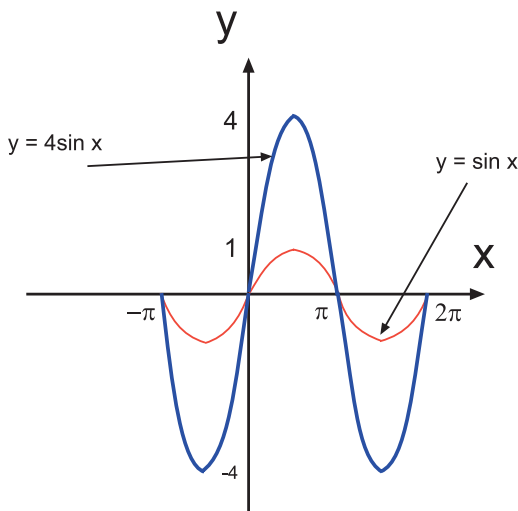
y = k f (x) calculator investigation

Either use a graphics calculator or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

5 min

   a) f (x) = sin (x) and g (x) = 3 sin (x)

   b) f (x) = cos (x) and g (x) = 2 cos (x)

   c) f (x) = x^3 and g (x) = $2x^3$

Again the activities lead to a straightforward rule.

Rule for graphing y = k f (x)

To obtain the graph of y = k f (x) scale the graph of y = f (x) vertically by a factor of k.

This type of transformation is known as a vertical scaling.

Therefore if k > 1 the scaling will in fact stretch the graph of f (x)

and if k < 1 the scaling will shrink the graph of f (x)

This rule works for all functions but this subsection has concentrated on demonstrating it through sin and cos graphs which show the scaling more clearly than linear or quadratic functions.

Vertical scaling exercise

Q63: Sketch the graph of y = $^1/_2$ cos x

Q64: What is the relationship between the graph of y = cos x and y = 3 cos x?

10 min

Q65: What is the relationship between the graph of y = sin x and y = $^1/_4$ sin x?

Q66: Using a graphics calculator explore the relationship between the two graphs
y = cos 2x and y = $^1/_2$ cos 2x

Q67: Explore the relationship between y = ln x and y = 3 ln x using a graphics calculator.

Q68: Explore the relationship between y = $^1/_2$e^x and y = 2e^x using a graphics calculator.

4.18.3   Type 3: y = f (x + k)

Example Sketch the graph of y = (x - 3)^2

Answer:

This is of the form f (x + k) where k = -3 and f (x) = x^2

Complete a table of values with x = -1 to 5 as column headings and calculate the values
for the rows y = x^2 and y = (x - 3)^2

values of x	-1	0	1	2	3	4	5
y = x^2	1	0	1	4	9	16	25
y = (x - 3)^2	16	9	4	1	0	1	4

The effect is a sideways shift of the graph.
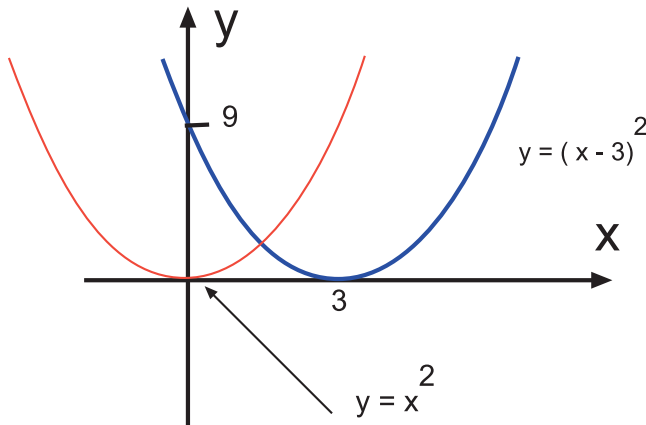
y = f (x + k) calculator investigation

5 min

Either use a graphics calculator or graph paper and sketch the graphs of y = f (x) for the
following functions. For each y = f (x) on the same screen or page, sketch the second
graph y = g (x) mentioned and determine a relationship between the functions:

   a) f (x) = x^2 and g (x) = (x - 2)^2

   b) f (x) = x^3 and g (x) = (x + 1)^3

   c) f (x) = 2x and g (x) = 2 (x - 3)

Horizontal translation exercise

Q69: Sketch the graph of y = 3(x - 2) and compare with graph of y = 3x

15 min

Q70: What is the relationship between y = cos x° and y = cos (x + 30)°

Q71: Where will the graph of y = (2x -1)^2 lie in relation to the graph of y = (2x)^2?

There is an web demonstration of this horizontal translation.

The rule this time needs a bit more care.

Rule for graphing y = f (x + k)

To obtain y = f (x + k) take y = f (x)

- For k > 0 slide the graph to the *left* by k units.

- For k < 0 slide the graph to the *right* by k units.

This type of transformation is called a horizontal translation or in trigonometric terms it is also called a phase shift.

Horizontal translation exercise

Q72: Sketch the graph of y = 3(x - 2) and compare with graph of y = 3x

Q73: Explore the relationship between y = cos 2x° and y = cos (2x - 45)° using a graphics calculator.

Q74: Explore the relationship between y = x^2 + 3 and y = x^2 + 4x + 7 using a graphics calculator.

Q75: Explore the relationship between y = $\sqrt{x}$ and y = $\sqrt{(x + 2)}$ using a graphics calculator.

Q76: What is the relationship between y = cos x° and y = cos (x + 30)°

Q77: Where will the graph of y = (2x -1)^2 lie in relation to the graph of y = (2x)^2?

Q78: Explore the relationship between y = $\sqrt{x}$ and y = $\sqrt{(x + 2)}$ using a graphics calculator.

4.18.4 Type 4: y = f (k x)

Example Sketch the graph of y = cos (2x) for x ∈ [-180°, 360°]

Answer :

The function is of the form y = f (k x) so f (x) = cos x

Then f (k x) = cos (k x). So k = 2

Complete a table of values with x = -180° to 360° as column headings and calculate the values for the rows y = cos x and y = cos (2x)

values of x	-180°	-90°	0	90°	180°	270°	360°
y = cos x	-1	0	1	0	-1	0	1
y = cos (2x)	1	-1	1	-1	1	-1	1

The effect is clearly seen on the diagram.

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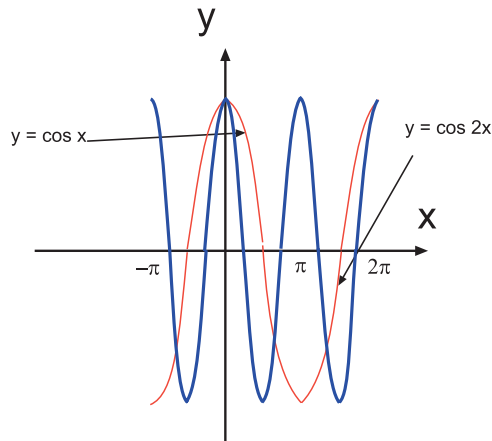
y = f (kx) calculator investigation

Either use a graphics calculator or graph paper and sketch the graphs of y = f (x) for the following functions. For each y = f (x) on the same screen or page, sketch the second graph y = g (x) mentioned and determine a relationship between the functions:

5 min

a) f (x) = sin (x) and g (x) = sin (3x)

b) f (x) = cos (x) and g (x) = cos (2x)

c) f (x) = x^3 and g (x) = $(2x)^3$

There is a demonstration of horizontal scaling on the web.

Another rule follows for this type of transformation but take care with this one.

Rule for graphing y = f (k x)

To obtain y = f (k x) scale the graph of y = f (x) horizontally by a factor of $^1/_k$

This type of transformation is called a horizontal scaling.

Therefore when k > 1, the graph of y = f (kx) will *squash* horizontally.

When k < 1, the graph will *stretch* horizontally.

Horizontal scaling exercise

Q79: What is the relationship between y = sin x and y = sin $^1/_3$ x?

15 min

Q80: Sketch the graph of y = tan 3x and compare with the graph of y = tan x

Q81: Using a graphics calculator explore and describe the relationship between the graph of y = cos x° and y = cos $^3/_4$x°

Q82: Using a graphics calculator explore and describe the relationship between the graph of y = $\sqrt{x}$ and y = $\sqrt{3x}$

Q83: Using a graphics calculator explore and describe the relationship between the graph of y = $\sqrt{x}$ and y = $\sqrt{3x}$

4.18.5 Modulus function

There is a particular function of x which uses part reflection in the x-axis. This function is called the modulus function.

The modulus of a real number is the value of the number (called its absolute value) regardless of the sign. So the modulus of 3 (written $| 3 |$) is 3 and the modulus of -3 (written $|-3|$) is also 3.

This is the basis of the definition of the modulus function.

The modulus function

For $x \in \mathbb{R}$ the modulus function of $f(x)$, denoted by $| f(x) |$ is defined by

$| f(x) | = f(x)$ if $f(x) \geqslant 0$
$\qquad - f(x)$ if $f(x) < 0$

Rule for graphing y = $| f(x) |$

To sketch the graph of a modulus function $| f(x) |$, first sketch the graph of the function $y = f(x)$

Take any part of it that lies below the x-axis and reflect it in the x-axis.

The modulus function $y = | f(x) |$ is the combined effect of the positive part of the original function and the new reflected part.

Example : The graph of the modulus

Sketch the graph of $f(x) = | x^2 - x - 6 |$

The graph of $y = x^2 - x - 6$ crosses the x-axis at $x = -2$ and $x = 3$

It has a minimum turning point at $(^1/_2, -^{25}/_4)$
Reflect the curve between $x = -2$ and $x = 3$

Keep the remainder of the curve $y = f(x)$
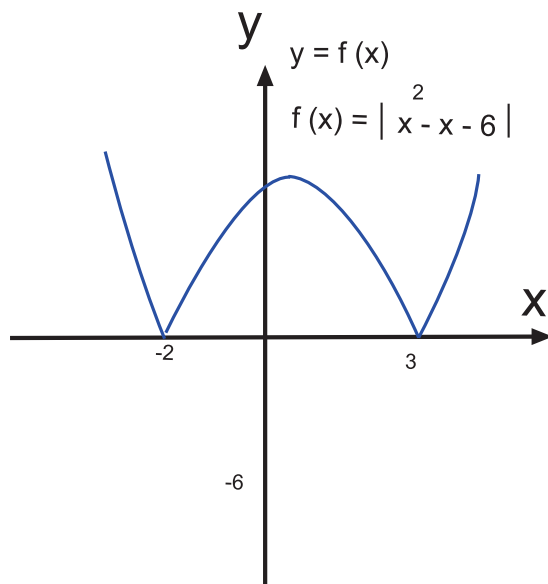
Q84: Sketch the graph of $y = \left| 2x^2 - x - 10 \right|$

Q85: Sketch the graph of $y = \left| \tan x \right|$ for $x \in [-270°, 270°]$

4.19 Summary

At this stage the following topics and techniques should be understood:

- One-to-one and Onto functions.

- Inverse function sketching.

- Odd and Even functions.

- Critical and Stationary points.

- Derivative tests.

- Concavity.

- Rational function sketching.

- Transformation of graphs.

4.20 Proofs

Learning Objective

Develop a proof for an onto or one-to-one function

Earlier in the unit the terms **one-to-one** and **onto** were defined.

Recall that a function f : A → B is a one-to-one function,
if whenever f (s) = f (t) then s = t where s ∈ A and t ∈ A.

An onto function is one in which the range is equal to the codomain.

When the unit explained odd and even functions it was stated that although the graphs of the functions could show that the property existed, a proof was still needed to be able to state that the function was odd or even.

In the same way it is necessary to show, by proof, that functions are one-to-one and/or onto before inverse functions can be explored further.

Proof of One-to-One functions - Strategy

The technique is to start with two elements in the domain.

Now suppose that f (s) = f (t)

If the function is one-to-one then it is possible to reach the conclusion that s = t

Example Prove that the function f (x) = $3x^3$ + 4 is a one-to-one function.

Answer :

Let s and t be elements in the domain of the function.

Suppose	f (s)	= f (t)
Then	$3s^3$ + 4	= $3t^3$ + 4
So	s^3	= t^3
Thus	s	= t

If f (s) = f (t) then s = t for all elements s, t belonging to the domain of the function f.

The function is a one-to-one function.

Proof of Onto functions - Strategy

This is much harder to establish than showing that a function is one-to-one. The proof is an existence proof. For functions with formulas the strategy is as follows.

Take a general element t which belongs to the codomain. Solve t = f (s) for s in the domain. This gives a possible domain element. By substituting the value of this element into the function and showing that it is a suitable domain element completes the proof.

Example Prove that the function f : ℝ → ℝ where f (x) = 3x + 4 is an onto function.

Answer :

Suppose that t is an element belonging to ℝ.

Let s	= $\frac{t - 4}{3}$	***
So f (s)	= f ($\frac{t - 4}{3}$)	
	= 3 ($\frac{t - 4}{3}$) + 4	
	= t	

Thus for every element t ∈ codomain of f there exists an element s in the domain of f such that f (s) = t

The function is an onto function.

*** To find the value of s, solve the equation t = 3s + 4 for s.

If, on the other hand, the task is to show that a function is not one-to-one (or onto) an example which shows this to be untrue is all that is required.

Example Determine whether the function f (x) = x^2 is a one-to-one function.

Answer :

Take the value of s = -1 and t = 1. These both give f (x) = 1 so here f (s) = f (t) does not mean that s = t. The function is not one-to-one.

Q86: Prove that f (x) = 2x - 4 for x $\in \mathbb{R}$ is a one-to-one function.

Q87: Prove that the function g (y) = $2y^5$ for y $\in \mathbb{R}$ is a one-to-one function.

Q88: Prove that the function h (z) = $\frac{1}{z+2}$ for z $\in \mathbb{R}$, z $\neq$ -2 is a one-to-one function.

Q89: Prove that the function f : $\mathbb{R} \to \mathbb{R}$ where f (x) = 3x is an onto function.

Q90: Prove that the function g : $\mathbb{R}^+ \to \mathbb{R}^+$ where g (y) = y^3 is an onto function.

Q91: Prove that the function h : {z $\in \mathbb{R}$; z $\neq$ 1} $\to$ {z $\in \mathbb{R}$; z $\neq$ 0}

where h (z) = 1 + $\frac{1}{z}$ is an onto function.

Q92: Show that the function f (x) = $\frac{1}{2x^2}$ is not one-to-one.

Q93: Show that the function f : $\mathbb{R}^+ \to \mathbb{R}$ where f (x) = $2\sqrt{x}$ is not onto.

4.21   Extended Information

> **Learning Objective**
>
> Display a knowledge of the additional information available on this subject

LEIBNIZ

Gottfried von Leibniz is attributed with first using the term 'function' but in those days (1694) the term was used to denote the slope of the curve.

He was a very famous mathematician and is best known for his work on calculus. He also developed the binary system of arithmetic.

EULER

In 1749 Leonhard Euler defined a function in terms of two related quantities, which is more in keeping with the modern definition. Euler made a considerable contribution to analysis and his name will appear in other sections of the course. He made contributions in the fields of geometry, calculus and number theory.

FOURIER

Joseph Fourier modified Euler's definition by noting that the domain of a function was important. He was an oustanding teacher and developed the theory of heat.

DIRICHLET

Lejeune Dirichlet introduced the concept of a correspondence relationship. This is similar to the definition of an onto function. He was interested in algebraic number theory and was also considered to be the founder of the theory of Fourier series (and not Fourier as one might expect).

4.22   Review Exercise

Review exercise

There is a web exercise for you to try if you wish.

Try to answer two questions in the twenty minutes given.

20 min

Q94:

a) For the function $y = \frac{x^2 - x - 2}{x + 2}$, $x \neq$ -2, $x \in \mathbb{R}$ write down the equation of the vertical asymptote.

b) For the function $y = \frac{x^2 - x - 2}{x + 2}$, $x \neq$ -2, $x \in \mathbb{R}$ show that the graph has a non vertical asymptote and find its equation.

c) Sketch the curve of $y = \frac{x^2 - x - 2}{x + 2}$. Show clearly all the important features.

Q95:

a) For the function $f(x) = \frac{-4x^2 + 13x - 28}{x - 3}$, $x \neq 3$, $x \in \mathbb{R}$ state the equation of the vertical asymptote.

b) For the function $f(x) = \frac{-4x^2 + 13x - 28}{x - 3}$ show that there is a non vertical asymptote and find its equation.

c) Sketch the graph of $\frac{-4x^2 + 13x - 28}{x - 3}$ and show all the important features clearly.

Q96: Sketch the curve of the function $y = \frac{x - 5}{x^2 - 16}$, $x \in \mathbb{R}$, $x \neq \{-4, 4\}$ indicating clearly your workings and the relevant important features of the graph.

Q97: Sketch the graph of the function $y = \frac{2x^2 - 2x - 2}{x - 2}$ and show all relevant important features of the graph clearly with appropriate justification.

4.23   Advanced Review Exercise

Advanced review exercise

There is an advanced review exercise on the web with randomised questions for you to try if you prefer.

30 min

Q98: Sketch the graph of the function $\frac{x^2 + 6x + 9}{x + 2}$. Show clearly all the important features.

Q99: Let the function f $(x) = \frac{2x^3 - 20x^2 + 64x - 37}{(x - 4)^2}$ with $x \neq 4$

a) The graph of $y = f(x)$ crosses the y-axis at $(0, a)$. State the value of a

b) Write down the vertical asymptote.

c) Show algebraically that there is a non-vertical asymptote and state its equation.

d) Find the coordinates and nature of the stationary point.

e) Show that f $(x) = 0$ has a solution in the interval $0 < x < 1$

f) Sketch the graph of the function. Show clearly all the important features.

4.24   Set review exercise

Set review exercise

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers.

15 min

Q100: Sketch the graph of $y = \frac{-6x^2 - 19x - 10}{5x^2 - x - 4}$ by taking the following steps:

a) Find where the graph crosses the y-axis and state the y value.

b) Find where the graph crosses the x-axis and state the smaller x value.

c) If there is a horizontal asymptote for this graph, find it and state the value of y

d) If there is a slant asymptote for this graph find it in the form $y = ax + b$ and state the right hand side of the equation.

e) What are the equations of the vertical asymptotes for this function?
Find the asymptote which lies closer to the y-axis in the form $x = a$ and state the value of a

f) Find the vertical asymptote which lies further away from the y-axis in the form $x = b$ and state the value of b

g) Now find any turning points and sketch the graph.

Q101: Sketch the graph of $y = \frac{1}{4x - 2}$ by taking the following steps:

a) Where does the graph cross the y-axis? Give the y value.

b) If the graph has a horizontal asymptote find it and give the value of y

c) If the graph has a slant asymptote find it in the form $y = ax + b$ and state the RHS side of the equation.

d) If the graph has a vertical asymptote find it and give the value of x

e) If there is a turning point find it and state the x value.

f) Now sketch the graph.

Q102: Sketch the graph of $y = \frac{4x - 2}{x^2 + 5x + 6}$ by taking the following steps:

a) If the graph crosses the y-axis find the point and state the y coordinate.

b) If the graph crosses the x-axis find the point and state the x coordinate.

c) If there is a horizontal asymptote for this type of function, find it and give the value of y

d) If there is a slant asymptote, find it in the form y = ax + b and give the RHS of the equation.

e) What are the equations of the vertical asymptotes for this function? Find the asymptote which lies closer to the y-axis in the form x = a and state the value of a

f) Find the vertical asymptote which lies further away from the y-axis in the form x = b and state the value of b

g) Find any turning points and sketch the graph.

Topic 5

Systems of Linear Equations

Contents

Prerequisite knowledge

- *A sound knowledge of solving two simultaneous equations algebraically is required for this unit.*

Learning Objectives

- *Use matrix methods to solve systems of linear equations.*

Minimum Performance Criteria:

- *Use Gaussian elimination to solve a 3 x 3 system of linear equations.*

5.1   Revision exercise

This exercise should help identify any areas of weakness in techniques which are required for the study of this unit. Some revision may be necessary if any of the questions seem difficult.

Revision exercise

10 min

There is a web version of this exercise if you would like to try it.

Q1: Solve the two equations x + 4y = 7 and x - y = 2 simultaneously.

Q2: Solve the two equations 3a + b = 10 and a + 3b = -2 simultaneously.

Q3: Solve the two equations -3s - 4t = -5 and 2s + t = 0 simultaneously.

5.2   Introduction

The two equations 5x - y = 7 and x + 2y = 8 have unique values for x and y that satisfy both equations. The equations can be solved either by algebraic means or by drawing the graphs of each equation and finding the point of intersection. This can be achieved since the equations are equations of lines in a plane.

What happens, however, with systems of equations in three unknowns... or four... or more?

It is true to say that a graphical approach can give the solution by finding the point of intersection of the lines (for two unknowns) or possibly planes (for three unknowns) for some systems. This still leaves many larger systems to solve and it would be convenient to find an algebraic way of solving such systems.

This unit will begin to explore the idea of matrices and how they can be used in various situations to provide information on several related linear equations. That is, information on systems of linear equations.

5.3 Matrix structure

Learning Objective

Identify elements in a matrix

What is a matrix?

Matrix

A matrix is a rectangular array of numbers.

A few examples will make this much clearer.

$$\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \text{ or } \begin{pmatrix} 45 & 3 \end{pmatrix} \text{ are all matrices.}$$

The first matrix has four rows and one column. It has four elements.

This matrix is said to have order 4×1.
The order is the number of rows $\times$ the number of columns.

Q4: How many rows, columns and elements does the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ have?

In general terms, an $m \times n$ matrix will have m rows, n columns and mn elements.

It can be shown as
$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \updownarrow \text{ m rows}$$
$$\leftrightarrow$$
$$\text{n columns}$$

This matrix A is written as $A = \left(a_{ij} \right)_{m \times n}$

a_{ij} is the element in the i-th row and the j-th column.

a_{23} is the element in the second row and the third column.

Example What is the value of the element a_{12} in the matrix shown
$$\begin{pmatrix} 2 & 7 & 3 & 6 \\ 1 & -5 & 8 & -11 \\ 13 & 9 & 0 & 4 \\ 12 & 21 & -6 & 10 \end{pmatrix}$$

Answer :

7. It is the element in the first row and the second column.

Matrix elements and entries exercise

There are two small web exercises covering the same ideas if you prefer to try them.

10 min

Q5: In the matrix shown $\begin{pmatrix} 2 & 7 & 3 & 6 \\ 1 & -5 & 8 & -11 \\ 13 & 9 & 0 & 4 \\ 12 & 21 & -6 & 10 \end{pmatrix}$ what values do these elements have:

1. a_{23}
2. a_{32}
3. a_{41}
4. a_{44}

Q6: In the matrix shown $\begin{pmatrix} 3 & -9 & 6 & 0 \\ 7 & 1 & 8 & -4 \\ 13 & -2 & 5 & -1 \\ 2 & 4 & -7 & 11 \end{pmatrix}$ what values do these elements have:

1. a_{31}
2. a_{13}
3. a_{43}
4. a_{14}

Q7: Using the matrix $\begin{pmatrix} 2 & 7 & 3 & 6 \\ 1 & -5 & 8 & -11 \\ 13 & 9 & 0 & 4 \\ 12 & 21 & -6 & 10 \end{pmatrix}$

state the values of i and j for which a_{ij} denotes the following elements:

1. 13
2. 21
3. -11

Q8: Using the matrix $\begin{pmatrix} 3 & -9 & 6 & 0 \\ 7 & 1 & 8 & -4 \\ 13 & -2 & 5 & -1 \\ 2 & 4 & -7 & 11 \end{pmatrix}$

state the values of i and j for which a_{ij} denotes the following elements:

1. 6
2. 2
3. -1

When a matrix has the same number of rows and columns, it is given a special name.

