SCHOLAR Study Guide

SQA Advanced Higher Mathematics Unit 2

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First published 2001 by Heriot-Watt University.

This edition published in 2009 by Heriot-Watt University SCHOLAR.

Distributed by Heriot-Watt University.

SCHOLAR Study Guide Unit 2: Advanced Higher Mathematics

1. Advanced Higher Mathematics

ISBN 978-1-906686-04-8

Printed and bound in Great Britain by Graphic and Printing Services, Heriot-Watt University, Edinburgh.

Acknowledgements

Thanks are due to the members of Heriot-Watt University's SCHOLAR team who planned and created these materials, and to the many colleagues who reviewed the content.

We would like to acknowledge the assistance of the education authorities, colleges, teachers and students who contributed to the SCHOLAR programme and who evaluated these materials.

Grateful acknowledgement is made for permission to use the following material in the SCHOLAR programme:

The Scottish Qualifications Authority for permission to use Past Papers assessments.

The Scottish Government for financial support.

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Topic 1

Further Differentiation

Contents

Learning Objectives

- *Use further differentiation techniques.*

Minimum Performance Criteria:

- *Differentiate an inverse trigonometrical function (involving the chain rule).*

- *Find the derivative of a function defined implicitly.*
- *Find the first derivative of a function defined parametrically.*

1.1 Inverse functions

Learning Objective

Differentiate inverse functions

Prerequisites

You should be able to perform differentiations involving a variety of variables. Try these revision questions.

Revision questions

5 min

Q1: When $a = b^3$ find $^{da}/_{db}$

Q2: When $x = \sqrt{y}$ find $^{dx}/_{dy}$

Q3: When $y = x^2$ find $^{dy}/_{dx}$

Q4: Can you spot any connection between the last two questions?

1.1.1 Differentiating inverse functions

Learning Objective

Recall the method that allows you to differentiate any inverse function

Suppose that f is a one-to-one and onto function. For each $y \in B$ (codomain) there is exactly one element $x \in A$ (domain) such that $f(x) = y$. The **inverse function** is defined as $f^{-1}(y) = x$

In order to find the derivative of an inverse function we need to establish the result that

$$\frac{dy}{dx} = \frac{1}{{}^{dx}/_{dy}}$$

Consider $y = f^{-1}(x)$ where $f^{-1}(x)$ denotes the inverse function (but not $^{1}/_{f(x)}$).

Then we have $f(y) = x$

Can we find $^{dy}/_{dx}$?

If we differentiate $f(y) = x$ with respect to x then we have

$$\frac{d}{dx}f(y)) = \frac{d}{dx}(x) = 1$$

Now from the chain rule we have

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dy}\left(f(y)\right)\frac{dy}{dx}$$

So we can rewrite

$$\frac{d}{dx}(f(y)) = \frac{d}{dx}(x) = 1$$

$$\text{as}\quad \frac{d}{dy}(f(y)) \times \frac{dy}{dx} = 1$$

Now, since $f(y) = x$

we can now write

$\frac{dx}{dy} \times \frac{dy}{dx} = 1$

thus

$\frac{dy}{dx} = \frac{1}{dx/dy}$

It may be useful to have this result written in the following alternative notation with $y = f^{-1}(x)$

Notice then dy/dx becomes $d/dx\,(f^{-1}(x))$

and for $x = f(y)$, $dx/dy = f'(y)$

therefore we have

$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(y)}$

Some examples may help to make this clearer.

Examples

1. Consider the function $f(x) = x^3$, it has inverse $f^{-1}(x) = x^{1/3}$

Find the derivative of this inverse function.

Answer

Let $y = f^{-1}(x) = x^{1/3}$

then $f(y) = y^3 = x$

and $f'(y) = 3y^2$

Now, using the rule for differentiating an inverse function, we have

$$\begin{aligned}\frac{d}{dx}(f^{-1}(x)) &= \frac{1}{f'(y)}\\ &= \frac{1}{3y^2}\end{aligned}$$

Since $y = x^{1/3}$ this last equation can be rewritten in terms of x as

$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{3\left(x^{1/3}\right)^2} = \frac{1}{3x^{2/3}}$

Notice that this is the same result as we would obtain from calculating

$\frac{d}{dx}(x^{1/3}) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

so our rule works!

2. A function is defined as $f(x) = 3e^{2x} + 4$

Find the derivative of the inverse function.

Solution

Let $y = f^{-1}(x)$

then $f(y) = 3e^{2y} + 4 = x$

and $f'(y) = 6e^{2y}$

Again, using the rule for differentiating an inverse function, we have

$$\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(y)} = \frac{1}{6e^{2y}} = \frac{1}{2x - 8}$$

3.

A function is defined as $f(x) = x^3 + 4x + 2$

If $y = f^{-1}(x)$, find the derivative of the inverse, giving your answer in terms of y

Also find the equation of the tangent to the inverse at the point (7,1).

Solution

We define $y = f^{-1}(x)$

then $f(y) = y^3 + 4y + 2$ in terms of y

Now $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(y)} = \frac{1}{3y^2 + 4}$

At the point (7, 1) the gradient of the tangent to the inverse is

$\frac{1}{3 \times 1^2 + 4} = \frac{1}{7}$

The equation of the tangent line is then

$y - b = m(x - a)$

$y - 1 = {}^1/_7 (x - 7)$

$7y - 7 = x - 7$

$x - 7y = 0$

The following strategy may help you with the next exercise

1. Let $y = f^{-1}(x)$ and therefore $x = f(y)$
2. Find $f'(y) = {}^{dx}/_{dy}$ by differentiating with respect to y
3. To find the derivative of the inverse function use the formula ${}^{dy}/_{dx} = 1 \div {}^{dx}/_{dy}$
4. Eliminate y from your answer if required.

Now try the questions in Exercise 1.