Square matrix

A square matrix has the same number of rows and columns.

The matrix $\begin{pmatrix} 4 & 7 \\ 3 & 5 \end{pmatrix}$ is a square matrix of order 2.

There are several other names that should also be noted.

A $1 \times n$ matrix $(\ a_{11} \quad a_{12} \quad ... \quad a_{1n} \ )$ is called a row matrix of dimension n (also termed a vector).

A $m \times 1$ matrix $\begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \end{pmatrix}$ is called a column matrix (or vector) of dimension m.

A 1×1 matrix (a_{11}) is called a scalar.

5.4 Matrices and simultaneous equations

Learning Objective

Form matrices to represent a system of equations

Matrices are used in a wide range of contexts. They will be used in this topic as a systematic way of solving systems of simultaneous (linear) equations.

To illustrate this, look at the following example of solving a system of two equations in two unknowns.

Example Solve the two equations $2x + 3y = 8$ and $3x - y = 1$ simultaneously for x and y.

Answer:

$$\begin{cases} 2x + 3y = 8 & (1) \\ 3x - y = 1 & (2) \end{cases}$$

becomes

$$\begin{cases} 2x + 3y = 8 & (1) \\ 9x - 3y = 3 & (3) \qquad 3 \times (2) \end{cases}$$

adding (1) and (3) gives $11x = 11$

so $x = 1$

Substitute $x = 1$ into equation (1) to give

$2 + 3y = 8 \Rightarrow y = 2$

Geometrically, the equations (1) and (2) define two lines in the plane.

The intersection of the two lines is the point (1, 2) given by the solution of the system of equations.
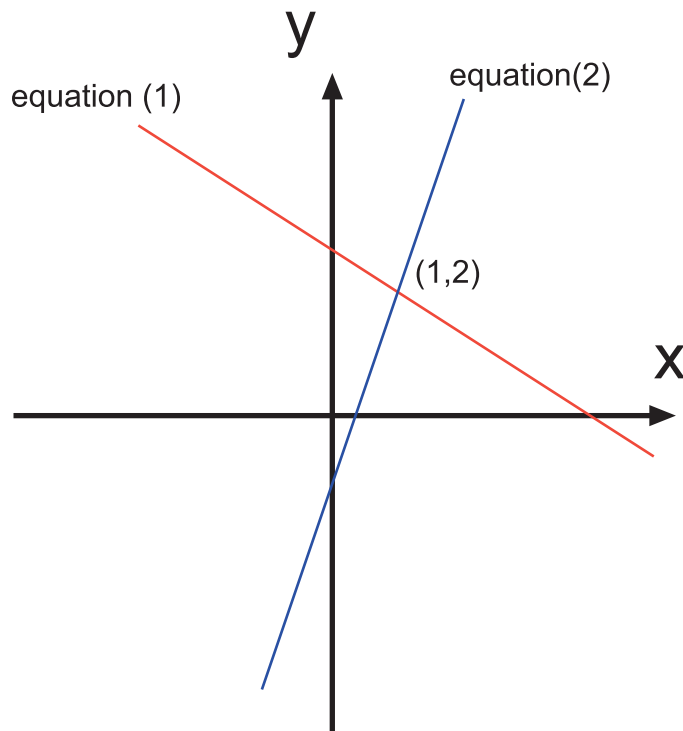
Algebraically the following tricks were used to solve these equations:

1. Multiply (or divide) both sides of an equation by some constant.

2. Add (or subtract) two equations.

By using matrices, these tricks can be developed into a systematic approach to solving larger systems of linear equations with more unknowns.

A matrix can be formed by taking the coefficients of the unknowns in a system of equations.

Example

$3x + 4y$ $= 10$
$5x - 3y$ $= 7$

Take the coefficients only as shown in bold

$\mathbf{3}x + \mathbf{4}y$ $= 10$
$\mathbf{5}x - \mathbf{3}y$ $= 7$

Set up the matrix $\begin{pmatrix} 3 & 4 \\ 5 & -3 \end{pmatrix}$

By constructing another matrix from the right hand side of the system of equations and a third from the unknowns it is possible to set up a matrix equation.

Formation of a matrix

There is an animated version of the formation of a matrix on the web.

5 min

Example Consider a collection of 3 equations in 3 unknowns x, y and z

2x - y + z $\quad\quad$ = 5
x - 3y + 2z $\quad\quad$ = 2
2x + y + 4z $\quad\quad$ = -3

- Take only the coefficients of the terms on the left hand side and form a matrix.

  2x **-1**y **+1**z $\quad\quad$ = 5
  1x **- 3**y **+ 2**z $\quad\quad$ = 2
  2x **+1**y **+ 4**z $\quad\quad$ = -3

  This is called a coefficient matrix. Say, $A = \begin{pmatrix} 2 & -1 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 4 \end{pmatrix}$

- Put the unknowns x, y and z into a column matrix. Say, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

- Write the numbers from the right hand side of the equation in another column matrix.

  Say, $B = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix}$

So now the system of three equations can be written as a single matrix equation

AX = B

This type of equation will be studied in greater depth in the second unit on matrices.

Another useful matrix construction is the augmented matrix. This is formed by using the coefficient matrix A and adjoining the column matrix B. It is written as

$$(A \mid B) = \left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & -3 & 2 & 2 \\ 2 & 1 & 4 & -3 \end{array} \right)$$

Augmented matrix formation

There is an animated version of the formation of an augmented matrix on the web.

5 min

Example Write down the coefficient matrix and the augmented matrix for the system of equations

3x - 2y + z $\quad\quad$ = 12
2x - z $\quad\quad$ = 6
y + z $\quad\quad$ = 3

Answer:

When there are missing terms such as a 'y' term in the second equation remember that this means that 'y' has a coefficient of zero.

The coefficient matrix is $\begin{pmatrix} 3 & -2 & 1 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

The augmented matrix is $\left(\begin{array}{ccc|c} 3 & -2 & 1 & 12 \\ 2 & 0 & -1 & 6 \\ 0 & 1 & 1 & 3 \end{array} \right)$

Forming matrices exercise

Q9: Write down the coefficient matrix and the augmented matrix for the system of three equations in three unknowns

4x - 3y + z   = -1

2x + y + 3z   = 7

x + 4y + 2z   = 8

Q10: Write down the coefficient matrix and the augmented matrix for the system of three equations in three unknowns

3x + z   = 11

2x - y + z   = 6

x + 4y   = 14

5.5   Elementary row operations

Learning Objective

Perform elementary row operations on matrices

The equations shown earlier

2x - y + z   = 5

x - 3y + 2z   = 2

2x + y + 4z   = -3

had an augmented matrix of $\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & -3 & 2 & 2 \\ 2 & 1 & 4 & -3 \end{array} \right)$

Notice that each row of the matrix represents one of the original equations.

The order of the equations and the rows of the augmented matrix can be changed around without altering the system of equations.

So along with the two tricks mentioned earlier for solving simultaneous equations there are now three ways of manipulating a matrix to solve a system of equations.

Elementary row operations

The three ways in which a matrix can be manipulated to solve a system of equations are called elementary row operations. They are:

- Interchange two rows.

- Multiply one row by a non zero constant.

- Change one row by adding a multiple of another row.

Here is an example of each type of elementary row operation.

Examples

1. Interchanging rows

Interchanging rows 1 and 2

$$\begin{pmatrix} 2 & -1 & 1 & \Big| & 5 \\ 1 & -3 & 2 & \Big| & 2 \\ 2 & 1 & 4 & \Big| & -3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \xrightarrow[r3]{r1 \leftrightarrow r2} \begin{pmatrix} 1 & -3 & 2 & \Big| & 2 \\ 2 & -1 & 1 & \Big| & 5 \\ 2 & 1 & 4 & \Big| & -3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix}$$

2. Multiplying one row by a non zero constant

Multiplying row 3 by 3

$$\begin{pmatrix} 2 & -1 & 1 & \Big| & 5 \\ 1 & -3 & 2 & \Big| & 2 \\ 2 & 1 & 4 & \Big| & -3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \xrightarrow[3r3]{r1 \; r2} \begin{pmatrix} 2 & -1 & 1 & \Big| & 5 \\ 1 & -3 & 2 & \Big| & 2 \\ 6 & 3 & 12 & \Big| & -9 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix}$$

3. Change one row by adding a multiple of another row

Taking row 2 and adding 2 times row 3 to it

$$\begin{pmatrix} 2 & -1 & 1 & \Big| & 5 \\ 1 & -3 & 2 & \Big| & 2 \\ 2 & 1 & 4 & \Big| & -3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \xrightarrow[r3]{\substack{r1 \\ r2 + 2r3}} \begin{pmatrix} 2 & -1 & 1 & \Big| & 5 \\ 5 & -1 & 10 & \Big| & -4 \\ 2 & 1 & 4 & \Big| & -3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix}$$

5.6 Upper triangular matrices

Learning Objective

Convert a matrix to upper triangular form

Certain square matrices have a special property. Matrices with this property are called upper triangular.

Upper triangular matrix

A square matrix is upper triangular if all the entries below the main diagonal are zero.

Note that the main diagonal is the diagonal which stretches from top left to bottom right.

Examples

1.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 7 & -5 \\ 0 & 0 & 2 \end{pmatrix}$$

This is an upper triangular matrix. The main diagonal entries are 1, 7 and 2. All entries below these are zero.

2.

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 7 & -5 \\ 0 & 1 & 2 \end{pmatrix}$$

This matrix is not upper triangular. The entry a_{32} is below the main diagonal and is a one not a zero.

Using the elementary row operations mentioned earlier a square matrix can now be converted to an upper triangular form.

Note the two examples in the following activity and how the row operations are used to convert the matrix.

Examples

1. Convert $\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix}$ to upper triangular form.

Answer:

$$\begin{pmatrix} 2 & 4 \\ 3 & -1 \end{pmatrix} \begin{matrix} r1 \\ r2 \end{matrix} \xrightarrow[r2]{\frac{1}{2}r1} \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix} \begin{matrix} r1 \\ r2 \end{matrix} \xrightarrow[r2-3r1]{r1} \begin{pmatrix} 1 & 2 \\ 0 & -7 \end{pmatrix} \begin{matrix} r1 \\ r2 \end{matrix}$$

2. Convert $\begin{pmatrix} 0 & 1 & 2 \\ 4 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ to upper triangular form.

Answer:

$$\begin{pmatrix} 0 & 1 & 2 \\ 4 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix}$$

$$\begin{matrix} r3 \\ r2 \\ r1 \end{matrix} \begin{pmatrix} 1 & 1 & 0 \\ 4 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \qquad \text{Ensure } a_{11} \neq 0 \text{ by swapping rows}$$

$$\begin{matrix} r1 \\ r2-4r1 \\ r3 \end{matrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \qquad \begin{matrix} \text{Add a suitable multiple of r1 to r2 to ensure} \\ \text{that } a_{12} = 0 \end{matrix}$$

$$\begin{matrix} r1 \\ r2 \\ r3+r2 \end{matrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \qquad \begin{matrix} \text{Add a suitable multiple of r2 to r3 to ensure} \\ \text{that } a_{32} = 0 \end{matrix}$$

Notice that the rows are always renumbered on the right hand side of each matrix.

This example demonstrates the following strategy for converting a matrix to upper triangular form.

Strategy for converting a matrix to upper triangular form

- Start at the top left hand corner. If the entry is a zero, interchange rows to obtain a non zero entry in this cell (a_{11}).

- Keep this first row fixed and use elementary row operations to obtain zeros for the remaining elements in the first column.

- Move to the second row and examine the first non zero entry (a_{22}). If the entry is a zero interchange it with any row below it to obtain a non-zero entry in this cell.

- Keep this row (and the first row) fixed and use elementary row operations to obtain zeros for the remaining elements in this second column.

- Continue with this strategy until there are no further possible operations on the matrix.

Example : The strategy in operation

Convert the matrix $\begin{pmatrix} 0 & 2 & 7 \\ 1 & 2 & 1 \\ 2 & 6 & 5 \end{pmatrix}$ to upper triangular form using the strategy for elementary row operations.

$$\begin{pmatrix} 0 & 2 & 7 \\ 1 & 2 & 1 \\ 2 & 6 & 5 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix}$$

$$\begin{matrix} r1 \leftrightarrow r2 \\ \rightarrow \\ r3 \end{matrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 7 \\ 2 & 6 & 5 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \qquad a_{11} \text{ non - zero.}$$

$$\begin{matrix} r1 \\ \rightarrow \quad r2 \\ r3 - 2r1 \end{matrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 7 \\ 0 & 2 & 3 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \qquad \text{Other first column entries zero.}$$

$$\begin{matrix} r1 \\ \rightarrow \quad r2 \\ r3 - r2 \end{matrix} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 7 \\ 0 & 0 & -4 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \qquad \text{Other second column entries zero.}$$

Note that, as in this example, not all the steps may be necessary.

Upper triangular matrix exercise

There is a web exercise if you prefer it.

20 min

Q11: Identify the elementary row operations used in this example.

$$\begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Q12: Identify the elementary row operations used in this example.

$$\begin{pmatrix} 3 & 3 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 3 & 3 & 2 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

Q13: Use elementary row operations to convert $\begin{pmatrix} 4 & 25 \\ 1 & 5 \end{pmatrix}$ to upper triangular form.

Q14: Use elementary row operations on the matrix $\begin{pmatrix} 1 & -2 & 4 \\ -1 & 3 & 4 \\ 0 & 6 & 46 \end{pmatrix}$ to convert it to upper triangular form.

Q15: Use elementary row operations on the matrix $\begin{pmatrix} 1 & 3 & -5 \\ 11 & 35 & -50 \\ 2 & 7 & 9 \end{pmatrix}$ to convert it to upper triangular form.

Q16: Use elementary row operations to convert $\begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 9 \end{pmatrix}$ to upper triangular form.

5.7　Gaussian elimination

Learning Objective

Solve a system of equations using Gaussian elimination

Recall from the previous sections that systems of equations can be transformed into a matrix equation and that it is possible to find the upper triangular form for square matrices by using elementary row operations.

This can now be brought together to give actual solutions to a system of equations.

Gaussian elimination

The technique of using an augmented matrix formed from a system of original equations to find the unknowns is called Gaussian elimination.

The technique is best explained by the following example.

Example Solve the following system of equations for x, y and z

$2x - y + z \qquad = 5$

$x - 3y + 2z \qquad = 2$

$2x + y + 4z \qquad = -3$

Answer:

Form the augmented matrix first $\left(\begin{array}{ccc|c} 2 & -1 & 1 & 5 \\ 1 & -3 & 2 & 2 \\ 2 & 1 & 4 & -3 \end{array} \right)$

Use elementary row operations to give upper triangular form.

$$
\begin{array}{c}
\text{r1} \\
\text{2r2 - r1} \\
\text{r3}
\end{array}
\left(
\begin{array}{ccc|c}
2 & -1 & 1 & 5 \\
0 & -5 & 3 & -1 \\
2 & 1 & 4 & -3
\end{array}
\right)
\begin{array}{c}
\text{r1} \\
\text{r2} \\
\text{r3}
\end{array}
$$

$$
\rightarrow
\begin{array}{c}
\text{r1} \\
\text{r2} \\
\text{r3 - r1}
\end{array}
\left(
\begin{array}{ccc|c}
2 & -1 & 1 & 5 \\
0 & 5 & -3 & 1 \\
0 & 2 & 3 & -8
\end{array}
\right)
\begin{array}{c}
\text{r1} \\
\text{r2} \\
\text{r3}
\end{array}
$$

$$
\rightarrow
\begin{array}{c}
\text{r1} \\
\text{r2} \\
\text{5r3 - 2r2}
\end{array}
\left(
\begin{array}{ccc|c}
2 & -1 & 1 & 5 \\
0 & 5 & -3 & 1 \\
0 & 0 & 21 & -42
\end{array}
\right)
\begin{array}{c}
\text{r1} \\
\text{r2} \\
\text{r3}
\end{array}
$$

Take the upper triangular matrix and change this back into a system of equations using the entries as coefficients to give:

$2x - y + z = 5$

$5y - 3z = 1$ Note that there is no x term.

$21z = -42$ Note that there is no x or y term.

From the last equation $z = -2$

Substitute $z = -2$ in the second equation to give

$5y + 6 = 1 \Rightarrow y = -1$

Finally, substitute both $z = -2$ and $y = -1$ in the first equation to give

$2x + 1 - 2 = 5 \Rightarrow x = 3$

The solution to the system of equations is $x = 3$, $y = -1$ and $z = -2$

This can be written as $(x, y, z) = (3, -1, -2)$

The three equations in the last example represent the equations of three planes whose point of intersection is $(3, -1, -2)$.

Note that if there are two equations in two unknowns then a unique solution would represent the intersection at a point of two lines in a plane.

Gaussian elimination exercise

There is another exercise on the web if you prefer it.

20 min

Q17: Use Gaussian elimination to solve the system of equations

$x + y = 15$

$5x - 6y = -13$

Q18: Use Gaussian elimination to solve the system of equations

$x + y + z = 6$

$-x + y + 2z = 5$

$3x + 2z = 12$

Q19: Use Gaussian elimination to solve the system of equations

-2x + y - z   = 1
-x + y + 2z   = 11
3x + 2y   = -4

This practice of reducing a matrix to upper triangular form can be taken further until the matrix itself provides the answers without any need for back substitution into the original equations.

This technique is the Gauss-Jordan elimination and is explained by the following example.

Example Solve the following system of equations

-2x - 3y + z   = 1
4x - y + z   = 17
x + 3y - z   = -4

Answer:

The augmented matrix is $\begin{pmatrix} -2 & -3 & 1 & | & 1 \\ 4 & -1 & 1 & | & 17 \\ 1 & 3 & -1 & | & -4 \end{pmatrix}$

Elementary row operations can reduce this to $\begin{pmatrix} 1 & 3 & -1 & | & -4 \\ 0 & 3 & -1 & | & -7 \\ 0 & 0 & 2 & | & 8 \end{pmatrix}$

but with further row operations the matrix reduces to $\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 4 \end{pmatrix}$

If this is converted back to equations then the answers are immediate and are:

x = 3, y = -1 and z = 4

This type of reduced matrix is called a matrix in reduced echelon form.

Activity

Take the matrix from the example, $\begin{pmatrix} 1 & 3 & -1 & | & -4 \\ 0 & 3 & -1 & | & -7 \\ 0 & 0 & 2 & | & 8 \end{pmatrix}$ and continue with row operations to obtain $\begin{pmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 4 \end{pmatrix}$

5.8 Solutions of systems of equations

Learning Objective

Identify the types of solutions which exist in systems of equations

From the previous work in this unit it would appear that every system of equations has a solution and that the solution is unique.

However this is not true. There are some systems which do have a unique solution but others have no solution and a third category have an infinite number of solutions.

Before looking at this idea in more depth consider the case of two parallel lines.

Earlier it was stated that solutions of a system of equations represent an intersection of planes at a point or at lines in a plane. Here is a case where there is no intersection and so there is no solution to the system of equations by these lines.

Example : No solutions

Investigate any solutions of the two equations

$2x - y = 3$

$3y - 6x = -5$

Answer:

In matrix form these can be written as $\begin{pmatrix} 2 & -1 & | & 3 \\ -6 & 3 & | & -5 \end{pmatrix}$

but this gives $\begin{pmatrix} 2 & -1 & | & 3 \\ 0 & 0 & | & 4 \end{pmatrix}$ after using elementary row operations in an attempt to put it in upper triangular form.

In turn this would convert back to an equation $0x + 0y = 4 \Rightarrow 0 = 4$ which is impossible. Such an equation leads to an inconsistency - the equation is said to be inconsistent.

This means that there are no solutions to this system of equations.

In fact the lines represented by these two equations are parallel. They will never meet and therefore there can be no intersection point.

Note carefully that it is important to write the elements in the matrix keeping the unknowns in the same order. Check carefully the second row in the last example and compare with the equations given.

The second equation $3y - 6x = -5$ is rearranged to $-6x + 3y = -5$ before the matrix entries can be completed.

Another rearrangement of the equations gives

$y = 2x - 3$ and $3y = 6x - 5$

On a graph these show clearly that there can be no solution as they will never intersect.
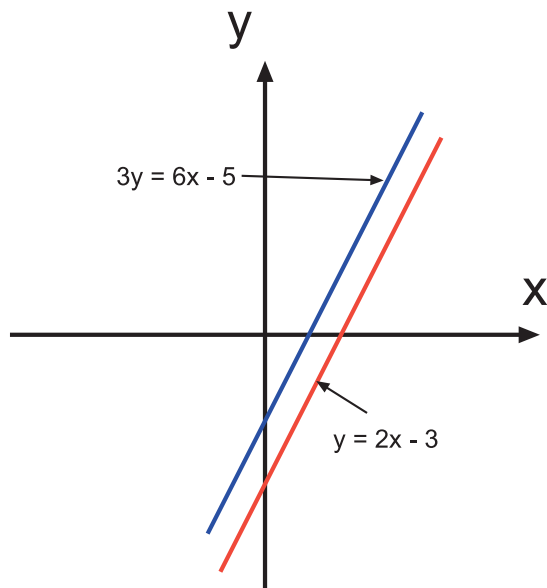
Inconsistent equations are found in a system of equations where the relationship in one equation between the unknowns is entirely different from the relationship in another.

For example, in the simple case of the two equations

$x + y = 3$

$x + y = -2$

These give two parallel lines so that the relationship between x and y is never the same. That is, there are no values for x and y which satisfy both equations.

For systems of three equations in three unknowns with no solutions (i.e. inconsistent equations) there are two possible geometrical interpretations:

- The intersections of the planes are three parallel lines. (in a triangle formation).
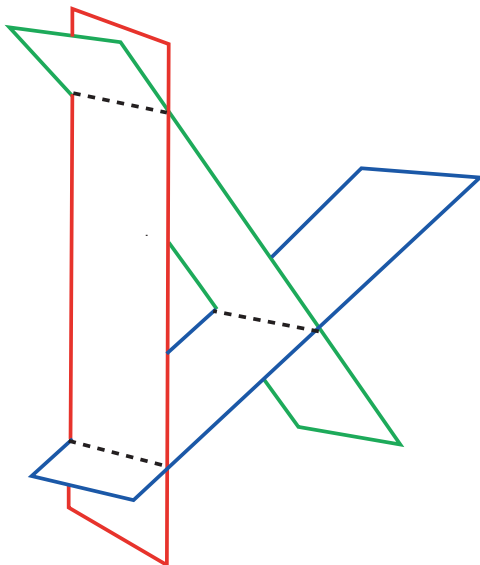
2d representation of the intersection of 3 planes
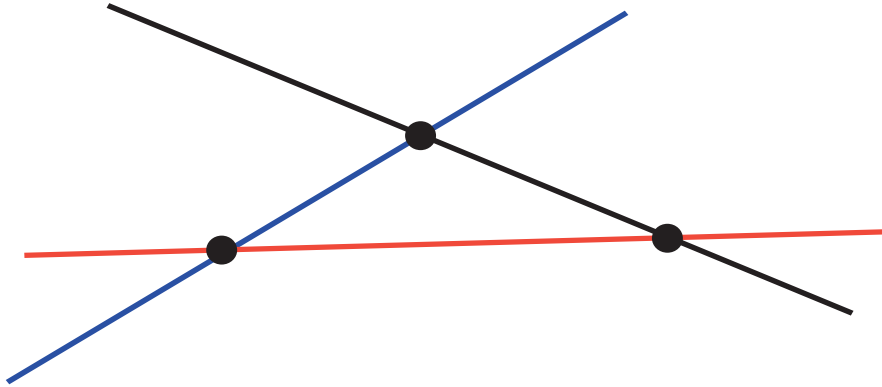
(The points of intersection of the planes are the points through which the parallel lines pass in the third dimension.)