Exercise 1

15 min

An on-line assessment is available at this point, which you might find helpful.

Q5: Find in terms of x the derivative of $f^{-1}(x)$ when

a) $f(x) = 5e^{3x}$
b) $f(x) = 2\ln(x) + 6$
c) $f(x) = \ln(3x + 1)$

Q6: Find ${}^{dy}/_{dx}$ when $x = \exp(y^2)$

Give your answer in terms of x

Q7: A function is defined as $f(x) = x^3 + 3x - 4$

a) If $y = f^{-1}(x)$, find the derivative of the inverse in terms of y
b) Calculate the equation of the tangent to the inverse at the point (10, 2)

Q8: A function is defined as $f(x) = 5x + \sin(3x)$

a) Find the derivative of the inverse in terms of y
b) Find the equation of the tangent to the inverse at the point (0, 0)

1.1.2 Differentiating inverse trigonometrical functions

Learning Objective

Calculate derivatives of the inverse sine, cosine and tangent functions

The derivative of the inverse sine function

For $x \in (-1, 1)$, $\sin^{-1} x$ is differentiable.

You should recall that $\sin^{-1} x$ is the inverse sine function, sometimes denoted as arcsin.

(See Unit1, Properites of Functions, for a reminder of this function).

Let $y = \sin^{-1} x$ then $x = \sin y$ (*)

Differentiating $x = \sin y$ with respect to y gives ${}^{dx}/_{dy} = \cos y$

so $\frac{dy}{dx} = \frac{1}{\cos y}$ $\left(\text{since } \frac{dy}{dx} = \frac{1}{{}^{dx}/_{dy}} \text{ as we saw earlier}\right)$

Now since $\sin^2 y + \cos^2 y = 1$

then $\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{(1 - \sin^2 y)} = \sqrt{(1 - x^2)}$ from (*)

(Note that we only consider the positive value of the square root in the above calculation because when $y = \sin^{-1}(x)$ for $x \in (-1, 1)$ then $-\pi/2 \leq y \leq \pi/2$ and between these values $\cos y \geq 0$

So we can write that

When $y = \sin^{-1}x$

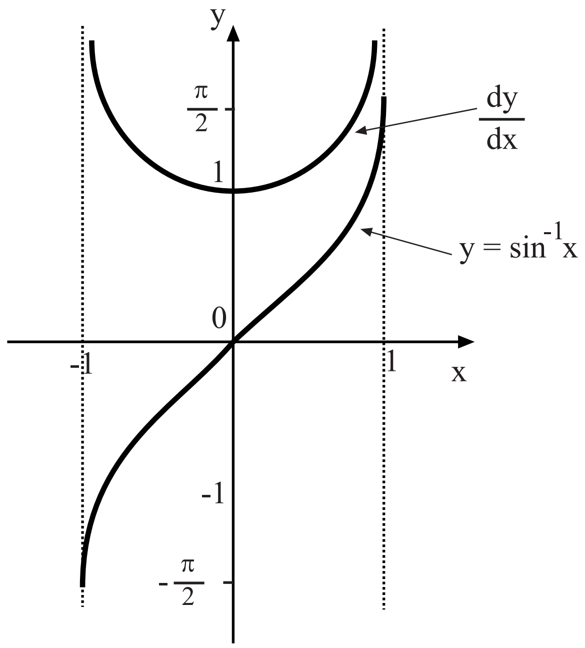
then $\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$

The graph for $y = \sin^{-1} x$ is shown here along with the graph of its derivative

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1 - x^2}}$$

(See Unit1, Properties of Functions, for a reminder of this.)



Notice that $^{dy}/_{dx} = 1$ at $x = 0$ and so the gradient of the curve $y = \sin^{-1} x$ is equal to 1 at $x = 0$

Also note that as $x \Rightarrow \pm 1$ then $^{dy}/_{dx} \Rightarrow \infty$

The derivative of the inverse cosine and tangent functions

In a similar way to the above it can be shown that

If $y = \cos^{-1}x$

then $\frac{dy}{dx} = \frac{-1}{\sqrt{1 - x^2}}$

If $y = \tan^{-1} x$

then $\frac{dy}{dx} = \frac{1}{1 + x^2}$

You might like to attempt to prove this for yourself.

Q9: Make a sketch of the curve $y = \cos^{-1} x$ for $x \in (-1, 1)$

By examining the gradient of the curve, on the same diagram sketch $^{dy}/_{dx}$

Q10: Make a sketch of the curve $y = \tan^{-1} x$

By examining the gradient of the curve, on the same diagram sketch $^{dy}/_{dx}$

Example Find $^{dy}/_{dx}$ when $y = \sin^{-1}(^x/_3)$

Answer

We need to use the chain rule to perform this differentiation.

Let $u = {}^{x}/_{3}$ then $y = \sin^{-1} u$

and so $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \frac{1}{\sqrt{1 - u^2}} \times \frac{1}{3}$$

$$= \frac{1}{3\sqrt{1 - \left(\frac{x}{3}\right)^2}}$$

$$= \frac{1}{\sqrt{3^2 - x^2}}$$

We can generalise the above result to give

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{a^2 - x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$$

You might like to attempt to prove this for yourself.

Example Find ${}^{dy}/_{dx}$ when $y = \cos^{-1}(e^x)$

Answer

Again, we need to use the chain rule to perform this differentiation.

Let $u = e^x$ then $y = \cos^{-1} u$

and so

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{-1}{\sqrt{1 - u^2}} \times e^x$$

$$= \frac{-e^x}{\sqrt{1 - e^{2x}}}$$

Now try the questions in Exercise 2.

25 min

Exercise 2

An on-line assessment is available at this point, which you might find helpful.

These new rules can be combined with rules for differentiating other functions, along with the chain rule, the product rule and the quotient rule. Bearing this in mind, differentiate the following functions:

Q11: $y = \tan({}^{x}/_{5})$

Q12: $y = \cos^{-1} 4x$

Q13: $y = \tan^{-1} 3x$

Q14: $y = \cos^{-1}(2x - 1)$

Q15: $y = \tan^{-1}(1 + 3x)$

Q16: $y = \cos^{-1}(1 - x^2)$

Q17: $y = x \tan^{-1}x$

Q18: $y = \tan^{-1}(\exp(x))$

Q19: $y = \tan^{-1}(3 \cos x)$

Q20: $y = \tan^{-1}(\sqrt{x - 1})$

Q21: $y = \sin^{-1}(\ln 3x)$

Q22: $y = \frac{\tan^{-1}x}{1 + x^2}$

1.2 Implicit differentiation

1.2.1 Implicit or explicit?

Learning Objective

Distinguish between implicitly and explicitly defined functions

It is easier to understand what an implicit function is if we first consider what is meant by an explicit function.

Consider two variables x and y.

For two variables x and y, y is an **explicit function** of x if it is a clearly defined function of x

This means that we can write y as an expression in which the only variable is x and we obtain only one value for y

Thus y = 3x = 1 and y = sin (2x - 3) are examples of y as an explicit function of x.

Q23: By rearranging the equation $xy = 4x^3 - x^2 + y$ show that y can be expressed as a clearly defined function of x (make y the subject of the equation)

For two variables x and y, y is an **implicit function** of x if it is not an explicit or clearly defined function of x. However, if x is given any value then the corresponding value of y can be found.

These are examples of implicit functions:

$y^2 + 2xy - 5x^2 = 0$

$x^2 + y^2 = 1$

The second equation gives the equation of the unit circle, centre the origin and radius 1.

$x^2 + y^2 = 1 \Leftrightarrow y^2 = 1 - x^2$

$$y = \pm\sqrt{1 - x^2}$$

Notice that this gives y as two functions defined on the interval [-1, 1]

Now try the questions in Exercise 3.

10 min

Exercise 3

Identify the following functions as either implicit or explicit. If the function is explicit then rewrite y as a clearly defined function of x.

Q24: $y^2 - 2xy = 6x$

Q25: $xy + 2y = 3x$

Q26: $\ln x = ye^x$

Q27: $8y = 3x^2 + 3y^2 + 6x + 10$

Q28: $x^2 - \sin y + xy = 0$

Q29: $x^2 y = \sin 2x$

1.2.2 Differentiating implicit equations

Learning Objective

Learn the method used to differentiate implicit equations

Given an implicit equation in variables x and y we may wish to find $^{dy}/_{dx}$. In order to differentiate implicit equations we need to use the chain rule. The following examples may help you to understand this point.

Example Find $^{dy}/_{dx}$ in terms of x and y when $y^2 - 3x = 4x^2$

Answer

We differentiate both sides of this implicit equation term by term

Notice that we cannot differentiate y^2 directly with respect to x

So, using the chain rule we rewrite

$\frac{d}{dx}(y^2)$ as $\frac{d}{dy}(y^2)\frac{dy}{dx}$

then

$$\frac{d}{dx}(y^2 - 3x) = \frac{d}{dx}(4x^2)$$
$$\frac{d}{dx}(y^2) - \frac{d}{dx}(3x) = \frac{d}{dx}(4x^2)$$
$$\text{and } \frac{d}{dy}(y^2)\frac{dy}{dx} - \frac{d}{dx}(3x) = \frac{d}{dx}(4x^2)$$
$$2y\frac{dy}{dx} - 3 = 8x$$
$$\frac{dy}{dx} = \frac{8x + 3}{2y}$$

When an implicit equation has an xy term then we need to use the product rule

$$\text{So that } \frac{d}{dx}(xy) = \frac{d}{dx}(x)\, y + x\frac{d}{dx}(y)$$
$$= 1y + x\frac{dy}{dx}$$
$$= y + x\frac{dy}{dx}$$

This result is used in the following example.

Example

Find $^{dy}/_{dx}$ in terms of x and y when $3xy + 5y^2 - x^2 = 2y$

Answer

Again, we differentiate this implicit function term by term.

$$\frac{d}{dx}\left(3xy + 5y^2 - x^2\right) = \frac{d}{dx}(2y)$$
$$\frac{d}{dx}(3xy) + \frac{d}{dx}(5y^2) - \frac{d}{dx}(x^2) = \frac{d}{dx}(2y)$$
$$\frac{d}{dx}(3x)y + 3x\frac{dy}{dx} + \frac{d}{dy}(5y^2)\frac{dy}{dx} - \frac{d}{dx}(x^2) = 2\frac{dy}{dx}$$
$$3y + 3x\frac{dy}{dx} + 10y\frac{dy}{dx} - 2x = 2\frac{dy}{dx}$$
$$(3x + 10y - 2)\frac{dy}{dx} = 2x - 3y$$
$$\frac{dy}{dx} = \frac{2x - 3y}{3x + 10y - 2}$$

An implicit equation may include a trigonometric function.

Example Find $^{dy}/_{dx}$ for the function of x implicitly defined by $\sin(2x + 3y) = 7x$

Answer

Differentiating both sides of the equation with respect to x gives

$$\begin{aligned}
\frac{d}{dx}(\sin(2x + 3y)) &= \frac{d}{dx}(7x) \\
\cos(2x + 3y) \times \frac{d}{dx}(2x + 3y) &= 7 \\
\cos(2x + 3y) \times \left(\frac{d}{dx}(2x) + \frac{d}{dy}(3y)\frac{dy}{dx}\right) &= 7 \\
\cos(2x + 3y) \times \left(2 + 3\frac{dy}{dx}\right) &= 7 \\
2 \cos(2x + 3y) + 3 \cos(2x + 3y)\frac{dy}{dx} &= 7 \\
3 \cos(2x + 3y)\frac{dy}{dx} &= 7 - 2 \cos(2x + 3y) \\
\frac{dy}{dx} &= \frac{7 - 2 \cos(2x + 3y)}{3 \cos(2x + 3y)}
\end{aligned}$$

Now try the questions in exercise 4.