- Two of the planes are parallel (in a 'z' type of formation).

2d representation of the intersection
of 3 planes

(The points of intersection of the planes are the points through which the parallel lines pass in the third dimension.)

Example : Infinite solutions

Investigate solutions of the system of equations

x + y + z   = 1

2x + 2y + z   = 3

2x + 2y + 2z   = 2

Answer:

The augmented matrix gives

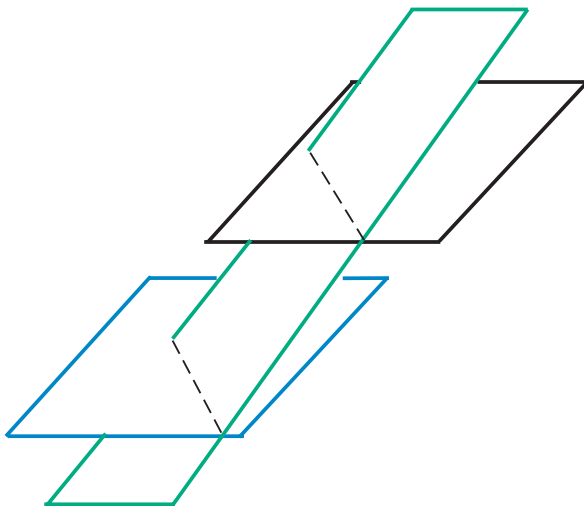$$\begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 2 & 2 & 1 & | & 3 \\ 2 & 2 & 2 & | & 2 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} \quad \begin{matrix} r1 \\ \rightarrow \ r2 - 2r1 \\ r3 - 2r1 \end{matrix} \quad \begin{pmatrix} 1 & 1 & 1 & | & 1 \\ 0 & 0 & -1 & | & 1 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \begin{matrix} r1 \\ r2 \\ r3 \end{matrix} .$$

Putting this back into equation form leaves z = -1 which when substituted in equation 1 gives x + y = 2

This leads to infinite solutions as the equation relates to a straight line. (How many points are on a line?)

In geometric terms it represents the intersection of two of the planes, not at a point but at a line. The third plane is coincident to (the same as) one of the others.

2 coincident planes
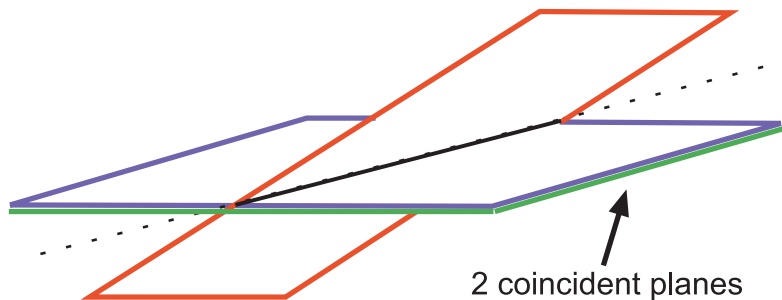
In this situation two of the equations are equivalent - one is a multiple of the other. In effect this leaves two equations in three unknowns and will lead to infinite solutions if the two planes do intersect.

For systems of three equations in three unknowns with infinite solutions there are three possible geometrical interpretations:

- The intersections of the planes is at a common line (like a star).

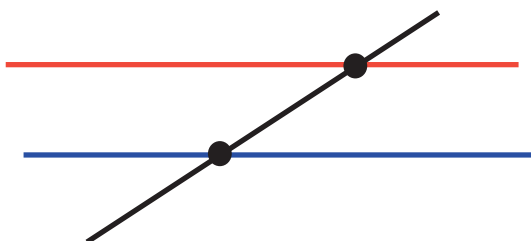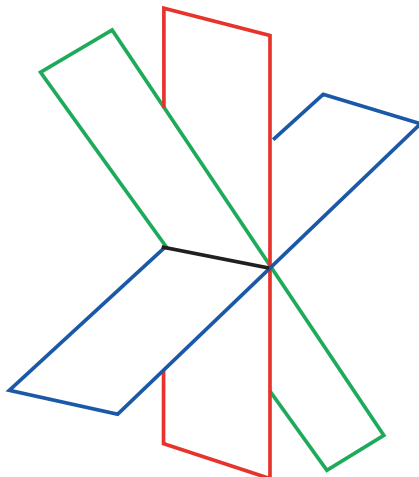The point of intersection of the planes is the point through which the line of intersection passes in the third dimension.

- Two of the planes intersect at a line and the third plane is coincident with one of the other two (as a cross like the previous example).
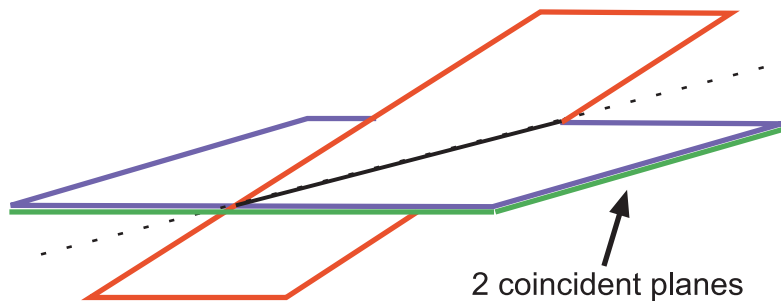
2 coincident planes

The point of intersection of the planes is the point through which the line of intersection passes in the third dimension.

- All three planes are coincident: the equations all represent the same plane (like a sheet of paper).
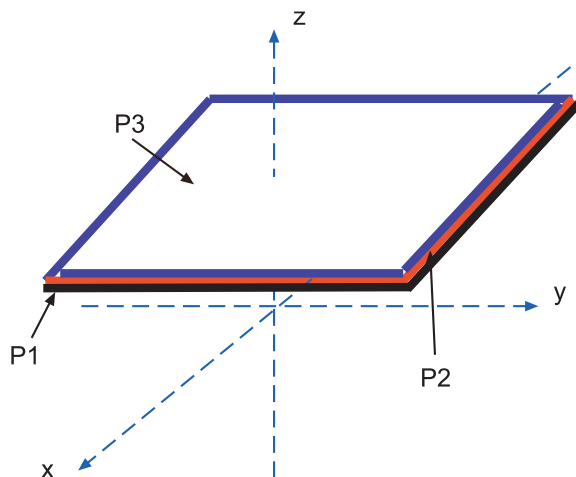
Any point on any plane represents a solution to the system.

One of the examples showed a system where two planes intersect in a line. Looking at the matrix, it reduces to leave one zero row. If the matrix reduces to leave two zero rows then the system of equations all represent the same plane.

A general rule is that if there is a zero on the leading diagonal after reduction to upper triangular form then there is no unique solution to the system of equations.

In the next topic on matrices the reasons for this will be explained in more detail.

For completeness, the following diagram shows a representation of three planes which intersect at a point. That is, the solution is unique.
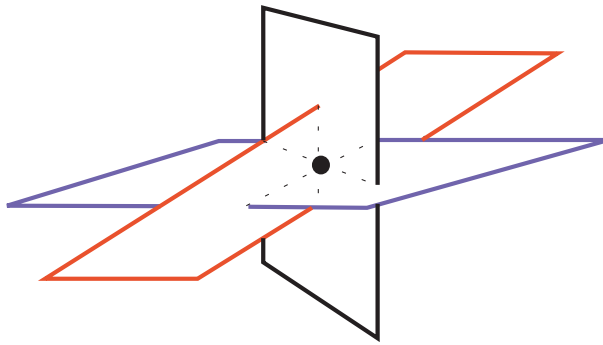
Q20: Investigate any solutions for the system of equations shown and interpret them geometrically.

3x - y + z $\quad$ = 1

-x + 2y + 3z $\quad$ = 0

3x - y + z $\quad$ = 3

Q21: Investigate any solutions for the system of equations shown and interpret them geometrically.

-3x + 4y - 6z $\quad$ = -2

x - y + 2z $\quad$ = 1

2x - 2y + 4z $\quad$ = 2

5.9 Ill-conditioning

Learning Objective

Identify when illconditioning may occur

Ill-conditioned

A system of equations is ill-conditioned when a small change in the right hand side of one of the equations causes a large change in the solution.

Consider the following system of equations

x + 0.99y $\quad$ = 1.99

0.99x + 0.98y $\quad$ = 1.97

This leads to the matrix

$$\begin{pmatrix} 1 & 0.99 & \vline & 1.99 \\ 0.99 & 0.98 & \vline & 1.97 \end{pmatrix} \begin{matrix} r1 \\ r2 \end{matrix} \quad \xrightarrow{\begin{matrix} r1 \\ r2 - 0.99r1 \end{matrix}} \quad \begin{pmatrix} 1 & 0.99 & \vline & 1.99 \\ 0 & -0.0001 & \vline & -0.0001 \end{pmatrix} \begin{matrix} r1 \\ r2 \end{matrix}$$

The solution is x = 1 and y = 1

Consider what happens with a small change to the original equations so that

x + 0.99y $\quad$ = 2.00

0.99x + 0.98y $\quad$ = 1.97

The matrix then becomes

$$\begin{pmatrix} 1 & 0.99 \\ 0.99 & 0.98 \end{pmatrix} \begin{matrix} 2.00 \\ 1.97 \end{matrix} \begin{matrix} r1 \\ r2 \end{matrix} \xrightarrow[\;r2 - 0.99r1\;]{r1} \begin{pmatrix} 1 & 0.99 \\ 0 & -0.0001 \end{pmatrix} \begin{matrix} 2 \\ -0.01 \end{matrix} \begin{matrix} r1 \\ r2 \end{matrix}$$

This leads to the solution x = -97 and y = 100

Such a small change as 0.01 in the equation value led to a huge change in the values of both unknowns.

Geometrically this occurs when the lines given by the two equations meet at a very small angle. A small change to the right hand side of one of the equations replaces one line by a parallel line very close by but gives an intersection that is considerably further away.

large change in intersection point

small change in parallel lines
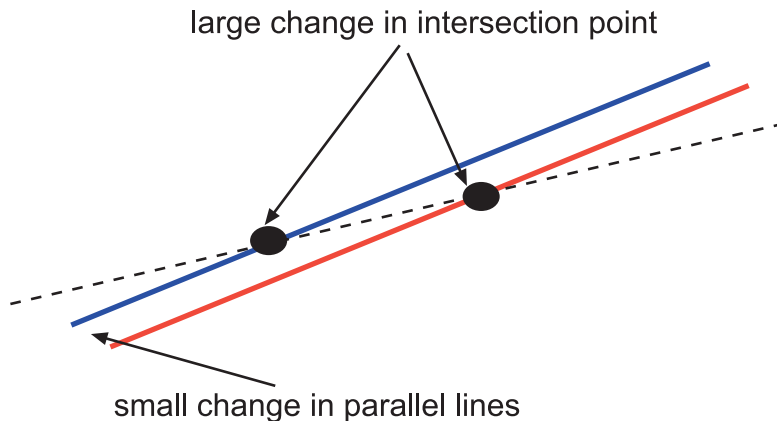
In the example there was a significant change in both unknowns because the gradient of the two lines represented by the equations was very close to 1 (a line with gradient exactly equal to 1 has the equation y = x + constant).

When the equations represent lines with a very large gradient (positive or negative), a small change in one of the ill-conditioned equations will give a large change in the y unknown only (these are lines which are almost vertical).

Q22: Compare and comment on the solutions to the two systems of equations shown

500x + y   = 497
1001x + 2y   = 995

500x + y   = 497
1001x + 2y   = 1000

Calculator activity - large gradients

Using a graphics calculator:

10 min

- Plot two equations with very large similar gradients and y-intercepts very close.

- Find the point of intersection of the two lines.

- Make a small change in the intercept value of one of the equations only.

- Plot these two graphs.

- Find the intersection again and compare with the previous result.

Conversely, when the equations represent lines with a very small gradient (positive or negative), a small change in one of the ill-conditioned equations will give a large change in the x unknown only (these are lines which are almost horizontal).

Q23: Compare and comment on the solutions to the two systems of equations shown

$x + 333y$ $= 343$
$3x + 1000y$ $= 1030$

$x + 333y$ $= 343$
$3x + 1000y$ $= 1034$

Calculator activity - small gradients

Using a graphics calculator:

- Plot two equations with very small similar gradients and y-intercepts very close.

- Find the point of intersection of the two lines.

- Make a small change in the intercept value of one of the equations only.

- Plot these two graphs.

- Find the intersection again and compare with the previous result.

This demonstrates the need to be aware of the dramatic effects that a small change in accuracy can have. In reality this could be extremely important for the correct functioning of equipment and machinery.

5.10 Summary

The following ideas have been covered in this topic:

- Systems of equations can be converted into matrix equations.

- Matrices can be used to solve systems of linear equations by means of elementary row operations.

- There are three types of elementary row operation:

  i) Interchanging two rows.

  ii) Multiplying one row by a constant.

  iii) Changing one row by adding a constant multiple of another.

- With these operations a square matrix can be put into upper triangular form.

- Gaussian elimination (or a form of it) can be used with upper triangular form to solve systems of equations.

- Solutions are not necessarily unique or may not exist.

- The accuracy of equations solutions is important and in some cases ill-conditioning can exist.

5.11 Extended information

Learning Objective

Demonstrate an awareness of the additional material available on this subject

SYLVESTER

Although the idea of matrices had been used for a long time it was James Joseph Sylvester (1814-1897), an Englishman, who first used the term 'matrix' to describe a rectangular array of numbers.

CAYLEY

Arthur Cayley, another English mathematician, identified and developed the algebraic properties of matrices. His work produced the basis of modern matrix techniques and is considered a joint founder of matrix theory along with Sylvester.

GAUSS

Carl Friedrich Gauss developed a means (called Gaussian elimination) of solving systems of linear equations which was named after him. He was a brilliant number theorist.

JORDAN

Camille Jordan and Gauss developed an extended method of solving a system of linear equations. Today this is called Gauss-Jordan elimination.

5.12   Review exercise

Review exercise

There is a randomised web exercise if you prefer it.

15 min

Q24: Use Gaussian elimination to find the solution of this system of equations

$2x + z = 16$
$4x - y + 2z = 32$
$3x + y + z = 25$

Q25: Use Gaussian elimination to find the solution of this system of equations

$2x + y - 4z = -6$
$x + 2y - z = -7$
$x + 3y + 2z = -5$

Q26: Use Gaussian elimination to find the solution of this system of equations

$2x - y + z = 3$
$3x + y + 4z = 2$
$4x - 5y + 2z = 0$

Q27: Use Gaussian elimination to find the solution of this system of equations

$4x - y + z = 1$
$-x + 2y + 3z = 0$
$3x - y + z = 0$

5.13   Advanced review exercise

Advanced review exercise

There is an advanced exercise on Gaussian elimination on the web if you prefer it.

20 min

Q28: Investigate the solutions for the system of equations

$-3x - y + z = 1$
$2x - 3y = 2$
$4x + 5y - 2z = -4$

Q29: Investigate the solutions for the system of equations

$4x + y + 5z = 2$
$-x + y - 2z = 7$
$3x - 3y + 6z = 21$

Q30: Show that the equations are inconsistent for $a = 15$

$x - 2y + z = 5$
$2x - 2y + 5z = 14$
$x + 2y + 7z = a$

Find the value of a for which the equations represent three planes which intersect at a line. Give the solutions to the system of equations for this value of a.

5.14 Set review exercise

Set review exercise

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers.

10 min

Q31: Solve the system of equations by Gaussian elimination.

-3x - 5y - 4z = -12

3x + 6y - z = -10

2x + 8y - 5z = -32

Q32: Solve the system of equations by Gaussian elimination.

7x - 2y - 5z = -1

2x + 7y - 5z = -15

3x + 4y - 2z = -5

Q33: Solve the system of equations by Gaussian elimination.

4x - 3y - z = 19

3x + 9y - 2z = -22

5x + 8y - z = -11

Topic 6

End of Unit Assessments and Extra Help

Contents

6.1   End of Unit Assessments

The first two tests are set to provide questions at level 'C' competency and cover work from the five topics for unit 1.

The third test gives questions at level 'A' or 'B'. This test does not cover all the topics in unit 1 as the questions for the topic on Algebra include techniques and material from unit 2. These questions will be included in the end of unit 2 tests.

6.2   Extra Help

If you feel that you would benefit from some extra help in differentiation or integration you may wish to try the following activities.

Extra Help: Differentiation

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

Extra Help: Integration

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

Glossary

anti-differentiation

> **Anti-differentiation** is the reverse process of differentiation.

asymptotes of rational functions

> For rational functions, a vertical asymptote is a vertical line with equation of the form x = k at which the function in question is undefined.

> The function values either increase rapidly towards + ∞ or decrease rapidly towards - ∞ as x gets closer and closer to the value k

> For rational functions, a horizontal asymptote is a horizontal line with equation of the form y = m for which the function value gets closer and closer to the value m as x tends towards + ∞ and / or - ∞

> A horizontal asymptote occurs when the degree of the numerator is less than or equal to the degree of the denominator.

> For rational functions, a slant or oblique asymptote is a line, neither horizontal nor vertical, with equation of the form y = ax + b for which the function value gets closer and closer to the line y = ax + b as x tends towards +∞ and / or towards - ∞

> A slant asymptote occurs when the degee of the numerator is exactly one greater than the degree of the denominator.

binomial coefficient formula 1

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

binomial coefficient formula 2

$$\binom{n}{r} = \binom{n}{n-r}$$

binomial coefficient formula 3

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

binomial theorem

> The Binomial Theorem states that if x, y $\in \mathbb{R}$ and n $\in \mathbb{N}$ then

> $(x + y)^n$

$$= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + ... + \binom{n}{r}x^{n-r}y^r + ... + \binom{n}{n}y^n$$

the closed interval [a, b]

> Note that the notation $[a,b] = \{x \in \mathbb{R} : a \leqslant x \leqslant b\}$ means **the closed interval** with end points a and b.

codomain

> For a function f : A $\rightarrow$ B, B is called the codomain of the function f.

concave downward

The graph of y = f (x) is concave downward in an interval if f (x) is a function which has a second derivative $\frac{d^2y}{dx^2} < 0$ for all x in the open interval.

concave upward

The graph of y = f (x) is concave upward in an interval if f (x) is a function which has a second derivative $\frac{d^2y}{dx^2} > 0$ for all x in the open interval.

constant of integration

In general if $^d/_{dx}$ (F (x)) = f (x) then F (x) is called an anti-derivative, or integral, of f (x) and we write $\int$ f (x)dx = F(x) + C and C is the **constant of integration**

continuous function

A continuous function f (x) is a function where at every point P on the domain

$\lim\limits_{x \to P}$ f (x) = f (P)

cosecant

The **cosecant** of x = cosec x = $^1/_{\sin x}$

cotangent

The **cotangent** of x = cot x = $^1/_{\tan x}$

critical point

A critical point is any point on a curve where the slope of the tangent to the curve is zero (parallel to the x-axis) or where the slope of the tangent to the curve is undefined (parallel to the y-axis).

definite integral

$\int\limits_a^b$ f (x)dx is a **definite integral** because the limits of integration are known and

$\int\limits_a^b$ f (x)dx = $[F(x)]_a^b$ = F(b) - F(a)

differentiation

The process of finding the derivative f $'$ (x) is called **differentiation**

discontinuous function

A function f (x) is discontinuous at a point P if f (x) is not defined at P or if

$\lim\limits_{x \to P}$ f (x) $\neq$ f (P)

dividend

The dividend in a long division calculation is the expression which is being divided. As a fraction it is the numerator.

divisor

The divisor is the expression which is doing the dividing.

It is the expression outside the division sign. As a fraction it is the denominator.

domain

For a function f : A → B, A is called the domain of the function f.

elementary row operations

The three ways in which a matrix can be manipulated to solve a system of equations are called elementary row operations. They are:

- Interchange two rows.
- Multiply one row by a non zero constant.
- Change one row by adding a multiple of another row.

even function

A function is even if f (-x) = f (x) for every value of x within the domain of the function.

The graph is symmetrical under reflection in the y-axis.

factorial n

formula one

n! = n × (n - 1)!

first derivative test

The first derivative test is a means of determining the nature of the turning point by finding the signs of the derivative to the left and to the right of the turning point.

function f

A function f from set A to set B is a rule which assigns to each element in A exactly one element in B. This is often written as f : A → B.

gaussian elimination

The technique of using an augmented matrix formed from a system of original equations to find the unknowns is called Gaussian elimination.

general indefinite integral of f (x)

For $\int f(x)dx = F(x) + C$, F (x) + C is the **general indefinite integral of f (x)**.

general term of $(x + y)^n$

The general term of $(x + y)^n$ is given by

$$\binom{n}{r} x^{n-r} y^r$$

global maximum and minimum turning points.

A global maximum point occurs when f is defined over a domain A and the value of the function at this point is greater than or equal to that at any other point within the domain.

A global minimum point occurs when f is defined over a domain A and the value of the function at this point is less than or equal to that at any other point within the domain.

ill-conditioned

A system of equations is ill-conditioned when a small change in the right hand side of one of the equations causes a large change in the solution.

image set or range

For a function f : A → B, the set C of elements in B which are images of the elements in A under the function f is called the image set or range of the function f. C is always contained in or equal to B. This is written C ⊆ B

improper rational function

Let P (x) be a polynomial of degree n and Q (x) be a polynomial of degree m.

If $n \geq m$ then $\frac{P(x)}{Q(x)}$ is an improper rational function.

For example $\frac{x^3 + 2x - 1}{x^2 + 4}$ or $\frac{x^2 + 2x - 1}{x^2 + 4}$

integrand

For $\int f(x)dx = F(x) + C$, f (x) is the **integrand**

integration

Integration is the method we use to find anti-derivatives.

inverse function

Suppose that f is a one-to-one and onto function. For each y ∈ B (codomain) there is exactly one element x ∈ A (domain) such that f (x) = y

The inverse function is denoted $f^{-1}(y) = x$

local maximum and minimum turning points.

A local maximum point occurs when a function has a greater value at that point than at any points close to it. It is not necessarily the greatest value of the function.

There can be more than one local maximum turning point.

A local minimum point occurs when a function has a lesser value at that point than at any points close to it. It is not necessarily the least value of the function.

There can be more than one local minimum turning point.

many-to-one

A function which maps more than one element in the domain to the same element in the range or image set is called a many-to-one or a many-one function.

The function is said be in many-to-one correspondence.

It is also common to say that such a function is not one-to-one.

matrix

A matrix is a rectangular array of numbers.

the modulus function

For x ∈ ℝ the modulus function of f (x), denoted by | f (x) | is defined by

$| f(x) | = f(x)$ if $f(x) \geqslant 0$
$\qquad\quad - f(x)$ if $f(x) < 0$

n factorial

n! (called n factorial) is the product of the integers n, n - 1, n - 2, ..., 2, 1

That is, n! = n $\times$ (n - 1) $\times$ (n - 2) $\times$. . . $\times$ 2 $\times$ 1 for n $\in \mathbb{N}$

non horizontal point of inflexion

A non horizontal point of inflexion occurs when $\frac{dy}{dx} \neq 0$ but $\frac{d^2y}{dx^2} = 0$

odd function

A function is odd if f (-x) = -f (x) for every value of x within the domain of the function.

The graph is symmetrical under 180° rotation about the origin.

It is said to have rotational symmetry of order 2

one-to-one

A function f : A $\to$ B is a one-to-one function if whenever f (s) = f (t)

then s = t where s $\in$ A and t $\in$ A.

The function is said to be in one-to-one correspondence.

onto

An onto function is one in which the range is equal to the codomain.

partial fractions

The process of taking a proper rational function and splitting it into separate terms each with a factor of the original denominator as its denominator is called expressing the function in partial fractions.

particular integral

For $\int$ f (x)dx = F(x) + C an anti-derivative given by a particular value of C is a **particular integral**.

polynomial of degree n.

If P (x) = $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + ... + a_2x^2 + a_1x^1 + a_0$

where $a_0, ..., a_n \in \mathbb{R}$

then P is a polynomial of degree n.

product rule

The **product rule** gives us a method to differentiate the product of two or more functions.

It states that when k (x) = f (x) g (x)

then k $'$ (x) = f $'$ (x) g (x) + f (x) g $'$ (x)

proper rational function

Let P (x) be a polynomial of degree n and Q (x) be a polynomial of degree m.

If n < m then $\frac{P(x)}{Q(x)}$ is a proper rational function. For example $\frac{x^2 + 2x - 1}{x^3 - 3x + 4}$

quotient

The quotient is the answer to the division **but not including** the remainder.

quotient rule

The **quotient rule** gives us a method that allows us to differentiate algebraic fractions.

It states that when $k(x) = \frac{f(x)}{g(x)}$

then

$k'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$

rational function

If $P(x)$ and $Q(x)$ are polynomials then $\frac{P(x)}{Q(x)}$ is called a rational function.

rule for graphing y = | f (x) |

To sketch the graph of a modulus function $|f(x)|$, first sketch the graph of the function $y = f(x)$

Take any part of it that lies below the x-axis and reflect it in the x-axis.

The modulus function $y = |f(x)|$ is the combined effect of the positive part of the original function and the new reflected part.

rule for graphing y = f (k x)

To obtain $y = f(kx)$ scale the graph of $y = f(x)$ horizontally by a factor of $\frac{1}{k}$

This type of transformation is called a horizontal scaling.

rule for graphing y = f (x) + k

To obtain the graph of $y = f(x) + k$ take the graph of $y = f(x)$.

- For $k > 0$ slide the graph UP the y axis by k units.
- For $k < 0$ slide the graph DOWN the y axis by k units.

This type of transformation is known as a vertical translation.

rule for graphing y = f (x + k)

To obtain $y = f(x + k)$ take $y = f(x)$

- For $k > 0$ slide the graph to the *left* by k units.
- For $k < 0$ slide the graph to the *right* by k units.

This type of transformation is called a horizontal translation or in trigonometric terms it is also called a phase shift.

rule for graphing y = k f (x)

To obtain the graph of $y = k f(x)$ scale the graph of $y = f(x)$ vertically by a factor of k.

This type of transformation is known as a vertical scaling.

secant

The **secant** of $x = \sec x = \frac{1}{\cos x}$

second derivative test

> The second derivative test of a function is a means of determining the nature of a turning point. A positive second derivative gives a minimum turning point and a negative second derivative gives a maximum turning point.

square matrix

> A square matrix has the same number of rows and columns.

standard number sets

> The standard number sets are:
>
> - $\mathbb{N}$ = {1, 2, 3, 4, 5, ...} the set of natural numbers.
> - $\mathbb{W}$ = {0, 1, 2, 3, 4, 5, ...} the set of whole numbers.
> - $\mathbb{Z}$ = {..., -3, -2, -1, 0, 1, 2, 3, ...} the set of integers.
> - $\mathbb{Q}$ = the set of all numbers which can be written as fractions, called the set of rational numbers.
> - $\mathbb{R}$ = the set of rational and irrational numbers, called the set of real numbers.

stationary point

> A stationary point is any point on a curve where the slope of the tangent to the curve is zero (parallel to the x-axis).
>
> The term stationary point can refer to:
>
> - A maximum turning point.
> - A minimum turning point.
> - A horizontal point of inflection.

strategy for sketching curves

> The **strategy for sketching curves** follows the following steps:
>
> - Identify whether the function is odd or even.
> - Identify crossing of x-axis.
> - Identify crossing of y-axis.
> - Look for turning points and points of inflection and their nature.
> - Consider the behaviour of the curve as $x \to \pm \infty$.
> - Check the behaviour where y is undefined i.e. is there a discontinuity?
> - Check, if necessary, the behaviour of x or y further as the asymptotes are approached.

upper triangular matrix

> A square matrix is upper triangular if all the entries below the main diagonal are zero.

zero factorial

> $0! = 1$

Hints for activities

Topic 2: Differentiation

The Sky's The Limit

Hint 1:

Calculate the limit for $\frac{1}{x}$ as $x \to \infty$ and so calculate $\lim_{x \to \infty} v(x)^2$
and then $\lim_{x \to \infty} v(x)$

Hint 2:

As $x \to \infty$ then $\frac{1}{x} \to 0$ and so $\lim_{x \to \infty} v(x)^2 = \frac{2k}{a}$
therefore $\lim_{x \to \infty} v(x) = \sqrt{\frac{2k}{a}}$

Answers to questions and activities

1 Algebra

Revision exercise (page 3)

Q1: $2x^2 - xy - 6y^2$

When expanding brackets remember that each term in the first bracket must multiply each term in the second bracket.

So 2x multiplies x to give $2x^2$

2x also multiplies -2y to give -4xy

3y multiplies x to give 3xy

3y multiplies -2y to give $-6y^2$

Q2: $4x^3 + 12x^2y - 15xy^2 + 4y^3$

Q3: $(x - 2)(x + 2)(2x^2 + 3)$

Try simple values of x that will make $2x^4 - 5x^2 - 12$ equal to zero. Then if x = n for some value n it follows that one factor is (x - n). For each value found there will be one factor. Note that not all polynomials factorise into linear factors.

Q4: $^{59}/_{40}$

When adding or subtracting fractions the denominator must be the same for each term.

The first step is therefore to find a common denominator and calculate new numerators for each term.

Finding a common denominator is the same as finding the lowest common multiple of the numbers. In this question the lowest common multiple of 8 and 5 is 40. The first term then becomes $^{35}/_{40}$ and the second term is $^{24}/_{40}$. The answer is then a simple addition of the numerators.

Q5: $^{9}/_{56}$

Use cancellation when possible before multiplying to make the calculation simpler.

Q6: 326

Q7: 334

Answers from page 4.

Q8: $6 (3 \times 2 \times 1)$

Q9: $24 (4 \times 3 \times 2 \times 1)$

Answers from page 5.

Q10: $8 \times 7!$

Q11: $12 \times 11!$

Answers from page 6.

Q12: 10 (5 × 4 ÷ 2)

Q13: 15 (6 × 5 ÷ 2)

Q14: 10 (5 × 4 ÷ 2)

Q15: $^7C_4$ = 35 ways

Answers from page 6.

Q16: $\begin{pmatrix} 7 \\ 3 \end{pmatrix}$

Q17: $\begin{pmatrix} 21 \\ 4 \end{pmatrix}$

Answers from page 7.

Q18: $\begin{pmatrix} 9 \\ 7 \end{pmatrix}$

Q19: $\begin{pmatrix} 15 \\ 12 \end{pmatrix}$

Answers from page 7.

Q20: 5

Q21: 7

Q22: 9

Answers from page 7.

Q23: 120

Q24: 1

Q25: 5040

Q26: 7 × 6!

Q27: 100 × 99!

Q28: 1 × 0!

Q29: 70

Q30: 15

Q31: 126

Q32: 35

Q33: $\binom{12}{7}$

Q34: $\binom{13}{1}$

Q35: $\binom{6}{2}$

Q36: $\binom{4}{3}$

Q37: $\binom{8}{5}$

Q38: $\binom{13}{10}$

Binomial coefficients interactive exercise (page 9)

Q39:

```
row 0                              1
row 1                         1         1
row 2                    1         2         1
row 3               1         3         3         1
row 4          1         4         6         4         1
row 5      1         5        10        10         5         1
row 6    1        6        15        20        15         6         1
row 7  1      7       21        35        35        21         7         1
```

Q40:

<div align="center">binomial coefficients</div>

row 4 $\quad\binom{4}{0}\quad\binom{4}{1}\quad\binom{4}{2}\quad\binom{4}{3}\quad\binom{4}{4}$

row 5 $\quad\binom{5}{0}\quad\binom{5}{1}\quad\binom{5}{2}\quad\binom{5}{3}\quad\binom{5}{4}\quad\binom{5}{5}$

row 6 $\quad\binom{6}{0}\quad\binom{6}{1}\quad\binom{6}{2}\quad\binom{6}{3}\quad\binom{6}{4}\quad\binom{6}{5}\quad\binom{6}{6}$

row 7 $\quad\binom{7}{0}\quad\binom{7}{1}\quad\binom{7}{2}\quad\binom{7}{3}\quad\binom{7}{4}\quad\binom{7}{5}\quad\binom{7}{6}\quad\binom{7}{7}$

			values of coefficients					
row 4		1	4	6	4	1		
row 5	1	5	10	10	5	1		
row 6	1	6	15	20	15	6	1	
row 7	1	7	21	35	35	21	7	1

Answers from page 11.

Q41: $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

Binomial expansion exercise (page 12)

Q42: $x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$

Q43:
$x^9 + 9x^8y + 36x^7y^2 + 84x^6y^3 + 126x^5y^4 + 126x^4y^5 + 84x^3y^6 + 36x^2y^7 + 9xy^8 + y^9$

Q44: $x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Q45: $x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7$

Q46:
$x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$

Further binomial expansion exercise (page 13)

Q47: $81x^4 - 108x^3y + 54x^2y^2 - 12xy^3 + y^4$

Q48: $-x^5 - 5x^4y - 10x^3y^2 - 10x^2y^3 - 5xy^4 - y^5$

Q49: $x^2 - 4xy + 4y^2$

Harder binomial expansion exercise (page 13)

Q50: $\frac{27}{64}x^3 + \frac{27}{16}x^2y + \frac{9}{4}xy^2 + y^3$

Q51: $x^2 - \frac{1}{2}xy + \frac{1}{16}y^2$

Q52: $x^4 + \frac{4}{3}x^3y + \frac{2}{3}x^2y^2 + \frac{4}{27}xy^3 + \frac{1}{81}y^4$

Mixed binomial expansion exercise (page 13)

Q53: $8x^3 - 12x^2y + 6xy^2 - y^3$

Q54: $x^4 - 12x^3y + 54x^2y^2 - 108xy^3 + 81y^4$

Q55: $x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$

Q56: $\frac{1}{8}x^3 - \frac{3}{4}x^2y + \frac{3}{2}xy^2 - y^3$

Q57: $x^4 + \frac{8}{3}x^3y + \frac{8}{3}x^2y^2 + \frac{32}{27}xy^3 + \frac{16}{81}y^4$

Q58: $\frac{1}{64}x^3 + \frac{1}{16}x^2y + \frac{1}{12}xy^2 + \frac{1}{27}y^3$

Finding coefficients exercise (page 15)

Q59: 240

Q60: 108

Q61: 5

Q62: -35

Q63: 24

Q64: 56

Q65: 1

Binomial applications exercise (page 17)

Q66: 8.365427

Q67: 0.1296

Partial fractions type 1a exercise (page 21)

Q68: $\frac{-1}{x+5} + \frac{3}{x-3}$

Q69: $\frac{-2}{-x+3} + \frac{5}{x+2}$

Q70: $\frac{2}{x+1} + \frac{-3}{x-1}$

Q71: $\frac{-1}{x+4} + \frac{-3}{x-2}$

Q72: $\frac{2}{x-3} + \frac{3}{x-2}$

Partial fraction type 1b exercise (page 22)

Q73: $\frac{3}{x+1} + \frac{-2}{(x+1)^2}$

Q74: $\frac{2}{x-2} + \frac{-3}{(x-2)^2}$

Q75: $\frac{1}{x-1} + \frac{-2}{(x-1)^2}$

Q76: $\frac{3}{x+2} + \frac{-1}{(x+2)^2}$

Q77: $\frac{2}{x+4} + \frac{3}{(x+4)^2}$

Partial fractions type 2a exercise (page 23)

Q78: $\frac{-9}{x-1} + \frac{13}{x-2} + \frac{1}{x+1}$

Q79: $\frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x-3}$

Q80: $\frac{1}{x+3} + \frac{1}{x-2} + \frac{-2}{x-3}$

Q81: $\frac{-1}{3(x+1)} + \frac{-5}{2(x+2)} + \frac{29}{6(x+4)}$

Partial fractions type 2b exercise (page 24)

Q82: $\frac{2}{x-1} + \frac{3}{x+2} + \frac{-5}{(x+2)^2}$

Q83: $\frac{-1}{x+1} + \frac{1}{x-2} + \frac{-1}{(x-2)^2}$

Q84: $\frac{4}{x-2} + \frac{2}{x+1} + \frac{-1}{(x+1)^2}$

Q85: $\frac{3}{x-1} + \frac{-2}{x+2} + \frac{1}{(x+2)^2}$

Partial fractions type 2c exercise (page 25)

Q86: $\frac{1}{x-2} + \frac{9}{(x-2)^2} + \frac{11}{(x-2)^3}$

Q87: $\frac{1}{x+1} + \frac{-4}{(x+1)^2} + \frac{6}{(x+1)^3}$

Q88: $\frac{2}{x+2} + \frac{-1}{(x+2)^2} + \frac{2}{(x+2)^3}$

Q89: $\frac{1}{x-1} + \frac{1}{(x-1)^2} + \frac{2}{(x-1)^3}$

Partial fractions type 2d exercise (page 26)

Q90: $\frac{2}{x+1} + \frac{3x-1}{x^2-3x+1}$

Q91: $\frac{2}{x-2} + \frac{-2x+1}{x^2-2x+3}$

Q92: $\frac{4}{x-2} + \frac{x+3}{x^2+2x+5}$

Q93: $\frac{2}{x-2} + \frac{-3x+1}{2x^2+x+1}$

Answers from page 27.

Q94: $x - 3 - \frac{1}{x+1} + \frac{8}{x+2}$

Q95: $x + \frac{1}{x+1} + \frac{2}{x+2}$

Partial fractions mixed exercise (page 27)

Q96: $\frac{3}{x+3} + \frac{4x+1}{2x^2+x+2}$

Q97: $\frac{-3}{-x-2} + \frac{4}{x+6}$

Q98: $\frac{2}{x+1} + \frac{2}{x-1} - \frac{1}{x-2}$

Q99: $\frac{3}{x+2} + \frac{-2}{x+1} + \frac{1}{(x+1)^2}$

Q100: $\frac{4}{x-1} + \frac{-2}{(x-1)^2}$

Q101: $\frac{1}{x-2} + \frac{1}{(x-2)^2} + \frac{-5}{(x-2)^3}$

Q102: $2x + 1 + \frac{1}{x-1} - \frac{3}{x-2}$

Answers from page 30.

Q103: $3x^3 - 16$ remainder 73

Q104: $x^3 - 2x^2 + 2x - 4$ remainder $9x - 5$

Q105: $x^2 - 4$ remainder $-2x + 21$

Q106: 768

Q107: $x + 5$ remainder $10x + 7$

Review exercise (page 35)

Q108: 35

Q109: $x^4 - 8x^3y + 24x^2y^2 - 32xy^3 + 16y^4$

Q110: 216

Q111: 108

Q112: $\frac{3}{x-1} + \frac{2}{x-2}$

Q113: $\frac{-1}{x+3} - \frac{5}{x+1}$

Q114: $\frac{3}{x-1} - \frac{2}{x+2} - \frac{3}{(x+2)^2}$

Q115: $\frac{2}{x-2} + \frac{3}{(x-2)^2} + \frac{4}{(x-2)^3}$

Advanced review exercise (page 36)

Q116: $x^6 - 12x^4 + 60x^2 - 160 + 240x^{-2} - 192x^{-4} + 64x^{-6}$

or $x^6 - 12x^4 + 60x^2 - 160 + \frac{240}{x^2} - \frac{192}{x^4} + \frac{64}{x^6}$

Q117: 2160

Q118: 4032x

Q119: $4x + \frac{4}{x-2} - \frac{3}{(x-2)^2}$

Q120: $2x^2 + 4x + 6 - \frac{3}{x-1} - \frac{2}{x-2}$

Q121: $-3x + 5 - \frac{1}{x+1} - \frac{2}{x^2 - 3x + 5}$

Set review exercise (page 36)

Q122: The answer to this question is available on the web.

Q123: The answer to this question is available on the web.

Q124: The answer to this question is available on the web.

Q125: The answer to this question is available on the web.

2 Differentiation

Revision Exercise 1 (page 42)

Q1:

a)   f' (t) $= -12t^{-5} + 2$

b)   f' (x) $= 10x - 3 \sin x$

c)   f' (x) $= -\dfrac{5}{6\sqrt{x^3}}$

d)   f' (w) $= \dfrac{3}{2}\sqrt{w} + 1$

e)   f' (u) $= 2u - \dfrac{14}{u^3}$

f)   f' (x) $= 20(4x + 3)^4$

g)   f' $(\theta) = 5 \cos\left(5\theta + \dfrac{\pi}{4}\right)$

Q2: $y + 3x = \pi/2$

Q3:
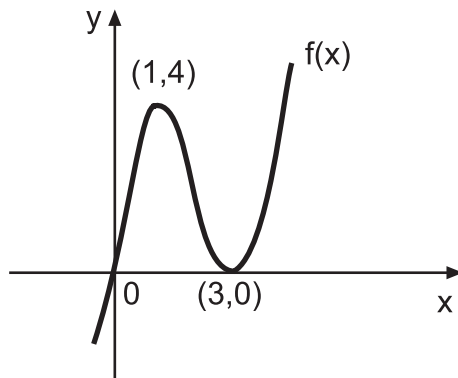
Exercise 2 (page 44)

Q4:

f (x) $= x^3$

f (x + h) $= (x + h)^3$

$$f'(x) = \lim_{h \to o} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$
$$= \lim_{h \to o} \frac{h(3x^2 + 3xh + h^2)}{h}$$
$$= 3x^2$$

Q5:

f (x) $= 3x^2$

f (x + h) $= 3(x + h)^2$

$$f'(x) = \lim_{h \to o} \frac{3x^2 + 6xh + 3h^2 - 3x^2}{h}$$

$$= \lim_{h \to o} \frac{h(6x + 3h)}{h}$$

$$= 6x$$

Q6:

$f(x) = \sqrt{x}$

$f(x + h) = \sqrt{x + h}$

$$f'(x) = \lim_{h \to o} \frac{\sqrt{x + h} - \sqrt{x}}{h}$$

$$= \lim_{h \to o} \frac{\sqrt{x + h} - \sqrt{x}}{h} \times \frac{\sqrt{x + h} + \sqrt{x}}{\sqrt{x + h} + \sqrt{x}}$$

$$= \lim_{h \to o} \frac{x + h - x}{h(\sqrt{x + h} + \sqrt{x})}$$

$$= \frac{1}{2\sqrt{x}}$$

Exercise 3 (page 47)

Q7:   a)

b) $x = 0$ and $x = 5$

Q8:   a) $\{x : x \in \mathbb{R}, x \geqslant -4\}$

b)
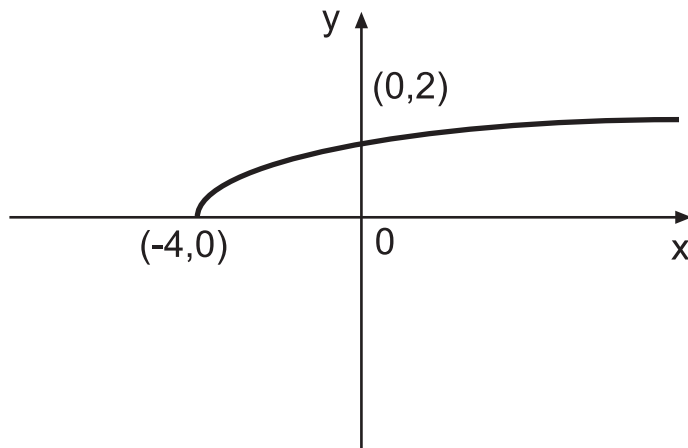
c) x = - 4

Exercise 4 (page 49)

Q9:

a) $\dfrac{dy}{dx} = 12x + 4$ $\qquad$ $\dfrac{d^2y}{dx^2} = 12$

b) $\dfrac{dy}{dx} = 8(2x + 9)^3$ $\qquad$ $\dfrac{d^2y}{dx^2} = 48(2x + 9)^2$

c) $\dfrac{dy}{dx} = -\,4\sin\left(4x + \dfrac{\pi}{4}\right)$ $\qquad$ $\dfrac{d^2y}{dx^2} = -\,16\cos\left(4x + \dfrac{\pi}{4}\right)$

Q10: x = 3

Q11: Proof, using

$\dfrac{dy}{dx} = 8x - 3$ $\quad$ and $\quad$ $\dfrac{d^2y}{dx^2} = 8$

Exercise 5 (page 50)

Q12:

a) a = 6t

b) 30 m/s^2

Q13:

a) 1.62 m/s

b) 1.57 seconds

c) -2.5 m/s^2

Q14:

Speed = 114 m/s, acceleration = 24 m/s^2

Q15:

a) $v = 3 \cos \left(3t - \dfrac{\pi}{6} \right)$ and at $t = 0, v = 2.598$

b) $a = -9 \sin \left(3t - \dfrac{\pi}{6} \right)$

c) $t = \dfrac{5}{9}\pi, \dfrac{11}{9}\pi$ and $\dfrac{17}{9}\pi$

Exercise 6 (page 51)

Q16:

$f'(x) = 12x^3 + 10x + 2$

$f''(x) = 36x^2 + 10$

$f'''(x) = 72x$

$f^{(4)}(x) = 72$

$f^{(n)}(x) = 0$ for $n \geqslant 5, n \in N$

Q17:

$\dfrac{dy}{dx} = 6\cos(2x)$

$\dfrac{d^2y}{dx^2} = -12\sin(2x)$

$\dfrac{d^3y}{dx^3} = -24\cos(2x)$

$\dfrac{d^4y}{dx^4} = 48\sin(2x)$

$\dfrac{d^{10}y}{dx^{10}} = -3 \times 2^{10}\sin(2x)$

Exercise 7 (page 52)