Exercise 4

20 min

An on-line assessment is available at this point, which you might find helpful.

For each of the following questions, find $^{dy}/_{dx}$ in terms of x and y

Q30: $y^2 - x^2 = 12$

Q31: $3x^2 + y^2 = 5x$

Q32: $5x^2 - 8x = 3y^2 + 2y$

Q33: $3x^2 - 3y^2 + 5y - 8x = 10$

Q34: $x^2 + xy + y^2 = 7$

Q35: $3 - 4x + 6y + 5xy = 0$

Q36: $x^3 + 10y^2 = 3xy + 4$

Q37: $x^2 - 3y^2 + 2x^2y + 5xy^2 = 6y - 9$

Q38: $3y = 2x^3 + \cos y$

Q39: $x \tan y = 2x - 3y$

Q40: $e^{\cos x} + e^{\sin y} = 2$

Q41: $4y^2 - \cos(3x + 2y) = 0$

1.2.3 Inverse trigonometric functions

Learning Objective

Be able to differentiate trigonometrical functions using implicit differentiation

We have already obtained results for differentiating $\sin^{-1} x$, $\cos^{-1} x$ and $\tan^{-1} x$

It is also possible to differentiate these functions as implicit equations. See the following examples.

Examples

1. Find $^{dy}/_{dx}$ when $y = \sin^{-1} x$ for $x \in (-1, 1)$

Answer

When $y = \sin^{-1} x$ then $\sin y = x$ (*)

Now, differentiating this equation implicitly, we obtain:

$$\frac{d}{dy}(\sin y)\frac{dy}{dx} = \frac{d}{dx}(x)$$

$$\cos y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Again, since $\sin^2 y + \cos^2 y = 1$

Then $\cos^2 y = 1 - \sin^2 y$

$$\cos y = \sqrt{1 - \sin^2 y}$$

$$= \sqrt{1 - x^2} \quad \text{from (*)}$$

(since $-\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$ we only consider the positive root.)

So we can write that when $y = \sin^{-1} x$ then

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - x^2}}$$

2. Find $^{dy}/_{dx}$ when $y = \tan^{x}/_{a})$

The following identity, which you might like to prove for yourself, will help

$1 + \tan^2 y = \sec^2 y$

Answer

When $y = \tan^{-1} (^{x}/_{a})$

then $\tan y = {^{x}/_{a}}$ (**)

Differentiating this equation implicitly we obtain

$$\frac{d}{dy}(\tan y)\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{a}\right)$$

$$\sec^2 y\frac{dy}{dx} = \frac{1}{a}$$

$$\frac{dy}{dx} = \frac{1}{a\sec^2 y}$$

$$= \frac{1}{a(1+\tan^2 y)} \quad (\text{from } 1+\tan^2 y = \sec^2 y)$$

$$= \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} \quad (\text{from } (**))$$

$$= \frac{a}{a^2+x^2}$$

So we can write that when $y = \tan^{-1}\left(\frac{x}{a}\right)$

then $\frac{dy}{dx} = \frac{1}{a\sec^2 y} = \frac{a}{a^2+x^2}$

Notice that these are the same results as stated previously in the section on inverse functions.

Using implicit differentiation, find the derivatives of the following inverse trigonometric functions. Show all your working.

Q42: $y = \cos^{-1}(3x)$

Q43: $y = \sin^{-1}\left(\frac{3x^2}{2}\right)$

1.2.4 Tangents

Learning Objective

Use implicit differentiation to find the equation of the tangent to a curve

Implicit differentiation may be needed when finding the equation of a tangent to a curve. Look at this example.

Example A circle has equation $(x - 2)^2 + (y - 3)^2 = 25$ which expands to give

$x^2 - 4x + y^2 - 6y = 12$.

The point (5, 7) lies on the circumference of this circle.

Find the equation of the tangent to the circle at this point.

Answer

The tangent is given by the straight line formula

$y - b = m(x - a)$

$y - 7 = m(x - 5)$

The gradient of the tangent, m, is given by $^{dy}/_{dx}$ at (5, 7). See how this is calculated below

$$\frac{d}{dx}(x^2 - 4x + y^2 - 6y) = \frac{d}{dx}(12)$$

$$2x - 4 + 2y\frac{dy}{dx} - 6\frac{dy}{dx} = 0$$

$$(2y - 6)\frac{dy}{dx} = 4 - 2x$$

$$\frac{dy}{dx} = \frac{4 - 2x}{2y - 6}$$

$$= \frac{2 - x}{y - 3}$$

Now substituting in (5, 7) for x and y

we get $^{dy}/_{dx} = -^3/_4$

this gives the gradient of the tangent to the circle at (5, 7)

We can now write the equation as

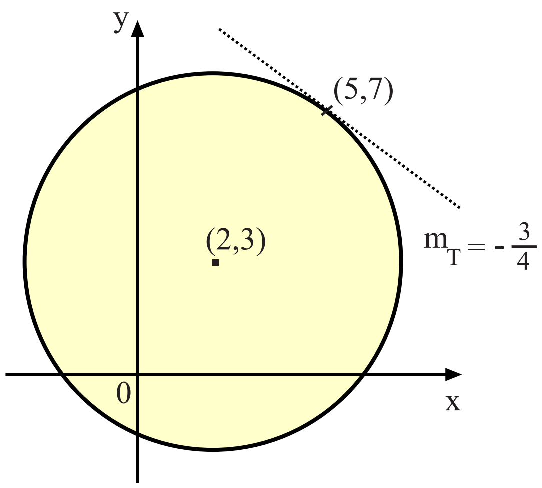
$y - b = m(x - a)$

$y - 7 = -^3/_4\ (x - 5)$

$4y - 28 = -3x + 15$

So $3x + 4y - 43 = 0$ is the equation of the tangent to the circle at the point (5, 7)

There is an alternative method to solve this type of question, that involves using the properties of a circle.