Q18:

a)
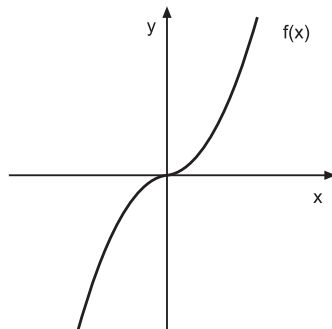
b) f'(x) = $\begin{cases} -2x & \text{for} \quad x < 0 \\ 2x & \text{for} \quad x \geqslant 0 \end{cases}$
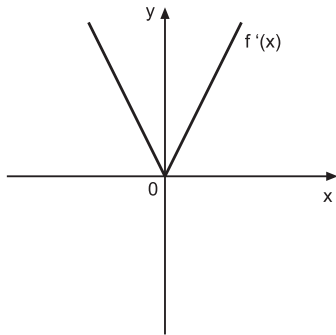
c) f ' (x) is not differentiable at x = 0

Q19:

a) f ' (x) = 5 (x + 2)$^{2/3}$ and ' ' (x) = 10 ÷ 3 (x + 2)$^{1/3}$

b) x = -2

Exercise 8 (page 55)

Q20: $^{dy}/_{dx}$ = 3x^2(3x + 1)(5x + 1)

Q21: $^{dy}/_{dx}$ = sin x + x cos x

Q22: $^{dy}/_{dx}$ = 2x(2x + 3)^5(8x + 3)

Q23: $^{dy}/_{dx}$ = 2x [cos (2x + 1) - x sin (2x + 1)]

Q24: $^{dy}/_{dx}$ = 2x (x^3 + 9)(4x^3 - 9x + 9)

Q25: $\frac{dy}{dx} = \frac{2x^2 + 4}{\sqrt{x^2 + 4}}$

Q26: $^{dy}/_{dx}$ = 3 cos 3x cos x - sin 3x sin x

Q27: $^{dy}/_{dx}$ = 3x^2 sin x cos 2x + x^3 cos x cos 2x - 2x^3 sin x sin 2x

Q28: y = x

Q29: Minima at (0, 0) and (3, 0)

Maximum at (1, 16)

Exercise 9 (page 58)

Q30: $\frac{dy}{dx} = \frac{6}{(x+2)^2}$

Q31: $\frac{dy}{dx} = \frac{7}{(3x+2)^2}$

Q32: $\frac{dy}{dx} = \frac{3x^2+2x+2}{(x^2-x+1)^2}$

Q33: $\frac{dy}{dx} = \frac{2(x\cos 2x - \sin 2x)}{x^3}$

Q34: $\frac{dy}{dx} = \frac{2x-3}{(2x-1)^{3/2}}$

Q35: $\frac{dy}{dx} = \frac{-\cos^2 x}{x^4}\left(3\sin x + 2x\cos x\right)$

Q36: $\frac{dy}{dx} = \frac{4x(\cos x - x\sin x)}{\cos^3 x}$

Q37: Proof

Q38: Proof

Q39: Proof

Exercise 10 (page 60)

Q40: 2.708333

Q41: 2.718279

Q42: 2.718282

Exercise 11 (page 61)

Q43: $^{dy}/_{dx} = 3e^{3x}$

Q44: $^{dy}/_{dx} = 2x \exp(x^2 + 4)$

Q45: $\frac{dy}{dx} = 6x\exp(x^2) - \frac{10}{\exp(2x)}$

Q46: $\frac{dy}{dx} = 2x^2\exp(2x)[3 + 2x]$

Q47: $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$

Q48: $\frac{dy}{dx} = \frac{xe^x - 3e^x - 6}{x^4}$

Q49: $\frac{dy}{dx} = \frac{-3\cos 4x - 4\sin 4x}{e^{3x}}$

Q50: Gradient = 5

Q51: Minimum at $(-1, -^1/_e)$

Exercise 12 (page 63)

Q52: $^{dy}/_{dx} = ^1/_x$

Q53: $\frac{dy}{dx} = \frac{-10(x + 3)}{x^2 + 6x}$

Q54: $\frac{dy}{dx} = \cot(2x)$

Q55: $\frac{dy}{dx} = \frac{2 + x}{x}$

Q56: $\frac{dy}{dx} = \frac{1}{x}$ - tanx

Q57: $\frac{dy}{dx} = \frac{-3}{x(2x-3)}$

Q58: $\frac{dy}{dx} = \frac{e^x}{x}(x \ln x + 1)$

Q59: $\frac{dy}{dx} = \frac{1 - x \ln x}{xe^x}$

Q60: $\frac{dy}{dx} = \tan x$

Q61: $\frac{dy}{dx} = x^2(3\ln 2x \sin 3x + \sin 3x + 3x \ln 2x \cos 3x)$

Exercise 13 (page 64)

Q62:

a) Maximum 10, minimum -4

b) Maximum 7, minimum 0

c) Maximum 7, minimum -4

Q63:

a) Maximum 9, minimum 0

b) Maximum 9, minimum 0

Q64:

a) Maximum 3, minimum -3

b) Maximum 3, minimum -1

Q65: Maximum f (3) = 36 and Minimum f ($^1/_3$) = -18.52

Q66:

Maximum $\quad f\left(\frac{\pi}{8}\right) = 3$

Minimum $\quad f\left(\frac{\pi}{2}\right) = -2.12$

Q67:

Maximum f (2) = 25 and Minimum f (-4) = -125

Q68:

Maximum f (3) = 3 and Minimum f (1) = 1

Q69:

Maxima at f (-2) and f (0) = -3 and Minimum f (-3) = -5

Q70:

Maximum f (3) = 4 and Minimum f (1) = -2

Q71:

Maximum f (2) = 4 and Minimum f (0) = 0

Exercise 14 (page 67)

Q72:

0.303

Q73:

a) $V = 24 \cos \theta (\sin \theta + 2)$

b) $\theta = 0.37$ radians.

This is a maximum as V"$(0.37) = -77 < 0$

c) 52 800 litres

Q74:

a) $r = 1$ cm

b) $t = 1$ second

c) 0.5 cm/second

d) $3 - r = \dfrac{2}{1 + t}$

i.e. $r = 3 - \dfrac{2}{1 + t}$

So as $t \to \infty, \quad r \to 3$ but does not reach 3 cm.

Q75:

a)   Hint: The angle at C is $\frac{3}{4}\pi - \theta$

Then use the sine rule to find side b

b) $\theta = \frac{3}{8}\pi$

Review exercise in differentiation (page 74)

Q76: f'(x) $= 3x^3(4\ln x + 1)$

Q77: f'(x) $= \dfrac{-5}{(4x + 1)^2}$

Q78: f'(x) $= $ - sin x exp(cos x)

Advanced review exercise in differentiation (page 74)

Q79:

a)   y'(x) $= xe^{-x^3}(2$ - $3x^3)$

b)   g'(x) $= \dfrac{2\cos x + 1}{(2 + \cos x)^2}$

c)   h'(x) $= $ - 2xsin(x^2)sin(3x) + 3cos(x^2)cos(3x)

d)   f'(x) $= \dfrac{1 - \ln(x + 4)}{(x + 4)^2}$

Q80:

f' (x) = $\frac{-6}{(x-2)^2}$ which is always negative.

Q81:

f' (x) = (x - 1)(x + 1)e^x

Since $e^x > 0$ for all $x \in \mathbb{R}$

Then f (x) is decreasing and f' (x) < 0 for $-1 < x < 1$

Q82:

$\frac{3\pi}{4}$

Q83:

$\frac{dy}{dx} = \frac{1}{x^2}$ (- cos(kx) - xksin(kx))

$\frac{d^2y}{dx^2} = \frac{1}{x^4} \left(2x^2ksin(kx) + 2xcos(kx) - x^3k^2cos(kx)\right)$

then substitute into the equation.

Q84:

a) $H = \frac{100}{\pi} - \frac{h}{3}$

b) Base $= 9\pi$

Curved surface of cylinder $= 600 - 2\pi h$

Curved surface of cone $= 3\pi\sqrt{h^2 + 9}$ (since $l^2 = h^2 + 9$)

Proof then follows ...

c) $h = \sqrt{7.2} = 2.7$

$h = 2.7$ is a minimum since $\frac{d^2s}{dh^2} > 0$ at $h = 2.7$

Set Review Exercise in differentiation (page 75)

Q85: This answer is only available on the course web site.

Q86: This answer is only available on the course web site.

Q87: This answer is only available on the course web site.

3 Integration

Revision exercise 1 (page 81)

Q1: $-3x^2 - 2x + {}^3/_4x^4 + C$

Q2: $-\frac{3}{2x^2} - \frac{1}{3}x^3 + C$

Q3: $-\sqrt{(5 - 2t)} + C$

Q4: ${}^1/_3u^3 + 2u - {}^1/_u + C$

Q5: $\frac{3}{2}\sin\left(2\theta + \frac{\pi}{4}\right) + C$

Q6: 10

Q7: $-4{}^5/_8$

Q8: 2

Q9: $-5{}^1/_2$ and $5{}^1/_2$

Q10: $\frac{3\sqrt{3}-1}{3}$

Q11: $6{}^1/_3$ units^2

Exercise 2 (page 85)

Q12: $-\frac{1}{\sqrt{2x - 5}} + C$

Q13: $\frac{1}{2}\ln|2s| + C$ or $\frac{1}{2}\ln|s| + C$

Q14: $-2e^{-4u} + C$

Q15: $2\sin(3x - 7) + C$

Q16: ${}^1/_4\tan(4x + 1) + C$

Q17: $-{}^1/_6\exp(-6t) + C$

Q18: $2\ln|5t + 1| + C$

Q19: $\frac{5}{3}e^{3x+1} + C$

Q20: $e^4 - e^{-2} \approx 54.46$

Q21: $4\ln({}^7/_3) \approx 3.389$

Q22: $-{}^1/_3$

Q23: $3\ln 7 \approx 5.838$

Q24: $\pi/4 - {}^1/_2$

Q25: $e^3 + 2e^{-1} - 2 \approx 18.821$

Exercise 3 (page 86)

Q26: x + ln | x - 1 | + C

Q27: t = 3 ln | t + 2 | + C

Q28: $^1/_2$ - ln | 4 + 2x | + C

Exercise 4 (page 87)

Q29: $\frac{5}{12}(x^2 + 6)^6$ + C

Q30: $\left(1 + x^2\right)^{3/2}$ + C

Q31: $-\frac{1}{4}\cos^4 x$ + C

Q32: -3 cos (x^2 + 3) + C

Q33: 4 ln | x^2 + 1 | + C

Q34: $\frac{1}{3}\exp(x^3 + 5)$ + C

Exercise 5 (page 88)

Q35: $\frac{1}{6}(x + 5)^5(x - 1)$ + C

Q36: $\frac{4}{3}\sqrt{2x - 1}(x + 1)$ + C

Q37: $\frac{2}{63}(3x - 2)^6(9x + 1)$ + C

Exercise 6 (page 89)

Q38: 31031

Q39: 32$^1/_2$

Q40: $^1/_{16}$

Q41: $^6/_5$

Q42: 6$^2/_3$

Q43: e^2 - 1 ≈ 6.389

Q44: $^1/_6$

Q45: ln 2 ≈ 0.693

Exercise 7 (page 91)

Q46: $\pi/2$

Q47: $\pi/12$

Q48: $\pi/24$

Q49: $\frac{\sqrt{3}}{27}\pi$

Q50: $\frac{25}{4}\left(1 + \frac{\pi}{2}\right)$

Q51: $\frac{1}{8}\left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right)$

Q52: π

Q53: $^3/_2$ - $2\pi/3$

Exercise 8 (page 94)

Q54: $\frac{1}{45}\left(3x^3 - 1\right)^5 + C$

Q55: $2\sin(2x - 5) + C$

Q56: $2\exp(x^3 + 5) + C$

Q57: $\frac{1}{3}\ln\left| x^3 - 3 \right| + C$

Q58: $-\frac{3}{2}\sin(1 - x^2) + C$

Q59: $2\sqrt{x^2 - 3x + 1} + C$

Q60: 8

Q61: $\frac{1}{2}\left(e^6 - e^1\right) \approx 200.36$

Q62: 0

Q63: $4^1/_3$

Q64: $\ln\left(\frac{3}{15}\right) \approx -1.609$

Q65: 89.1

Answers from page 98.

Q66: 0. Because the areas above and below the x-axis cancel each other out.

Exercise 9 (page 98)

Q67: 8 units^2

Q68:
a) -5, -1 and 3
b) 128

Q69:
Coordinates of p are (-1, 0)
Shaded area = 7.568

Q70: 8 $\sqrt{2}$

Exercise 10 (page 101)

Q71: 5$^1/_3$ units^2

Q72:
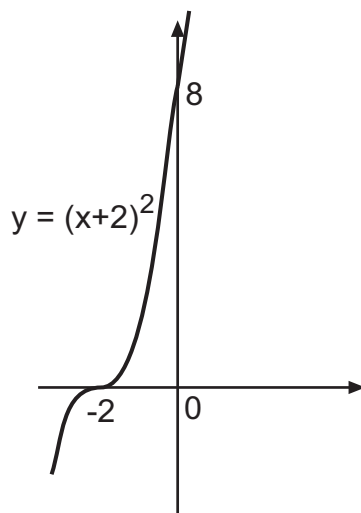
Area = 2$^3/_4$ units^2

Q73: 2 units^2

Q74: 36 units^2

Exercise 11 (page 104)

Q75: $81\frac{1}{15}\pi$ units^3

Q76: $34\frac{2}{15}\pi$ units^3

Q77: $\frac{\pi^2}{2}$

Q78: $26\frac{2}{3}\pi$ units^3

Exercise 12 (page 105)

Q79: $V = \int_a^b \pi x^2 dy$ for $x = f(y)$

Q80: 3π units^3

Q81: $18\frac{3}{5}\pi$ units^3

Exercise 13 (page 107)

Q82:

  a) s (t) = 20t + $^5/_2$ t^2
  b) 120 metres

Q83:

  a) v 9t) = 200t - t^2
  b) h (t) = 100t^2 - $^1/_3$t^3
  c) 666$^2/_3$km

Q84:

  a) The speed increases to a maximum and then falls back down to zero
  b) The particle moves to the left. Its speed increases and then falls back down to zero
  c) 2.25 units to the left
  d) 3$^1/_{12}$ units
  e) 6 m/s^2

Review exercise in integration (page 110)

Q85: $\ln(x^2 + 4) + C$

Q86: $\frac{1}{4}$exp(4x) + C

Q87: - $\frac{1}{5}$cos^{5}x + C

Advanced review exercise in integration (page 111)

Q88: 1

Q89: $^3/_4$

Q90: 3 - π

Set review exercise in integration (page 111)

Q91: This answer is only available on-line.

Q92: This answer is only available on-line.

Q93: This answer is only available on-line.

4 Properties of Functions

Revision exercise (page 115)

Q1: 6x

Q2: $-3x^{-3}$

Q3: $\frac{-2x^2 + 18}{(x + 3)^4}$ using the quotient rule.

Domain and range exercise (page 120)

Q4: $\{x \in \mathbb{R} : x \geq 2\}$ Note that if x was less than two f (x) is the square root of a negative number.

Q5: $\{y \in \mathbb{R} : y \neq 0\}$ The condition $y \neq 0$ is needed since $\frac{1}{0}$ is not defined.

Q6: $\{z \in \mathbb{R} : z \geq 0\}$. Again z cannot be negative or the square root of a negative number is required and this does not belong to the reals.

Q7: $\{ f (x) \in \mathbb{Q} : 0 < f (x) \leq 1\}$

Q8: $\{ g (y) \in \mathbb{N}\}$ or $\{ g (y) \in \mathbb{Z}^+\}$ or $\{ g (y) \in \mathbb{Z} : g (y) > 0\}$

Q9: $\{h (w) \in \mathbb{Z}^+\}$

One-to-one and onto function exercise (page 122)

Q10: yes

Q11: yes

Q12: no (For example y = -2 and y = 2 both give g (y) = -4)

Q13: no (w = 0 and w = 4 both give k (w) = 0)

Q14: yes

Q15: yes

Q16: no The range is {-4, -3, -2, -1, 0}. This does not equal the codomain.

Answers from page 126.

Q17:
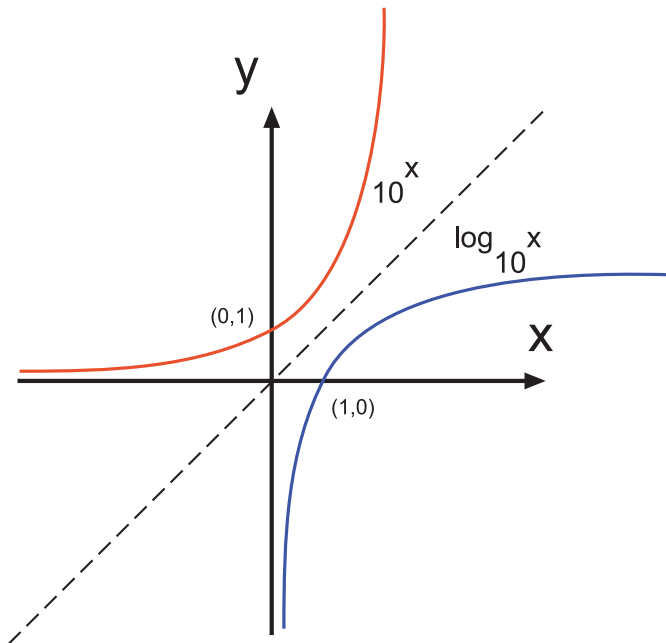
Q18:
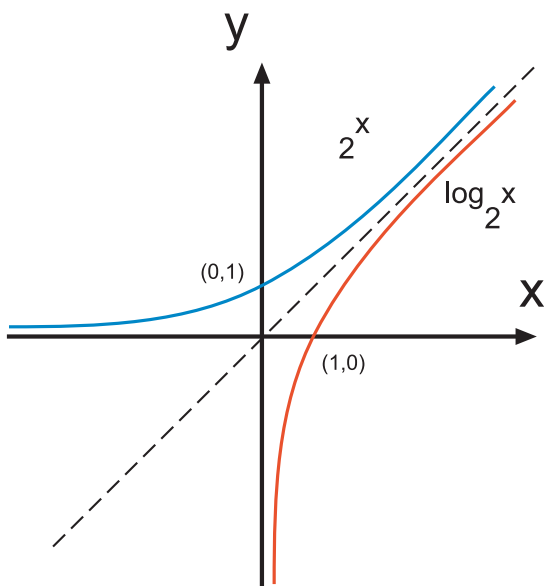
Q19: $f^{-1}(x) = \frac{8-x}{2}$ where $x \in \mathbb{R}$. The drawing below is the screen dump from a graphics calculator showing the line $y = x$, the original function and a bold line denoting the inverse function.
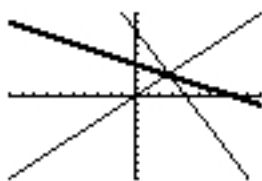
Q20: g $^{-1}$(x) = 2 - $\frac{1}{x}$ where x $\in$ $\mathbb{R}$, x $\neq$ 0. The diagram shows a sketch of this inverse function using a graphics calculator.
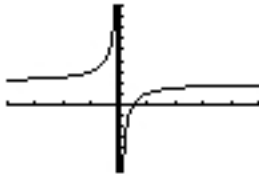
Q21: h^{-1} (x) = $\frac{x+6}{4}$ where x $\in$ $\mathbb{R}$, x > 6. The diagram shows the inverse function sketch from a graphics calculator. The calculator image shows a dotted line of the original function and a solid line for its inverse.
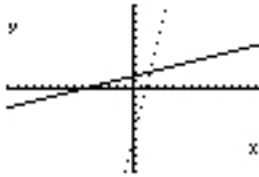
Answers from page 127.

Q22:

Use a horizontal line and there is only one intersection at any point.

The function is one-to-one.

The domain generates a range of [-1,1]

This is equal to the codomain shown.

The function is onto.

A possible restricted domain for cos x is [0, π]

A possible restricted domain for tan x is $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

The domain of sin^{-1} x is [-1, 1]

y = tan $^{-1}$ x
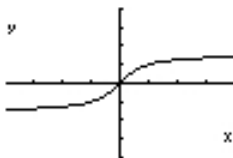
This inverse would be obtained from y = tan x with the domain of $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

Note the horizontal asymptotes at y = 1 and at y = -1
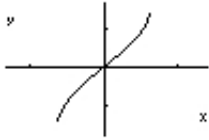
y = cos^{-1} x

This inverse was produced using a domain for y = cos x of [0, π]

Note that it touches the x-axis at x = 1

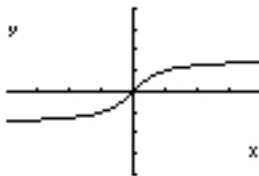**Answers from page 127.**

Q23:

y = sin^{-1} x

a sin x: A general domain would be $\left[\frac{k\pi}{2}, \frac{k\pi}{2} + \pi\right]$ where k is an odd integer.

y = cos^{-1} x

cos bx: A general domain would be $\left[\frac{k\pi}{b}, \frac{(k+1)\pi}{b}\right]$ where k is an integer.

y = tan^{-1} x

tan cx: A general domain would be $\left[\frac{k\pi}{2c}, \frac{k\pi}{2c} + \frac{\pi}{c}\right]$ where k is an odd integer.

Q24:

x	sin^{-1} x	tan^{-1} x	cos^{-1} x
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$	********	$\frac{\pi}{6}$
1	$\frac{\pi}{2}$	$\frac{\pi}{4}$	0
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	********	$\frac{\pi}{4}$
$\sqrt{3}$	********	$\frac{\pi}{3}$	********
$\frac{1}{2}$	$\frac{\pi}{6}$	********	$\frac{\pi}{3}$

Q25:

x	sin^{-1} x	tan^{-1} x	cos^{-1} x
$\frac{\sqrt{3}}{2}$	60°	********	30°
1	90 °	45°	0 °
$\frac{1}{2}$	30°	********	60°
$\sqrt{2}$	45°	********	45°
$\sqrt{3}$	********	60°	********

Q26: (0,1)

Q27: These are inverses of each other.

Answers from page 130.