Now try the questions in Exercise 5.

Exercise 5

An on-line assessment is available at this point, which you might find helpful.

20 min

For these questions, find:

a) the gradient of each curve at the given point;

b) the equation of the tangent to the curve at that point.

Q44: $3x^2 + 5y^2 = 17$ at (-2, 1)

Q45: $x^2y^2 = 16$ at (1, 4)

Q46: $2x \sin y + x^2 = 1$ at $(1, \pi)$

Q47: $x^2 + 4xy - y^2 = 16$ at (2, 2)

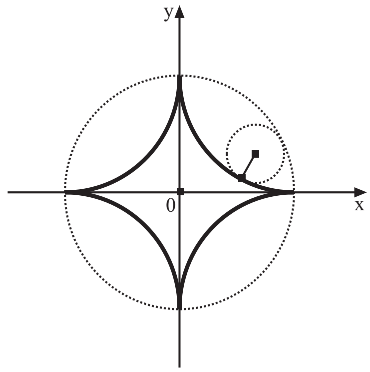
Q48: $y^2 = x^3 - 2xy + 9$ at (1, -4)

Q49: $\tan^{-1}y + 3y = 2x^2 - 8$ at (2, 0)

Q50: Astroid

$x^{2/3} + y^{2/3} = 2^{2/3}$ defines a famous curve called the astroid.

This curve is drawn in an interesting way. Imagine that a circle of radius 1/2 is rolling inside another circle of radius 2. The astroid is the curve traced out by a point on the circumference of the smaller circle.



Implicit differentiation can help us to understand the shape of the astroid.

a) Calculate $^{dy}/_{dx}$ for $x^{2/3} + y^{2/3} = 2^{2/3}$

b) Determine the gradient of the tangent to the curve at the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Write down the coordinates of the four points on the curve of the astroid where the derivative cannot be defined.

Astroid

10 min

Learning Objective

Analyse the astroid curve by implicit differentiation

There is an activity on the web that you may like to try at this point.

1.2.5 The second derivative

Learning Objective

Be able to find the second derivative for an implicitly defined function

Like regular differentiation, we can also use implicit differentation to find the second derivative.

The **second derivative** of y with respect to x is obtained by differentiating twice.

We write $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Study the following example.

Example Find $\frac{d^2y}{dx^2}$ when 3x - 6y + xy = 10

We have to calculate $^{dy}/_{dx}$ first before we can find $\frac{d^2y}{dx^2}$

$$\frac{d}{dx}(3x - 6y + xy) = \frac{d}{dx}(10)$$
$$\frac{d}{dx}(3x) - \frac{d}{dx}(6y) + \frac{d}{dx}(xy) = \frac{d}{dx}(10)$$
$$\frac{d}{dx}(3x) - \frac{d}{dy}(6y)\frac{dy}{dx} + \frac{d}{dx}(x)y + x\frac{d}{dy}(y)\frac{dy}{dx} = \frac{d}{dx}(10)$$
$$3 - 6\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$$
$$(x - 6)\frac{dy}{dx} = - 3 - y$$
$$\frac{dy}{dx} = \frac{- 3 - y}{x - 6}$$

Now to find $\frac{d^2y}{dx^2}^{-1}$ it is easier if we write

$\frac{dy}{dx} = \frac{- 3 - y}{x - 6}$ as

$\frac{dy}{dx} = (- 3 - y)(x - 6)^{-1}$ t

Use the product rule on the expression $(-3 - y)(x - 6)^{-1}$

with f = (-3 - y) then $f' = -^{dy}/_{dx}$

with $g = (x - 6)^{-1}$ then $g' = -(x - 6)^{-2}$

and remember that $^{d}/_{dx}(fg) = f'g + fg'$

Then we can write

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left((- 3 - y)(x - 6)^{-1}\right)$$
$$\frac{d^2y}{dx^2} = - \frac{dy}{dx}(x - 6)^{-1} - (- 3 - y)(x - 6)^{-2}$$
$$= \frac{3 + y}{(x - 6)^2} + \frac{3 + y}{(x - 6)^2}$$
$$= \frac{6 + 2y}{(x - 6)^2}$$

Note that in the second last line $\frac{- 3 - y}{x - 6}$ was substituted for $^{dy}/_{dx}$

Now try the questions in Exercise 6.

25 min

Exercise 6

An on-line assessment is available at this point, which you might find helpful.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when:

Q51: $3x^2 - 4y^2 = 12$

Q52: $y^2 - x^2 = 2x$

Q53: $x^2 + 2xy + y^2 = 1$

Q54: $xy + y^2 = 1$

1.2.6 Logarithmic differentiation

Learning Objective

Be able to apply logarithmic differentiation to appropriate functions

Sometimes, taking the natural logarithms of an expression makes it easier to differentiate. This can be the case for extended products and quotients or when the variable appears as a power.

natural logarithm

The examples shown here may make this clearer.

Examples

1. Find $^{dy}/_{dx}$ when $y = 5^x$

Answer

We take **natural logarithms** of both sides of the equation so that

$\ln y = \ln 5^x$

$\ln y = x \ln 5$

Now using implicit differentiation we differentiate both sides of the equation

$\frac{1}{y}\frac{dy}{dx} = \ln 5$ (ln5 is a constant)

The next step is to multiply both sides by y, then

$^{dy}/_{dx} = y \ln 5$

$= 5^x \ln 5$

So when $y = 5^x$ then $^{dy}/_{dx} = 5^x \ln 5$

2. Find $\frac{dy}{dx}$ when $y = \frac{x^3}{\sqrt{x+4}}$

Answer

We 'take logs' of both sides of the equation so that

$\ln(y) = \ln\left(\frac{x^3}{\sqrt{x+4}}\right)$

$\ln(y) = \ln(x^3) - \ln(x+4)^{1/2}$

$\ln(y) = 3\ln x - \frac{1}{2}\ln(x+4)$

Now using implicit differentiation we differentiate both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} - \frac{1}{2(x+4)}$$
$$= \frac{5x+24}{2x(x+4)}$$

The next step is to multiply both sides by y, then

$$\frac{dy}{dx} = \frac{5x+24}{2x(x+4)} \times y$$
$$= \frac{5x+24}{2x(x+4)} \times \frac{x^3}{\sqrt{x+4}} \quad (y \text{ is replaced})$$
$$= \frac{x^2(5x+24)}{2(x+4)^{3/2}}$$

So when $y = \frac{x^3}{\sqrt{x+4}}$

then $\frac{dy}{dx} = \frac{x^2(5x+24)}{2(x+4)^{3/2}}$

Now try the questions in Exercise 7.

Exercise 7

25 min

An on-line assessment is available at this point, which you might find helpful.

For each of the following questions use logarithmic differentiation to find $^{dy}/_{dx}$ in terms of x

Q55: $y = 4^x$

Q56: $y = 3^{x^2}$

Q57: $y = 2x^x$

Q58: $y = \frac{(2x+3)^2}{\sqrt{x+1}}$

Q59: $y = \frac{2^x}{2x+1}$

Q60: $y = \sqrt{\frac{3+x}{3-x}}$

Q61: $y = (1 + x)(2 + 3x)(x - 5)$

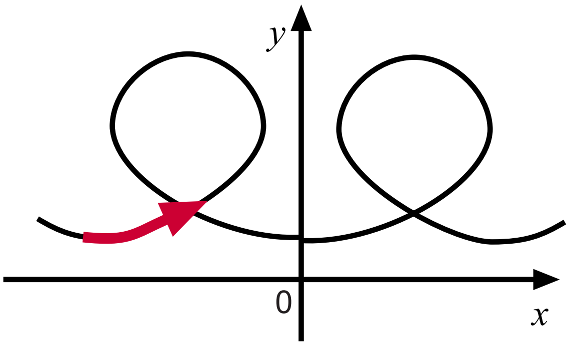
Q62: $y = \frac{x^2\sqrt{7x-3}}{1+x}$

1.3 Parametric equations

1.3.1 Parametric curves

Learning Objective

Be able to sketch a curve from parametric equations



When a curve is traced out over time, maybe sometimes crossing itself or doubling back on itself, then that curve cannot be described by expressing y directly in terms of x

The curve is not the graph of a function.

To cope with this difficulty we can describe each position along the curve at time t by

x = f (t)

y = g (t)

Equations like these are called parametric equations.

Parametric equations are equations that are expressed in terms of an independent variable.
They are of the form

x = f (t)

y = g (t)

where t is the independent variable.

for x and y and the variable t is called a parameter.

A **parameter** is a variable that is given a series of arbitrary values in order that a relationship between x and y may be established. The variable may denote time, angle or distance.

The parameter may not always represent time, it might instead denote an angle about the origin or a distance along the curve. Study the following examples in order to understand this more clearly.

Examples

1. A curve is defined by the following parametric equations

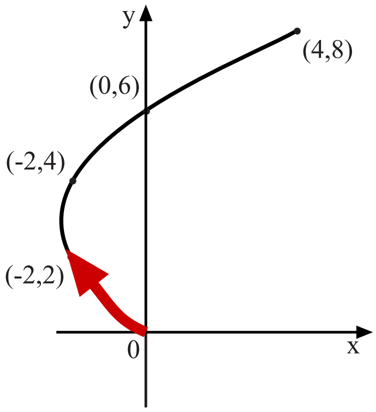
$x = t^2 - 3t$

$y = 2t$

Notice that we can find coordinates on the curve by giving the **parameter** *t* various values as in the following table.

t	0	1	2	3	4
x	0	- 2	- 2	0	4
y	0	2	4	6	8

This then allows us to plot the points (x, y) and make a sketch of the curve as shown here.



The arrow indicates direction or **orientation** of the curve as *t* increases.

Note that it is sometimes possible to eliminate *t* and obtain the **cartesian equation** of the curve.

In our example we have the parametric equations $x = t^2 - 3t$ and $y = 2t$

From $y = 2t$ we obtain $t = {}^{y}/_{2}$

Now, substituting this value of t into $x = t^2 - 3t$

$$x = \left(\frac{y}{2}\right)^2 - 3\left(\frac{y}{2}\right)$$

$$x = \frac{y^2}{4} - \frac{3y}{2}$$

$$4x = y^2 - 6y$$

Therefore $4x = y^2 - 6y$ is the cartesian equation of the curve.

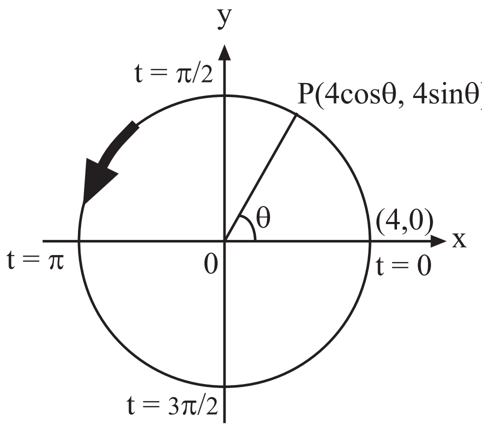
2. The parametric equations $x = 4\cos\theta$, $y = 4\sin\theta$, for $0 \leq \theta \leq 2\pi$ describe the position P (x, y) of a particle moving in the plane.