Q28:

1. f (x) = cos x
    f (-x) = cos (-x) = cos x
    Since f (x) = f (-x) the function is even.

note the reflection in the y-axis
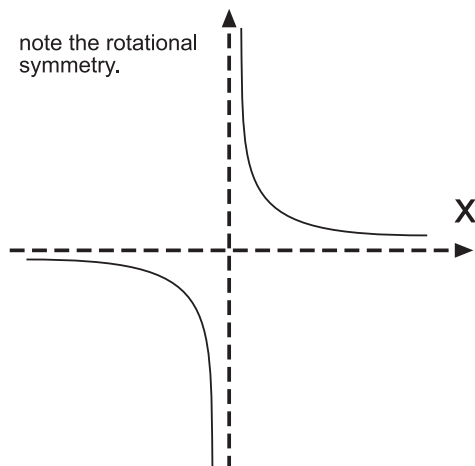
y = cos x
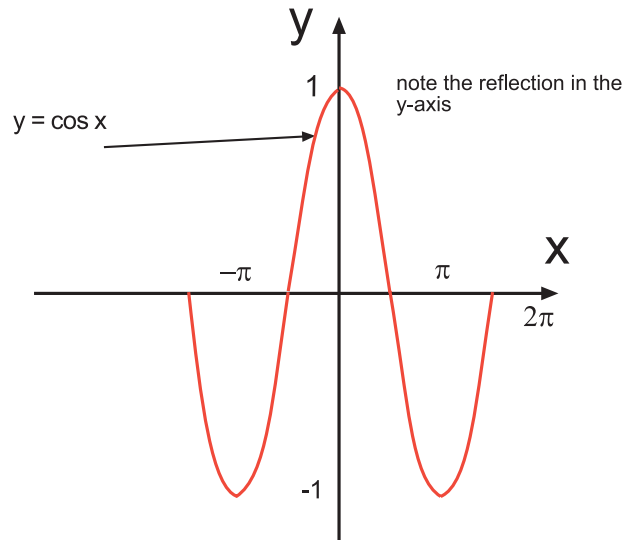
2. g (x) = $\frac{1}{2x-1}$
    g (-x) = $\frac{1}{-2x-1} \neq$ g (x) or - g (x)
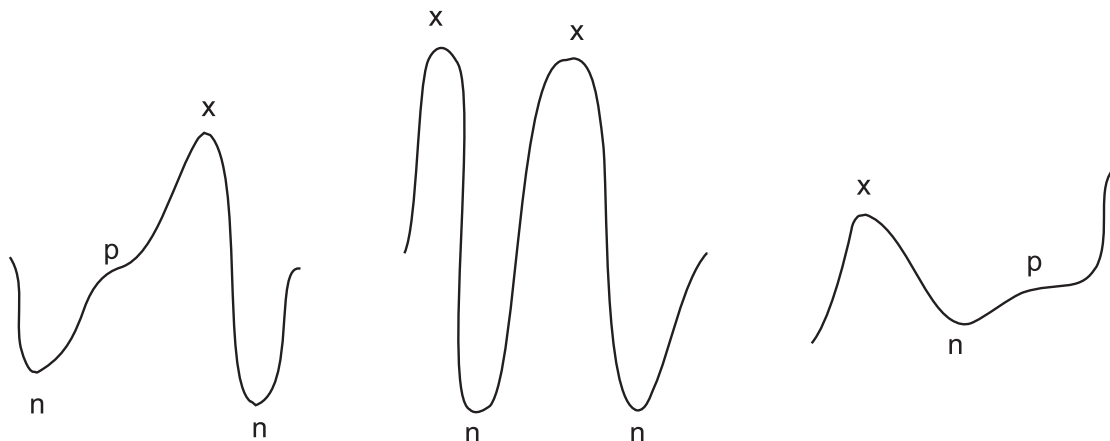   The function is neither odd or even.

3. k (x) = $\frac{1}{x}$
    k (-x) = $\frac{1}{-x}$ = - $\frac{1}{x}$
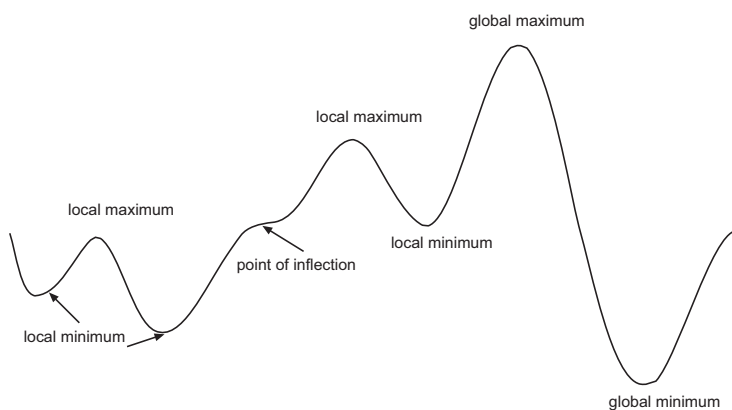   k (x) = -k (x) so the function is odd.

note the rotational symmetry.

Answers from page 131.

Q29:

Local and global maxima and minima exercise (page 132)

Q30:

First derivative test exercise (page 133)

Q31: $(2, {}^{-16}/_3)$ is a minimum and $(-2, {}^{16}/_3)$ is a maximum.

Q32: $(-1, -7)$ is a minimum.

Q33: $(0, 0)$ is a minimum.

Turning points exercise (page 135)

Q34: $(2, {}^{-16}/_3)$ is a minimum turning point.

The second derivative of $2x = 4$ at $x = 2$

$(-2, {}^{16}/_3)$ is a maximum turning point as the second derivative of $2x = -4$ at $x = -2$

Q35: It is a minimum turning point as the second derivative has a value of 8 regardless of any input from x.

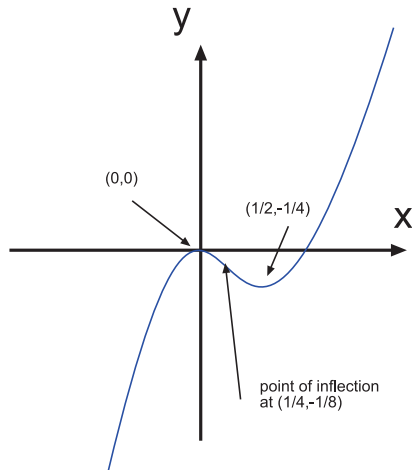**Q36:** It is a minimum since the second derivative = 6

Q37:

At turning points $\frac{dy}{dx} = 12x^2 - 6x = 0 \Rightarrow x = 0$ or $x = {}^1/_2$

$\frac{d^2y}{dx^2} = 24x - 6$

At $x = 0$, $\frac{d^2y}{dx^2} = $ -ve value $\Rightarrow$ maximum turning point.
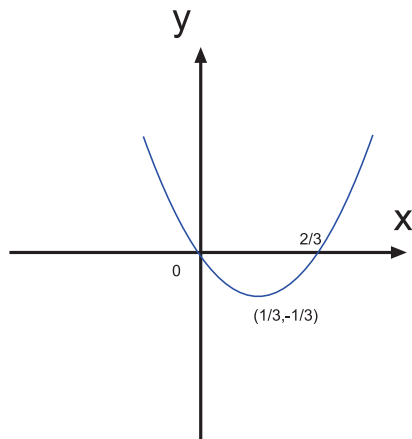
At $x = {}^1/_2$, $\frac{d^2y}{dx^2} = $ +ve value $\Rightarrow$ minimum turning point.

Q38:

At turning points $\frac{dy}{dx} = 6x - 2 = 0 \Rightarrow x = {}^1/_3$. The turning point is $({}^1/_3, {}^{-1}/_3)$

$\frac{d^2y}{dx^2} = 6$ which is positive and the turning point is a minimum.

Q39:

$\frac{dy}{dx} = 3\cos x = 0$

$\Rightarrow \cos x = 0$

$\Rightarrow x = \frac{\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, -\frac{3\pi}{2}$ at the turning points.

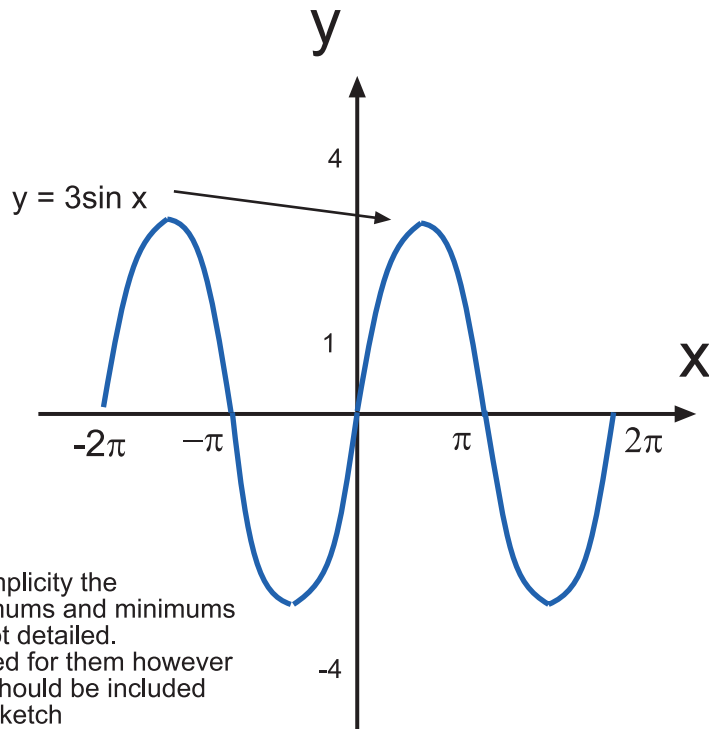$\frac{d^2y}{dx^2} = -3\sin x$

At $x = -\frac{3\pi}{2}$, $\frac{d^2y}{dx^2} = $ -ve and the turning point is a maximum.

At x = $-\frac{\pi}{2}$, $\frac{d^2y}{dx^2}$ = +ve and the turning point is a minimum.

At x = $\frac{\pi}{2}$, $\frac{d^2y}{dx^2}$ = -ve and the turning point is a maximum.

At x = $\frac{3\pi}{2}$, $\frac{d^2y}{dx^2}$ = +ve and the turning point is a minimum.

y = 3sin x

for simplicity the
maximums and minimums
are not detailed.
If asked for them however
they should be included
on a sketch

Answers from page 137.

Q40:

    1. d

    2. b

    3. c

    4. a

Concavity exercise (page 138)

Q41:

$\frac{dy}{dx} = 12x^2 - 6x$

$\frac{d^2y}{dx^2} = 24x - 6$

$\frac{d^2y}{dx^2} = 0$ when x = $\frac{1}{4}$ and at this point $\frac{dy}{dx} \neq 0$

There is a non horizontal point of inflexion at $(\frac{1}{4}, -\frac{1}{8})$

Q42:

$\frac{dy}{dx} = 6x - 2$

$\frac{d^2y}{dx^2} = 6$

There is no point on this curve where $\frac{d^2y}{dx^2} = 0$. There is no point of inflexion.

Q43:

$\frac{dy}{dx} = 3\cos x$

$\frac{d^2y}{dx^2} = -3\sin x$

$\frac{d^2y}{dx^2} = 0 \Rightarrow -3\sin x = 0 \Rightarrow x = 0, \pi, 2\pi, -\pi, -2\pi$ and at these values $\frac{dy}{dx} \neq 0$

These points are all points of inflexion.

Type 1 exercise (page 145)

Q44:

$\frac{16}{x^2}$

|	
Symmetry	The function is even and is symmetrical about the y-axis.
Crosses x-axis	If y = 0, $\frac{16}{x^2} = 0 \Rightarrow 16 = 0$ which is impossible.
	The graph does not cross the x-axis.
Crosses y-axis	x = 0 $\Rightarrow$ y = $\frac{16}{0}$ which is undefined.
	The graph does not cross the y-axis.
Turning Points	$\frac{dy}{dx} = \frac{-32}{x^3} = 0$ for turning points.
	But this implies that -32 = 0 which is not true.
	There are no turning points.
	$\frac{d^2y}{dx^2} = \frac{96}{x^4} = 0$ at a point of inflection.
	This implies that 96 = 0 which is not true.
	There are no points of inflection.
As $x \to \pm \infty$	As $x \to \pm \infty$ $y \to 0$
	There is a horizontal asymptote at y = 0
discontinuity	y is undefined at x = 0
-y undefined	There is a vertical asymptote at x = 0
	As $x \to 0^+$, $y \to +\infty$
	As $x \to 0^-$, $y \to +\infty$
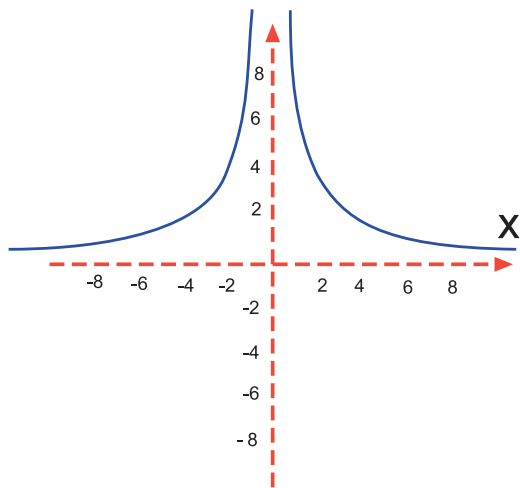
Q45:

$y = \frac{3}{x}$

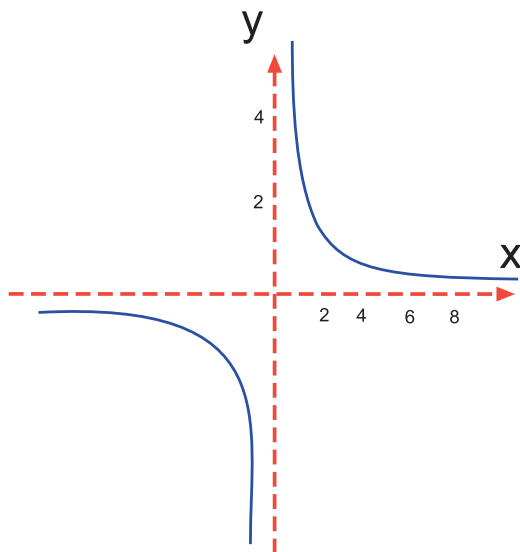|	
Symmetry	The function is odd.
	It has a rotational symmetry of order 2
Crosses x-axis	$y = 0 \Rightarrow \frac{3}{x} = 0$ which is not possible.
	It does not cross the x-axis.
Crosses y-axis	$x = 0 \Rightarrow y = \frac{3}{0}$ which is undefined.
	It does not cross the y-axis.
Turning Points	$\frac{dy}{dx} = \frac{-3}{x^2} \neq 0$ at any point.
	There is no turning point.
	$\frac{d^2y}{dx^2} = \frac{6}{x^3} \neq 0$ at any point.
	There is no point of inflection.
As $x \to \pm \infty$	As $x \to \pm \infty$, $y \to 0$
	There is a horizontal asymptote at $y = 0$
discontinuity	When $x = 0$, y is undefined.
-y undefined	There is a vertical asymptote at $x = 0$
	As $x \to 0^{+}$, $y \to + \infty$
	As $x \to 0^{-}$ $y \to - \infty$

Q46:

$y = \frac{4}{3x - 2}$

|	
Symmetry	The function is neither odd nor even.
Crosses x -axis	$y = 0 \Rightarrow 4 = 0$ which is not possible.
	It does not cross the x-axis.
Crosses y-axis	$x = 0 \Rightarrow y = -2$
	It crosses the y-axis at (0, -2)
Turning Points	$\frac{dy}{dx} = \frac{-12}{(3x - 2)^2} \neq 0$ anywhere.
	There are no turning points.
	$\frac{d^2y}{dx^2} = \frac{72}{(3x - 2)^3} \neq 0$ anywhere.
	There are no points of inflection.
As $x \to \pm \infty$	As $x \to \pm \infty$, $y \to 0$
	There is a horizontal asymptote at $y = 0$
discontinuity	When $3x - 2 = 0$, y is undefined.
-y undefined	That is, at $x = \frac{2}{3}$
	There is a vertical asymptote at $x = \frac{2}{3}$
	As $x \to \frac{2}{3}^+$, $y \to + \infty$
	As $x \to \frac{2}{3}^-$, $y \to - \infty$
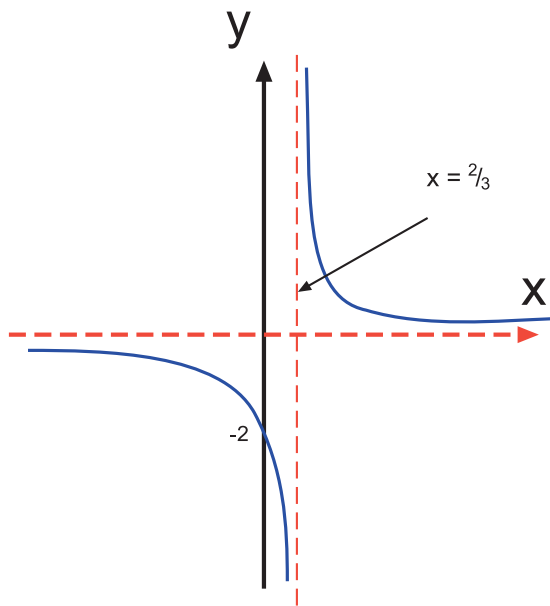
Type 2 exercise (page 147)

Q47: $y = \frac{3x - 4}{x + 3}$

Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow 3x - 4 = 0 \Rightarrow x = \frac{4}{3}$
	It crosses the x-axis at $(\frac{4}{3}, 0)$
Crosses y-axis	$x = 0 \Rightarrow y = \frac{-4}{3}$
	It crosses the y-axis at $(0, \frac{-4}{3})$
Turning Points	$\frac{dy}{dx} = \frac{13}{(x + 3)^2}$ but $\neq 0$ anywhere.
	There are no turning points.
	$\frac{d^2y}{dx^2} = \frac{-26}{(x + 3)^3} \neq 0$ anywhere.
	There are no points of inflection.
As $x \to \pm\infty$	$y \to \frac{3}{1} = 3$
	Divide top and bottom by x to see this.
	There is a horizontal asymptote at $y = 3$
discontinuity	y is undefined if $x + 3 = 0 \Rightarrow x = -3$
- y undefined	There is a vertical asymptote at $x = -3$
	As $x \to -3^+$, $y \to -\infty$
	As $x \to -3^-$, $y \to +\infty$
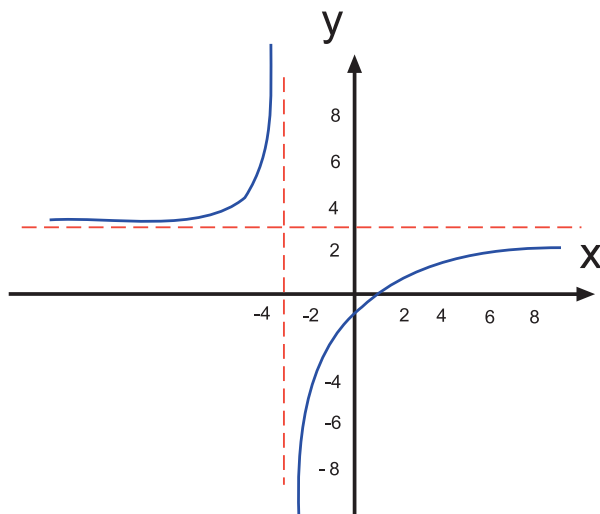
Type 3 exercise (page 150)

Q48: $y = \frac{x - 4}{x^2 - 4}$

Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow x = 4$
	The graph crosses the x-axis at (4, 0)
Crosses y-axis	$x = 0 \Rightarrow y = 1$
	The graph crosses the y-axis at (0, 1)

Turning Points

$\frac{dy}{dx} = \frac{-x^2 + 8x - 4}{(x^2 - 4)^2} = 0 \Rightarrow -x^2 + 8x - 4 = 0$

Using the quadratic formula gives x = 7.5 or 0.5

$\frac{d^2y}{dx^2} = \frac{(-2x + 8)(x^2 - 4)^2 - 2(2x)(x^2 - 4)(-x^2 + 8x - 4)}{(x^2 - 4)^4}$

At x = 0.5, $\frac{d^2y}{dx^2} = +$ ve value.

This point (0.5, 0.93) is a minimum.

At x = 7.5, $\frac{d^2y}{dx^2} = -$ ve value.

This point (7.5, 0.067) is a maximum.

As x → ± ∞   As x → ± ∞, y → 0

There is a horizontal asymptote at y = 0

Discontinuity   y is undefined if $x^2 - 4 = 0 \Rightarrow x = -2$ and x = 2

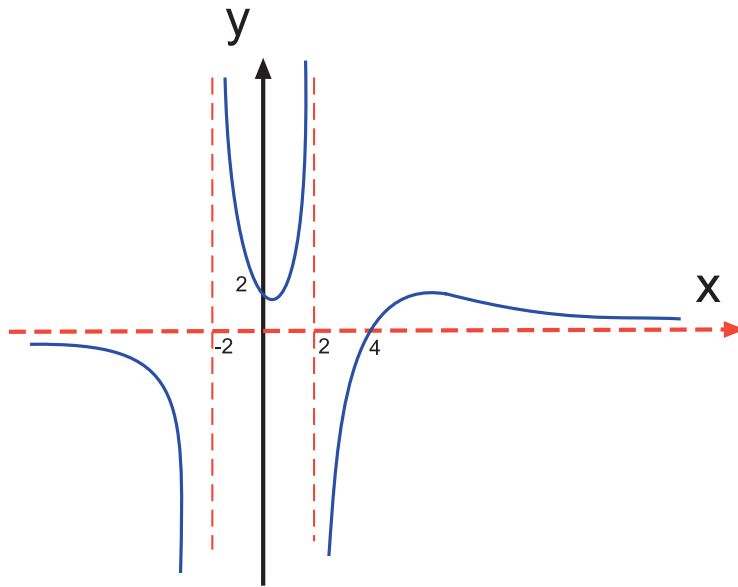-y undefined   There are vertical asymptotes at x = 2 and x = -2

As x → 2⁺, y → - ∞

As x → 2⁻, y → + ∞

As x → -2⁺, y → + ∞

As x → -2⁻, y → - ∞

Type 4 exercise (page 152)

Q49: $y = \frac{2x^2 - 4x - 1}{x^2 + 3x + 2}$

Symmetry　　　　　　　This function is neither odd nor even.

Crosses x-axis　　　　　$y = 0 \Rightarrow 2x^2 - 4x - 1 = 0$

　　　　　　　　　　Using the quadratic formula x = 2.2 or -0.22

　　　　　　　　　　The graph crosses the x-axis at x = 2.2 or -0.22

Crosses y-axis　　　　　$x = 0 \Rightarrow y = -^1/_2$

　　　　　　　　　　The graph crosses the y-axis at $(0, -^1/_2)$

Turning Points　　　　$\frac{dy}{dx} = \frac{(4x - 4)(x^2 + 3x + 2) - (2x + 3)(2x^2 - 4x - 1)}{(x^2 + 3x + 2)^2}$

　　　　　　　　　　$\frac{dy}{dx} = 0 \Rightarrow 2x^2 + 2x - 1 = 0$

　　　　　　　　　　$\Rightarrow x = 0.37$ or $x = -1.37$

　　　　　　　　　　$\frac{d^2y}{dx^2} = \frac{(4x + 2)(x^2 + 3x + 2) - (2x + 3)(2x^2 + 2x - 1)}{(x^2 + 3x + 2)^2}$

　　　　　　　　　　At x = -1.37, $\frac{d^2y}{dx^2}$ = -ve value.
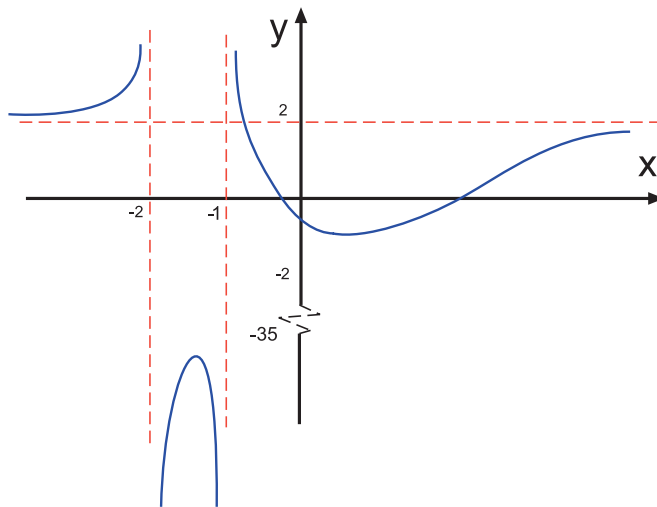
　　　　　　　　　　This point (-1.37, -35.3) is a maximum.