You can check that

$x^2 + y^2 = 16\cos^2\theta + 16\sin^2\theta = 16$

Thus the cartesian equation for the curve is $x^2 + y^2 = 16$, which is the equation of a circle centre the origin with radius 4.

The parameter θ is the radian measure of the angle that radius OP makes with positive x-axis.



We can find out the direction in which the particle is moving by calculating (x, y) coordinates for some values of θ.

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	4	0	- 4	0	4
y	0	4	0	- 4	0

From the table, we can see that as θ increases the particle moves anticlockwise around the circle starting and ending at (4, 0).

The **cartesian equation** of a curve is an equation which links the x-coordinate and the y-coordinate of the general point (x, y) on the curve.

Now try the questions in exercise 8.

Exercise 8

20 min

The following parametric equations give the position of a general point P (x, y) on a curve.

a) Identify the cartesian form of the curve.

b) Make a sketch of the curve and indicate the orientation of the curve. If appropriate indicate where the graph starts and stops.

Q63: $x = 2 \cos t$, $y = 2 \sin t$, for $0 \leq t \leq 2\pi$

Q64: $x = 2t$, $y = 4t^2$, for $-\infty < t < \infty$

Q65: $x = 2t - 3$, $y = 4t - 3$, for $-\infty < t < \infty$

Q66: $x = 3 \cos t$, $y = -3 \sin t$, for $0 \leq t \leq 2\pi$

Q67: $x = \cos(\pi - t)$, $y = \sin(\pi - t)$, for $0 \leq t \leq \pi$

Q68: $x = t$, $y = \sqrt{(1 - t^2)}$, for $0 \leq t \leq 1$

Q69: $x = 2t$, $y = 3 - 3t$, for $0 \leq t \leq 1$

Q70: $x = \cos^2 t$, $y = \sin^2 t$, for $0 \leq t \leq \pi/2$

1.3.2 Parametric differentiation

Learning Objective

Differentiate parametric equations

When a curve is determined by parametric equations we may wish to gain more information about the curve by calculating its derivative.

We can do this directly from the parametric equations.

Given that x = f (t) and y = g (t) then by the chain rule $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

Now using the fact that $\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}}$ we can see that $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{1}{\frac{dx}{dt}} = \frac{dy/dt}{dx/dt}$

and we can rewrite the formula for the derivative as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Study the following examples to see how this works.

Examples

1.

Find dy/dx in terms of the parameter t when $x = t^2 + 6$ and $y = 4t^3$

Answer

When $x = t^2 + 6$ then $dx/dt = 2t$

When $y = 4t^3$ then $dy/dt = 12t^2$

Now using the formula for parametric differentiation we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12t^2}{2t} = 6t$$

2. Find dy/dx in terms of the parameter t when $x = 2/t$ and $y = \sqrt{(t^2 - 3)}$

Answer

When $x = \frac{2}{t} = 2t^{-1}$

then $\frac{dx}{dt} = -2t^{-2} = \frac{-2}{t^2}$

When $y = \sqrt{t^2 - 3} = (t^2 - 3)^{1/2}$

then $\frac{dy}{dt} = \frac{1}{2}(t^2 - 3)^{-1/2}2t = \frac{t}{(t^2 - 3)^{1/2}}$

Now using the formula for parametric differentiation we have

$$\frac{dy}{dx} = \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}}$$

$$= \frac{t}{(t^2 - 3)^{1/2}} \Bigg/ \frac{-2}{t^2}$$

$$= \frac{t}{(t^2 - 3)^{1/2}} \times \frac{t^2}{-2}$$

$$= \frac{-t^3}{2\sqrt{t^2 - 3}}$$

Now try the questions in Exercise 9.

20 min

Exercise 9

An on-line assessment is available at this point, which you might find helpful.

In the following questions x and y are given in terms of the parameter t.

Find ${}^{dy}/_{dx}$ in terms of t

Q71: $x = 6t^2$, $y = t^3$

Q72: $x = 4 \cos t$, $y = 4 \sin t$

Q73: $x = 3t^2 + 4t - 5$, $y = t^3 + t^2$

Q74: $x = {}^{2}/_{t + 1}$, $y = {}^{4}/_{t - 1}$

In the following two questions find ${}^{dy}/_{dx}$ in terms of x

Q75: $x = 4 - 3t$, $y = {}^{3}/_{t}$

Q76: $x = \sqrt{t}$, $y = t^2 - 3$

1.3.3 Motion in a plane

Learning Objective

Be able to calculate velocity components, speed and the instantaneous direction of motion

When the position of a particle moving in a plane is given as the parametric equations x = f (t) and y = g (t) then the velocity of the particle can be split into components.

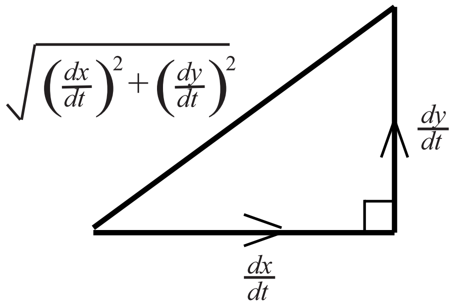
The **horizontal velocity component** is given by ${}^{dx}/_{dt}$

The **vertical velocity component** is given by ${}^{dy}/_{dt}$

The **speed** of the particle is then

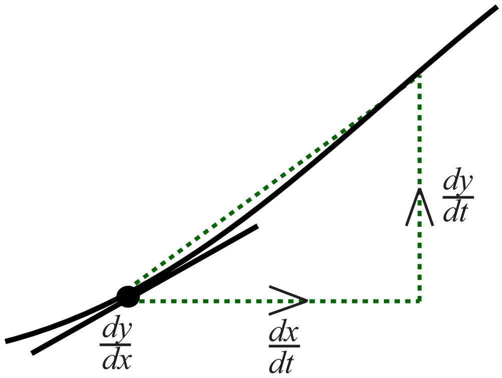
$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

(See diagram.)