　　　　　　　　　　At x = 0.37, $\frac{d^2y}{dx^2}$ = +ve value.

　　　　　　　　　　This point (0.37, -0.68) is a minimum.

As $x \to \pm \infty$	$x \to \pm \infty$, $y \to 2$. (divide through by x^2)
	There is a horizontal asymptote at $y = 2$
discontinuity	y is undefined when $x^2 + 3x + 2 = 0$
-y undefined	$\Rightarrow x = -2$ or $x = -1$
	There are vertical asymptotes at $x = -2$ and $x = -1$

As $x \to -1^+$, $y \to + \infty$

As $x \to -1^-$, $y \to - \infty$

As $x \to -2^+$, $y \to - \infty$

As $x \to -2^-$, $y \to + \infty$

Type 5 exercise (page 154)

Q50: $y = \frac{x^2 + 2x - 3}{x + 2}$

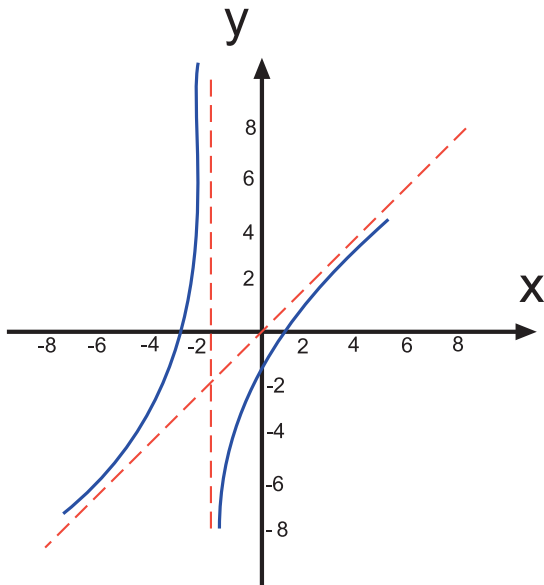|	
Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow x^2 + 2x - 3 = 0 \Rightarrow x = -3$ or $x = 1$
	The graph crosses the x-axis at $x = -3$ and $x = 1$
Crosses y-axis	$x = 0 \Rightarrow y = -^3/_2$
	The graph crosses the y-axis at $(0, -^3/_2)$
Turning Points	$\frac{dy}{dx} = \frac{(2x + 2)(x + 2) - (x^2 + 2x - 3)}{(x + 2)^2} = 0$
	$\Rightarrow x^2 + 4x + 7 = 0$
	This has no real roots and so there are no turning points.
As $x \to \pm \infty$	As $x \to \pm \infty$, $y \to x$
	There is a slant asymptote at $y = x$
discontinuity	y is undefined if $x + 2 = 0 \Rightarrow x = -2$
-y undefined	There is a vertical asymptote at $x = -2$
	As $x \to -2^-$, $y \to + \infty$
	As $x \to -2^+$, $y \to - \infty$

Sketching exercise using shortcuts (page 156)

Q51:

$y = \frac{3x^2 - 4}{x + 1}$

So a = 3, c = - 4, e = 1, f = 1 (Type 5)

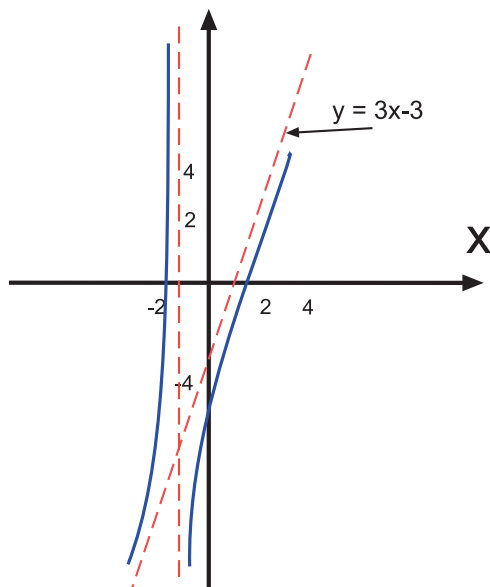|	
Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow 3x^2 - 4 = 0 \Rightarrow x = \frac{2}{\sqrt{3}}$ or $\frac{-2}{\sqrt{3}}$
	The graph crosses the x-axis twice.
Crosses y-axis	$x = 0 \Rightarrow y = -4$
	The graph crosses y at the point (0, -4)
Turning Points	$\frac{dy}{dx} = \frac{6x\,(x + 1) - (3x^2 - 4)}{(x + 1)^2} = 0$
	$\Rightarrow 3x^2 + 6x + 4 = 0$
	But this has no real roots and so there are no turning points.
Slant asymptote	There is a slant asymptote at y = 3x - 3
	(Divide the numerator by the denominator)
Vertical asymptote	There is a vertical asymptote at x = -1
	As $x \rightarrow -1^-$, $y \rightarrow + \infty$
	As $x \rightarrow - 1^+$, $y \rightarrow - \infty$

Q52:

$y = \frac{4x - 5}{x^2 - 1}$

b = 4, c = -5, d = 1, f = -1 (Type 3)

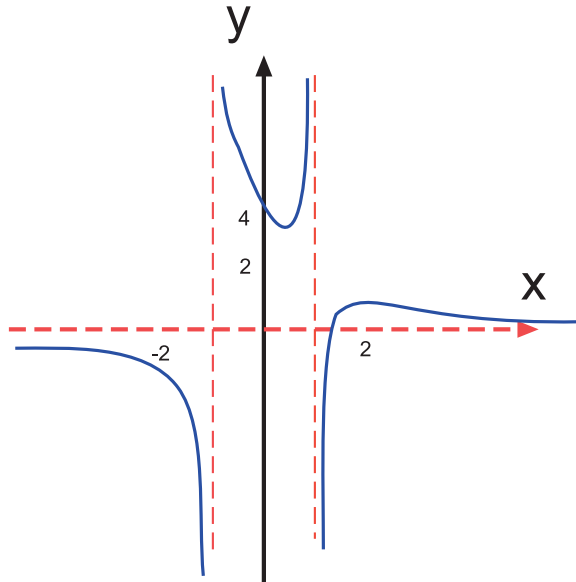|	
Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow x = \frac{5}{4}$
	The graph crosses the x-axis at $(\frac{5}{4}, 0)$
	$x = 0 \Rightarrow y = 5$
Crosses y-axis	
	The graph crosses the y-axis at (0, 5)
	$\frac{dy}{dx} = \frac{4 (x^2 - 1) - 2x (4x - 5)}{(x^2 - 1)^2} = 0$
	$\Rightarrow (2x - 1)(x - 2) = 0$
	$\Rightarrow x = 2$ or $x = \frac{1}{2}$
	There are turning points at $x = 2$ and at $x = \frac{1}{2}$
Turning Points	$\frac{d^2y}{dx^2} = \frac{(- 8x + 10)(x^2 - 1)^2 - [4x (x^2 - 1)](- 4x^2 + 10x - 4)}{(x^2 - 1)^4}$
	At $x = 2$, $\frac{d^2y}{dx^2} = (- \times +) = $ -ve value.
	The point is a maximum at (2, 1)
	At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = (+ \times +) = $ +ve value.
	The point is a minimum at $(\frac{1}{2}, 4)$
Horizontal asymptote	There is a horizontal asymptote at y = 0

There are vertical asymptotes at x = -1 and x = 1

As x → 1$^+$, y = - ∞

Vertical asymptote   As x → 1$^-$, y → + ∞

As x → -1$^-$, y →- ∞

As x → -1$^+$, y → + ∞

Q53:

$y = \frac{x^2 - 2x}{x^2 - 1}$

a = 1, b = -2, d = 1, f = -1 (Type 4)

Symmetry	The function is neither odd nor even.
Crosses x-axis	y = 0 ⇒ x = 0 or x = 2
	The graph crosses the x-axis at (0, 0) and (2, 0)
Crosses y-axis	x = 0 ⇒ y = 0
	The graph crosses the y-axis at (0, 0)

Turning Points $\frac{dy}{dx} = \frac{(2x - 2)(x^2 - 1) - 2x\,(x^2 - 2x)}{(x^2 - 1)^2} = 0$

$\Rightarrow x^2 - x + 1 = 0$

This quadratic has no real roots and so there are no turning points.

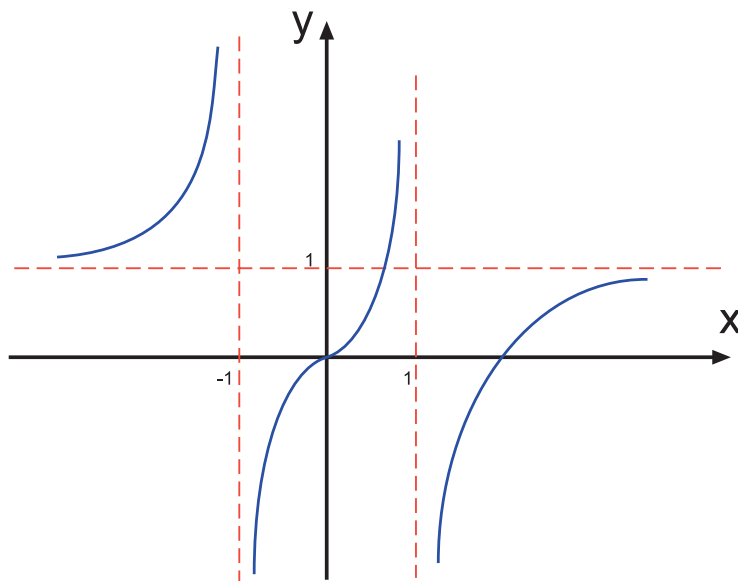Horizontal asymptote   There is a horizontal asymptote at y = 1

Vertical asymptote   There are vertical asymptotes at x = 1 and x = -1

As x → -1$^+$, y → - ∞

As x → -1$^-$, y → + ∞

As x → 1$^-$, y → + ∞

As x → 1$^+$, y → - ∞

Q54:

$y = \frac{2x^2 - 5x}{x^2 - 1}$

$a = 2, b = -5, d = 1, f = -1$ (Type 4)

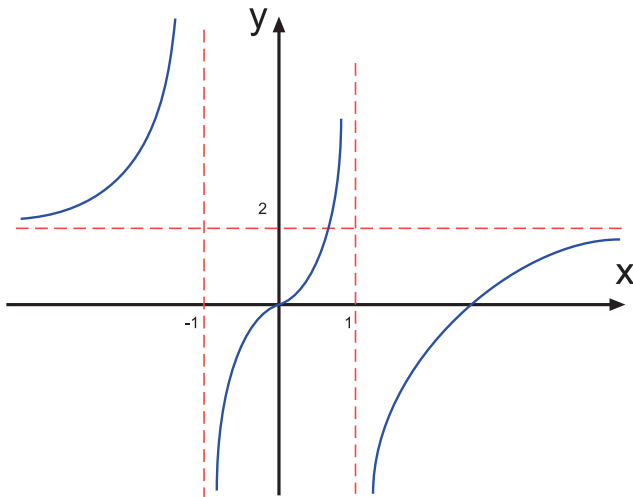|	
Symmetry	The function is neither odd nor even .
Crosses x-axis	$y = 0 \Rightarrow x = 0$ or $x = \frac{5}{2}$
Crosses y-axis	$x = 0 \Rightarrow y = 0$
Turning Points	$\frac{dy}{dx} = \frac{(4x - 5)(x^2 - 1) - 2x(2x^2 - 5x)}{(x^2 - 1)^2} = 0$
	$\Rightarrow 5x^2 - 4x + 5 = 0$
	This function has no real roots and so there are no turning points.
Horizontal asymptote	There is a horizontal asymptote at $y = 2$
Vertical asymptote	$x = 1$ and $x = -1$ are vertical asymptotes.
	As $x \to 1^-$, $y \to + \infty$
	As $x \to 1^+$, $y \to - \infty$
	As $x \to -1^+$, $y \to - \infty$
	As $x \to -1^-$, $y \to + \infty$

Q55:

$y = \frac{2x^2}{x - 2}$

$a = 2$, $e = 1$, $f = -2$ (Type 5)

Symmetry	The function is neither odd nor even .
Crosses x-axis	$y = 0 \Rightarrow x = 0$
Crosses y-axis	$x = 0 \Rightarrow y = 0$
Turning Points	$\frac{dy}{dx} = \frac{4x\,(x - 2) - 2x^2}{(x - 2)^2} = 0$
	$\Rightarrow 2x^2 - 8x = 0$
	$\Rightarrow x = 0 \text{ or } x = 4$
	When $x = 0$, $\frac{d^2y}{dx^2} = \frac{(- \times +)}{+}$ = -ve value.
	$\Rightarrow$ Maximum turning point at (0, 0)
	When $x = 4$, $\frac{d^2y}{dx^2} = \frac{(+ \times +)}{+}$ = +ve value.
	$\Rightarrow$ Minimum turning point at (4, 16)
Horizontal asymptote	There is none for a type 5 rational function.
	There is a vertical asymptote at $x = 2$
Vertical asymptote	As $x \to 2^+$, $y \to + \infty$
	As $x \to 2^-$, $y \to - \infty$
Slant asymptote	There is a slant asymptote at $y = 2x + 4$

Q56:

$y = \frac{x^2 - 3x + 3}{x - 1}$

a = 1, b = -3, c = 3, e = 1, f = -1 (Type 5)

Symmetry   The function is neither odd nor even.

Crosses x-axis $y = 0 \Rightarrow x^2 - 3x + 3 = 0$ which has no real solutions.

                     The graph does not cross the x-axis.

Crosses y-axis $x = 0 \Rightarrow y = -3$

Turning Points $\frac{dy}{dx} = \frac{(2x - 3)(x - 1) - (x^2 - 3x + 3)}{(x - 1)^2} = 0$

                     $\Rightarrow x^2 - 2x = 0$

                     $\Rightarrow x = 0$ or $x = 2$

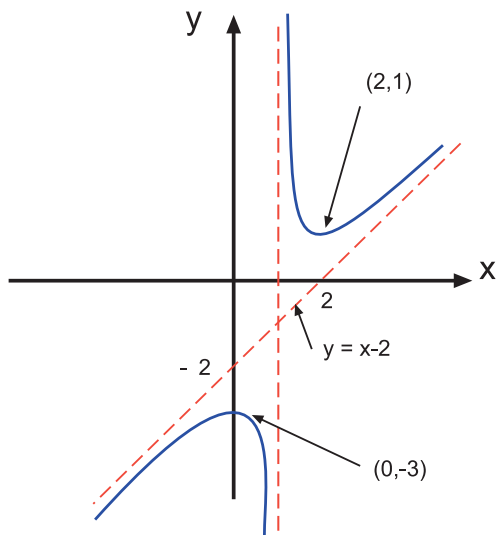   When $x = 0$, $\frac{d^2y}{dx^2} = \frac{(-\times+)}{+}$ = - ve value.

                     $\Rightarrow$ Maximum turning point at (0, -3)

                     When $x = 2$, $\frac{d^2y}{dx^2} = \frac{(+\times+)}{+}$ = + ve value.

                     $\Rightarrow$ Minimum turning point at (2, 1)

Horizontal asymptote	It is a type 5 rational function.
	There is no horizontal asymptote.
Vertical asymptote	There is a vertical asymptote at x = 1
	As x → 1⁺, y → + ∞
	As x → 1⁻, y → - ∞
Slant asymptote	There is a slant asymptote at y = x - 2

Vertical translation exercise (page 158)

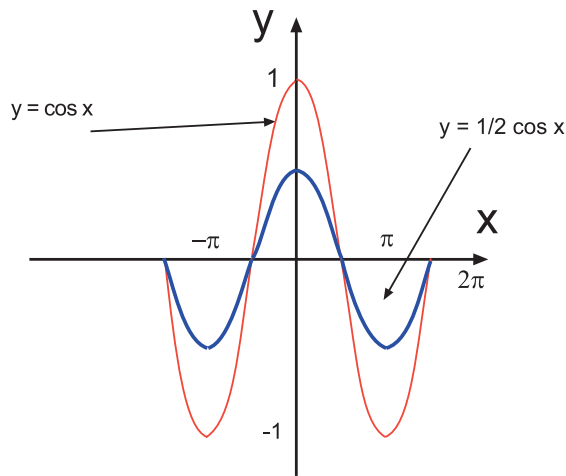**Q57:** The line y = 6x is moved downwards by 3 units.

Q58: The line y = 2x is moved upwards by 8 units.

Q59: The second graph moves down the y-axis by 3 units.

Q60: The second graph moves up the y-axis by 2 units.

Q61: The second graph moves up the y-axis by 4 units.

Q62: The second graph moves down the y-axis by 3 units (in this case by 3 radians).

Vertical scaling exercise (page 159)

Q63:

Q64: The first graph is scaled vertically by a factor of 3 to give the second graph.
(It stretches by a factor of 3)

Q65: The graph of sin x is scaled vertically by a factor of $^1/_4$ to give the second graph.
(It shrinks to one quarter of its size)

Q66: The graph of y = cos 2x is scaled vertically by a factor of $^1/_2$

(It shrinks to half the size)

Q67: The graph of y = ln x is scaled vertically by a factor of 3

Q68: The graph of y = $^1/_2$e^x is scaled vertically by a factor of 4

Horizontal translation exercise (page 160)

Q69: The graph of y = 3 (x - 2) sits two units along the x-axis to the right of the graph of
y = 3x

Q70: The graph of y = cos (x + 30)° sits 30° to the left of the graph of y = cos x°

Q71: The graph will lie $^1/_2$ a unit along the x-axis to the right of the graph of y = (2x)^2
y = (2x - 1)^2 = (2(x - $^1/_2$))^2 so k = -$^1/_2$

Horizontal translation exercise (page 161)

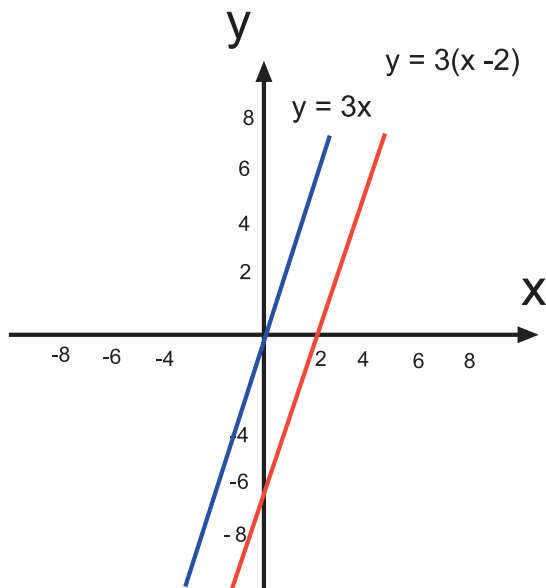**Q72:** The graph of y = 3 (x - 2) sits two units along the x-axis to the right of the graph of
y = 3x

Q73: The graph of $y = \cos(2x - 45)°$ sits $22.5°$ to the right of the graph of $y = \cos 2x$

$y = \cos(2x - 45) = \cos 2(x - 22.5)$ so $k = -22.5$

Q74: The graph of $y = x^2 + 4x + 7$ sits two units to the left of the graph of $y = x^2 + 3$

since $x^2 + 4x + 7 = (x + 2)^2 + 3$

Q75: The graph of $y = \sqrt{x + 2}$ moves to the left of the graph of $\sqrt{x}$ by 2 units.

Q76: The graph of $y = \cos(x + 30)°$ sits $30°$ to the left of the graph of $y = \cos x°$

Q77: The graph will lie $^1/_2$ a unit along the x-axis to the right of the graph of $y = (2x)^2$

$(y = (2x - 1)^2 = (2(x - ^1/_2))^2$ so $k = -^1/_2$)
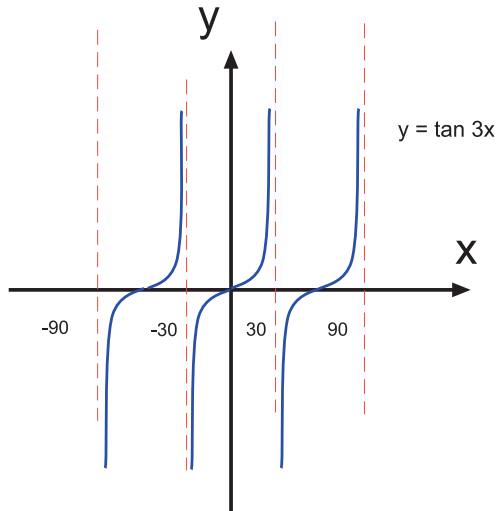
Q78: The graph of $y = \sqrt{x + 2}$ moves to the left of the graph of $\sqrt{x}$ by 2 units.

Horizontal scaling exercise (page 162)

Q79: The graph of $y = \sin x$ is scaled horizontally by a factor of 3 to give the graph of
$y = \sin ^1/_3 x$

It stretches horizontally.

Q80: The graph of $y = \tan x$ is scaled horizontally by a factor of $^1/_3$

Q81: The graph of $y = \cos x°$ is scaled horizontally by a factor of $^4/_3$ to give the graph of $y = \cos \,^3/_4 x°$

Q82: The graph of $y = \sqrt{x}$ has been scaled horizontally by a factor of $^1/_3$ to give the graph of $y = \sqrt{3x}$

Q83: The graph of $y = \sqrt{x}$ has been scaled horizontally by a factor of $^1/_3$ to give the graph of $y = \sqrt{3x}$

Answers from page 164.

Q84:

Q85: y = | tan x |
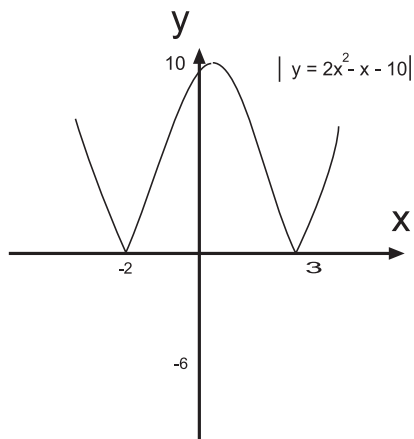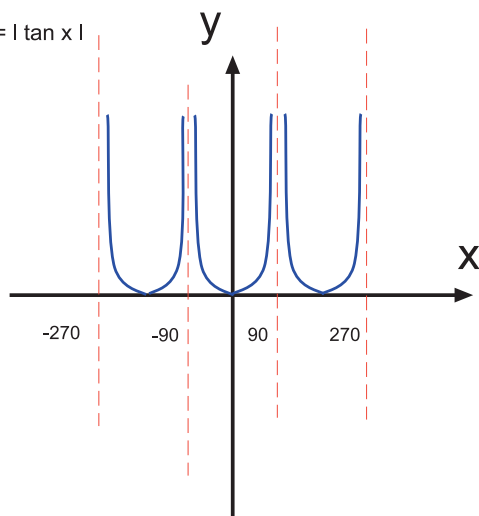
Answers from page 166.

Q86:

f (s)	= 2s - 4
f (t)	= 2t - 4
Let f (s)	= f (t)
Then 2s - 4	= 2t - 4
So s	= t

The function is one-to-one.

Q87:

g (s)	$= 2s^5$
g (t)	$= 2t^5$
Let g (s)	= g (t)
Then $2s^5$	$= 2t^5$
So s	= t

The function is one-to-one.

Q88:

h (s)	$= \frac{1}{s+2}$
h (t)	$= \frac{1}{t+2}$
Let h (s)	= h (t)
Then $\frac{1}{s+2}$	$= \frac{1}{t+2}$
So s	= t

The function is one-to-one.

Q89: Let $t \in \mathbb{R}$

Let s $\qquad = {}^{t}/_{3}$

f (s) $\qquad = f ({}^{t}/_{3})$

$\qquad\qquad\quad = 3 \times {}^{t}/_{3}$

$\qquad\qquad\quad = t$

The function is onto.

Q90: Let $t \in \mathbb{R}^{+}$

Let s $\qquad = \sqrt[3]{t}$

g (s) $\qquad = g \left(\sqrt[3]{t} \right)$

$\qquad\qquad\quad = \left(\sqrt[3]{t} \right)^{3}$

$\qquad\qquad\quad = t$

The function is onto.

Q91: Let $t \in \{z \in \mathbb{R}: z \neq 1, 3\}$

Let s $\qquad = \frac{1}{t-1}$

h (s) $\qquad = h \left(\frac{1}{t-1} \right)$

$\qquad\qquad\quad = 1 + \frac{1}{\frac{1}{t-1}} = 1 + \frac{t-1}{1}$

$\qquad\qquad\quad = t$

The function is onto.

Q92: Take s = -1 and t = 1 then f (s) = $^{1}/_{2}$ and f (t) = $^{1}/_{2}$. But s $\neq$ t and so the function is not one-to-one.
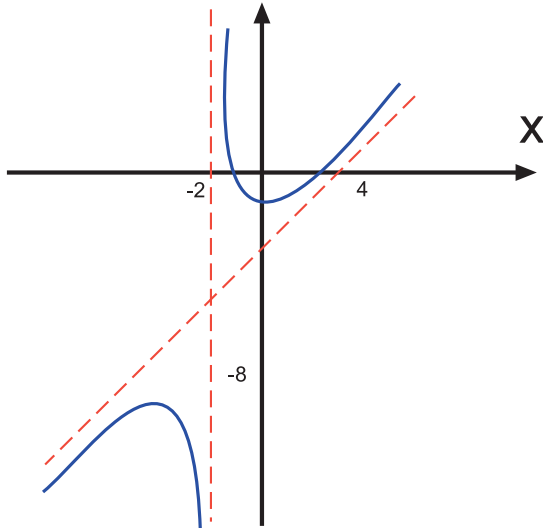
Q93: Let b = -4. There is no element $s \in \mathbb{R}^{+}$, the domain of f such that
f (s) = -4 as $f(s) = 2\sqrt{s} \geq 0$ for all s. Hence the function is not onto.

Review exercise (page 167)

Q94:

a) x = -2

b) y = x - 3

c) $y = \frac{x^{2} - x - 2}{x + 2}$

Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow x^2 - x - 2 = 0$
	$\Rightarrow x = 2$ or $x = -1$
	The graph crosses the x-axis twice.
Crosses y-axis	$x = 0 \Rightarrow y = -1$
Turning Points	$\frac{dy}{dx} = 1 - \frac{4}{(x+2)^2} = 0$
	$\Rightarrow x + 2 = \pm 2$
	$\Rightarrow x = 0$ or $x = -4$
	When $x = 0$, $\frac{d^2y}{dx^2} = +$ ve value.
	$\Rightarrow$ Minimum turning point at $(0, -1)$
	When $x = -4$, $\frac{d^2y}{dx^2} = -$ ve value.
	$\Rightarrow$ Maximum turning point at $(-4, -9)$
Horizontal asymptote	There is no horizontal asymptote for type 5.
Vertical asymptote	There is a vertical asymptote at $x = -2$
	As $x \to -2^+$, $y \to +\infty$
	As $x \to -2^-$, $y \to -\infty$
Slant asymptote	There is a slant asymptote at $y = x - 3$

Q95:

a) $x = 3$

b) $y = -4x + 1$

c) $\frac{-4x^2 + 13x - 28}{x - 3}$

Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow -4x^2 + 13x - 28 = 0$
	There are no real roots.
	The graph does not cross the x-axis.
Crosses y-axis	$x = 0 \Rightarrow y = \frac{28}{3}$
Turning Points	$\frac{dy}{dx} = -4 + \frac{25}{(x-3)^2} = 0$
	$\Rightarrow x - 3 = \pm\frac{5}{2}$
	$\Rightarrow x = \frac{11}{2}$ or $x = \frac{1}{2}$
	When $x = \frac{11}{2}$, $\frac{d^2y}{dx^2} = $ - ve value.
	$\Rightarrow$ Maximum turning point at $(\frac{11}{2}, -31)$
	When $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = $ + ve value.
	$\Rightarrow$ Minimum turning point at $(\frac{1}{2}, 9)$
Horizontal asymptote	There is no horizontal asymptote for type 5 rational function graphs.
	There is a vertical asymptote at $x = 3$
Vertical asymptote	As $x \rightarrow 3^+$, $y \rightarrow -\infty$
	As $x \rightarrow 3^-$, $y \rightarrow +\infty$
Slant asymptote	There is a slant asymptote at $y = -4x + 1$
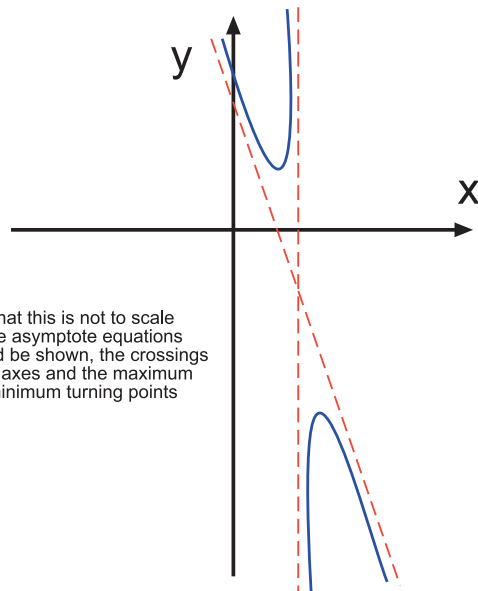
note that this is not to scale
but the asymptote equations
should be shown, the crossings
of the axes and the maximum
and minimum turning points

Q96: $y = \frac{x-5}{x^2-16}$

Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow x = 5$
	It crosses the x-axis at (5, 0)
Crosses y-axis	$x = 0 \Rightarrow y = \frac{5}{16}$
	It crosses the y-axis at $(0, \frac{5}{16})$

Turning Points $\quad \dfrac{dy}{dx} = \dfrac{(x^2 - 16) - 2x\,(x - 5)}{(x^2 - 16)^2} = 0$

$\Rightarrow (x - 8)(-x + 2) = 0 \Rightarrow x = 8 \text{ or } x = 2$

There are turning points at x = 2 and at x = 8

At x = 2, $\dfrac{d^2y}{dx^2} = (+ \times +) = $ +ve value.

The point is a minimum $(2, {}^1/_4)$

At x = 8, $\dfrac{d^2y}{dx^2} = (+ \times -) = $ -ve value.

The point is a maximum $(8, {}^1/_{16})$

Horizontal
asymptote $\qquad$ There is a horizontal asymptote at y = 0

Vertical asymptote $\quad$ There are vertical asymptotes at x = -4 and x = 4

As x → -4$^+$, y →+ ∞

As x → -4$^-$, y → - ∞

As x → 4$^-$, y →+ ∞

As x → 4$^+$, y → - ∞

the y-axis is not
to scale but the
turning points should
be annotated
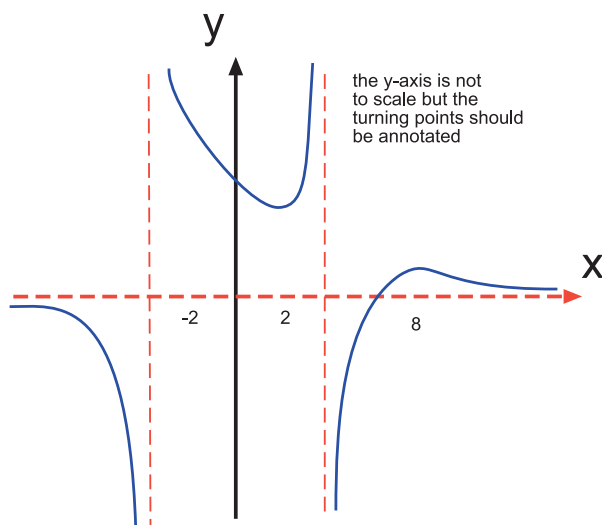
Q97: $y = \frac{2x^2 - 2x - 2}{x - 2}$

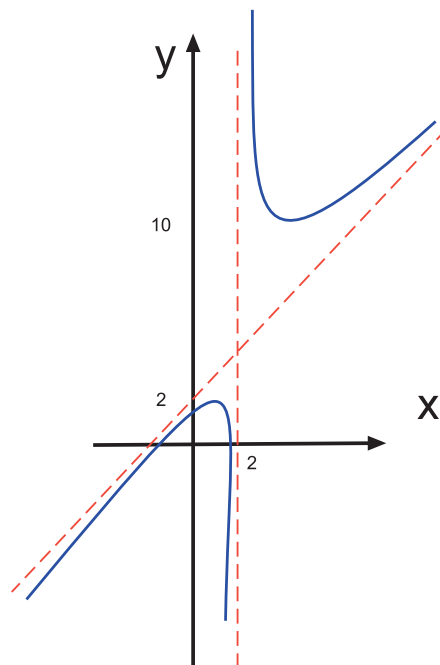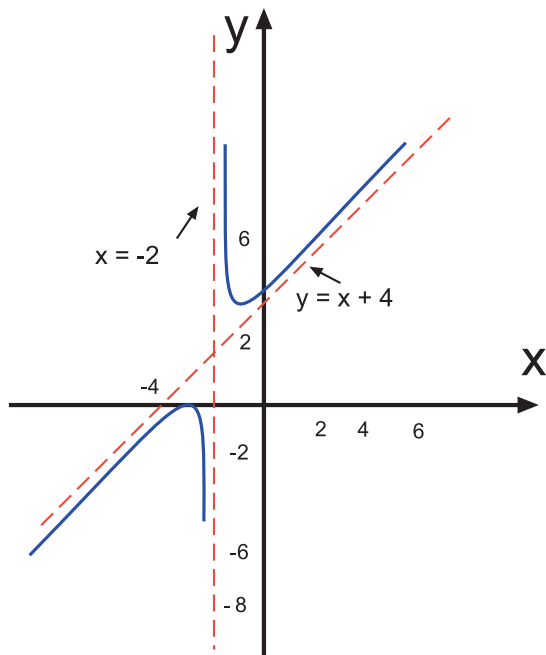|	
Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow 2x^2 - 2x - 2 = 0$
	This has no real roots.
	The graph does not cross the x-axis.
Crosses y-axis	$x = 0 \Rightarrow y = 1$
Turning Points	$\frac{dy}{dx} = 2 - \frac{2}{(x - 2)^2} = 0$
	$\Rightarrow x - 2 = \pm 1$
	$\Rightarrow x = 3$ or $x = 1$
	When $x = 1$, $\frac{d^2y}{dx^2} = -$ ve value.
	$\Rightarrow$ Maximum turning point at (1, 2)
	When $x = 3$, $\frac{d^2y}{dx^2} = +$ ve value.
	$\Rightarrow$ Minimum turning point at (3, 10)
Horizontal asymptote	There is no horizontal asymptote for type 5 rational function graphs.
Vertical asymptote	There is a vertical asymptote at $x = 2$
	As $x \to 2^+$, $y \to + \infty$
	As $x \to 2^-$, $y \to - \infty$
Slant asymptote	There is a slant asymptote at $y = 2x + 2$

Advanced review exercise (page 167)

Q98: $y = \frac{x^2 + 6x + 9}{x + 2}$

|	
Symmetry	The function is neither odd nor even.
Crosses x-axis	$y = 0 \Rightarrow x^2 + 6x + 9 = 0$
	$\Rightarrow x = -3$
	The graph crosses the x-axis at (-3, 0)
Crosses y-axis	$x = 0 \Rightarrow y = {}^9/_2$
	The graph crosses the y-axis at $(0, {}^9/_2)$
Turning Points	$\frac{dy}{dx} = 1 - \frac{1}{(x+2)^2} = 0$
	$\Rightarrow x + 2 = \pm 1$
	$\Rightarrow x = -3$ or $x = -1$
	When $x = -1$, $\frac{d^2y}{dx^2} = +$ ve value.
	$\Rightarrow$ Minimum turning point at (-1, 4)
	When $x = -3$, $\frac{d^2y}{dx^2} = -$ ve value.
	$\Rightarrow$ Maximum turning point at (-3, 0))
Horizontal asymptote	There is no horizontal asymptote for type 5.
Vertical asymptote	There is a vertical asymptote at $x = -2$
	As $x \Rightarrow -2^-$, $y \Rightarrow -\infty$
	As $x \Rightarrow -2^+$, $y \Rightarrow +\infty$
Slant asymptote	There is a slant asymptote at $y = x + 4$

Q99: $y = \frac{2x^3 - 20x^2 + 64x - 37}{(x - 4)^2}$

a) $x = 0 \Rightarrow y = {}^{-37}/_{16}$

The value of a is ${}^{-37}/_{16}$

b) There is a vertical asymptote at $x = 4$

c) Long division gives $f(x) = 2x - 4 + \frac{27}{(x-4)^2}$. The slant asymptote is $y = 2x - 4$

d) First of all

$\frac{dy}{dx} = 2 - \frac{54}{(x-4)^3}$

At a turning point, the derivative is zero

$\Rightarrow 2(x-4)^3 = 54$

$\Rightarrow (x-4) = 3 \Rightarrow x = 7$

The turning point is at (7, 13)

At $x = 7$, $\frac{d^2y}{dx^2}$ is positive. Therefore the turning point is a minimum.

e) $f(0) = {}^{-37}/_{16} =$ -ve value

$f(1) = 1 =$ +ve value. There is a change of signs and so $f(x)$ has a solution in the interval $0 < x < 1$

f) The graph is
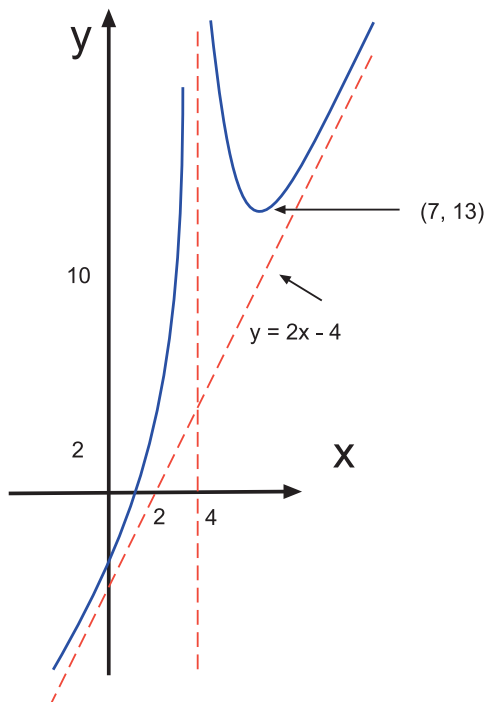
Set review exercise (page 168)

Q100: The answer to this question is only available on the web.

Q101: The answer to this question is only available on the web.

Q102: The answer to this question is only available on the web.

5 Systems of Linear Equations

Revision exercise (page 172)

Q1: x = 3 and y = 1

Q2: a = 4 and b = -2

Q3: s = -1 and t = 2

Answers from page 173.

Q4: It has 2 rows, 3 columns and 6 elements.

Matrix elements and entries exercise (page 173)

Q5:

1. 8
2. 9
3. 12
4. 10

Q6:

1. 13
2. 6
3. -7
4. 0

Q7:

1. i = 3 and j = 1
2. i = 4 and j = 2
3. i = 2 and j = 4

Q8:

1. i = 1 and j =3
2. i = 4 and j = 1
3. i = 3 and j = 4

Forming matrices exercise (page 178)

Q9: The coefficient matrix is $\begin{pmatrix} 4 & -3 & 1 \\ 2 & 1 & 3 \\ 1 & 4 & 2 \end{pmatrix}$ and the augmented matrix is $\begin{pmatrix} 4 & -3 & 1 & \vline & -1 \\ 2 & 1 & 3 & \vline & 7 \\ 1 & 4 & 2 & \vline & 8 \end{pmatrix}$

Q10: The coefficient matrix is $\begin{pmatrix} 3 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 4 & 0 \end{pmatrix}$ and the augmented matrix is

$$\begin{pmatrix} 3 & 0 & 1 & \vline & 11 \\ 2 & -1 & 1 & \vline & 6 \\ 1 & 4 & 0 & \vline & 14 \end{pmatrix}$$

Upper triangular matrix exercise (page 181)

Q11:

$$\begin{array}{cc} & \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 2 & 1 & 3 \end{pmatrix} & \begin{array}{c} r1 \\ r2 \\ r3 \end{array} \end{array}$$

$$\rightarrow \begin{array}{c} r1 \\ r2 \\ r3 + 2r1 \end{array} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & -1 & 5 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array}$$

$$\rightarrow \begin{array}{c} r1 \\ r2 \\ r3 - r2 \end{array} \begin{pmatrix} -1 & -1 & 1 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array}$$

Q12:

$$\begin{pmatrix} 3 & 3 & 2 \\ 1 & 1 & -1 \\ 2 & 2 & 2 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array} \quad \begin{array}{c} r1 \leftrightarrow r2 \\ \rightarrow \quad r1 \\ r3 \end{array} \begin{pmatrix} 1 & 1 & -1 \\ 3 & 3 & 2 \\ 2 & 2 & 2 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array}$$

$$\rightarrow \begin{array}{c} r1 \\ r2 - 3r1 \\ r3 \end{array} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 2 & 2 & 2 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array}$$

$$\rightarrow \begin{array}{c} r1 \\ r2 \\ r3 - 2r1 \end{array} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 0 & 5 \\ 0 & 0 & 4 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array}$$

Q13: The answer is not unique. This is only one possible solution with the steps which achieve it.

$$\begin{array}{c} r1 \\ 4r2 - r1 \end{array} \begin{pmatrix} 4 & 25 \\ 0 & -5 \end{pmatrix} \begin{array}{c} r1 \\ r2 \end{array}$$

Q14: The answer is not unique. This is only one possible solution with the steps which achieve it.

$$\begin{array}{c} r1 \\ r2 + r1 \\ r3 \end{array} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & 8 \\ 0 & 6 & 46 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array} \quad \begin{array}{c} r1 \\ \rightarrow \quad r2 \\ r3 - 6r2 \end{array} \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & 8 \\ 0 & 0 & -2 \end{pmatrix} \begin{array}{c} r1 \\ r2 \\ r3 \end{array}$$

Q15: The answer is not unique. This is only one possible solution with the steps which achieve it.

$$
\begin{array}{ll}
\text{r1} & \\
\text{r2 - 11r1} & \begin{pmatrix} 1 & 3 & -5 \\ 0 & 2 & 5 \\ 2 & 7 & 9 \end{pmatrix} \begin{array}{l} \text{r1} \\ \text{r2} \\ \text{r3} \end{array} \\
\text{r3} &
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{ll}
\text{r1} & \\
\text{r2} & \begin{pmatrix} 1 & 3 & -5 \\ 0 & 2 & 5 \\ 0 & 1 & 19 \end{pmatrix} \begin{array}{l} \text{r1} \\ \text{r2} \\ \text{r3} \end{array} \\
\text{r3 - 2r1} &
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{ll}
\text{r1} & \\
\text{r2} & \begin{pmatrix} 1 & 3 & -5 \\ 0 & 2 & 5 \\ 0 & 0 & 33 \end{pmatrix} \begin{array}{l} \text{r1} \\ \text{r2} \\ \text{r3} \end{array} \\
\text{2r3 - r2} &
\end{array}
$$

Q16: The answer is not unique. This is only one possible solution with the steps which achieve it.

$$
\begin{array}{ll}
\text{r1} \leftrightarrow \text{r2} & \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 9 \end{pmatrix} \begin{array}{l} \text{r1} \\ \text{r2} \\ \text{r3} \end{array} \\
\text{r3} &
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{ll}
\text{r1} & \\
\text{r2} & \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{pmatrix} \begin{array}{l} \text{r1} \\ \text{r2} \\ \text{r3} \end{array} \\
\text{r3 - 2r1} &
\end{array}
$$

$$
\rightarrow \quad
\begin{array}{ll}
\text{r1} & \\
\text{r2} & \begin{pmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{array}{l} \text{r1} \\ \text{r2} \\ \text{r3} \end{array} \\
\text{r3 + r2} &
\end{array}
$$

Gaussian elimination exercise (page 183)

Q17: $x = 7$ and $y = 8$ or $(7, 8)$

Q18: $x = 2$, $y = 1$ and $z = 3$ or $(2, 1, 3)$

Q19: $x = -2$, $y = 1$ and $z = 4$ or $(-2, 1, 4)$

Answers from page 190.

Q20: The system is inconsistent and there are no solutions.

The planes do not intersect at all.

Q21: There are infinite solutions as $x + 2z = 2$ with $y = 1$

Two planes intersect at a line. The equations $x - y + 2z = 1$ and $2x - 2y + 4z = 2$ represent the same plane.

Answers from page 191.

Q22: The solutions are $(1, -3)$ and $(6, -2503)$

This shows an ill-conditioned system of equations where a small change has led to a dramatic change in the solutions, especially for the y value.

Check the gradient of either line - it is very large.

Answers from page 192.

Q23: The solutions are (10, 1) and (-1322, 5)

This shows an ill-conditioned system of equations where a small change has led to a dramatic change in the solutions, especially for the x value.

Check the gradient of either line - it is very small.

Review exercise (page 194)

Q24: x = 9, y = 0 and z = -2

Q25: x = 3, y = -4 and z = 2

Q26: x = 4, y = 2 and z = -3

Q27: x = 1, y = 2 and z = -1

Advanced review exercise (page 194)

Q28: There are infinite solutions to this system of equations.

Q29: There is no solution to this system of equations. The system is inconsistent.

Q30: At a = 15 there are no solutions since the matrix reduces to 0 = 2

a = 13 gives the three planes intersecting at a line. The solutions of the system are infinite as 2y + 3z = 4 and x + 4z = 9

Set review exercise (page 195)

Q31: The answer is only available on the web.

Q32: The answer is only available on the web.

Q33: The answer is only available on the web.