The **instantaneous direction of motion** for the particle is given by $^{dy}/_{dx}$

(This concept is similar to the idea of the gradient of a curve at a point. See diagram.)



Now try the questions in Exercise 10.

Exercise 10

An on-line assessment is available at this point, which you might find helpful.

10 min

Q77: A particle is moving along a path determined by the parametric equations

x = 4t - 1, y = t^2 + 3t, where t represents time in seconds and distance is measured in metres.

a) How far is the particle from the origin at t = 0?

b) Calculate in terms of t
 - i the horizontal velocity component;
 - ii the vertical velocity component for the movement of the particle.

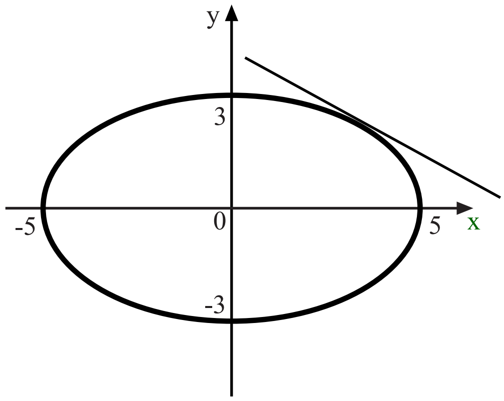
c) At t = 2, calculate:
 1. the horizontal velocity;
 2. the vertical velocity;
 3. the speed.

d)
 - i Derive a formula for the instantaneous direction of motion of the particle.
 - ii In which direction is the particle moving at t = 2? Give your answer in degrees measured relative to the positive direction of the x-axis.

Q78:

A particle is moving along the path of an ellipse as shown here. The equation of the ellipse is given by the parametric equations

x = 5 cos t, y = 3 sin t



a) Calculate the speed of the particle when t = $\pi/4$

b) What is the instantaneous direction of the particle at t = $\pi/4$?

1.3.4 Tangents

Learning Objective

Be able to calculate the tangent to a curve from parametric equations

When a curve is determined by a set of parametric equations it is still possible to find the equation of a tangent to the curve.

Consider the following example.

Example A curve is defined by the parametric equations $x = t^2 - 3$, $y = 2t^3$

Find the equation of the tangent to this curve at the point where t = 1

Answer

When $x = t^2 - 3$ then ${}^{dx}/_{dt} = 2t$

When $y = 2t^2$ then ${}^{dy}/_{dt} = 6t^2$

Now, using the formula for differentiating parametric equations, we have

$$\frac{dy}{dx} = \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}}$$
$$= \frac{6t^2}{2t}$$
$$= 3t$$

Therefore the gradient of the tangent at t = 1 is 3

We can calculate the corresponding (x, y) coordinates at t = 1 by substituting into the original parametric equations

$x = t^2 - 3 = -2$

$y = 2t^2 = 2$

We now use the straight line formula y - b = m(x - a) to give the equation of the straight line as

$y - 2 = 3\,(x + 2)$

$y = 3x + 8$

Now try the questions in exercise 11.

Exercise 11

15 min

An on-line assessment is available at this point, which you might find helpful.

The following parametric equations describe curves in the plane. Find the equation of the tangent at the given point on the curve.

Q79: $x = {}^{1}/_{(t + 1)}$, $y = 4t$, at $t = 1$

Q80: $x = \sqrt{(t^2 + 5)}$, $y = (t - 3)^2$, at $t = 2$

Q81: $x = 5 \cos t$, $y = 3 \sin t$, at $t = \pi/4$

Q82: $x = \sqrt{(2t + 5)}$, $y = (4t)^{1/3}$ at (3, 2)

Q83: Find the coordinates of the stationary point on the curve given by the parametric equations, $x = (t^2 + 3)^2$, $y = 4t^2 - 8t$

1.3.5 The second derivative

Learning Objective

Be able to calculate $\frac{d^2y}{dx^2}$ from parametric equations

We have seen previously that the **second derivative** of y with respect to x is obtained by differentiating twice and that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

We use this same method when dealing with **parametric equations**.

Make sure that you understand the following example.

Example Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t given that $x = \frac{2}{t}$ and $y = 2t^3 + 1$

Answer

When $x = \dfrac{2}{t} = 2t^{-1}$ then $\dfrac{dx}{dt} = -2t^{-2} = \dfrac{-2}{t^2}$

When $y = 2t^3 + 1$ then $\dfrac{dy}{dt} = 6t^2$

Now, using the formula for differentiating parametric equations, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}}\\ &= 6t^2 \Big/ \frac{-2}{t^2}\\ &= 6t^2 \times \frac{t^2}{-2}\\ &= -3t^4\end{aligned}$$

To find the second derivative we now need to differentiate ${}^{dy}/_{dx}$ with respect to x

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-3t^4\right)$$

We need to use the chain rule so that we can differentiate $-3t^4$ with respect to x.

$$\begin{aligned}\text{Therefore } \frac{d^2y}{dx^2} &= \frac{d}{dx}\left(-3t^4\right)\\ &= \frac{d}{dt}(-3t^4) \times \frac{dt}{dx}\\ &= \frac{d}{dt}(-3t^4) \Big/ \frac{dx}{dt} \quad \left(\text{since } \frac{dt}{dx} = \frac{1}{{}^{dx}/_{dt}}\right)\\ &= -12t^3 \Big/ \frac{-2}{t^2}\\ &= -12t^3 \times \frac{t^2}{-2}\\ &= 6t^5\end{aligned}$$

This gives us ${}^{dy}/_{dx} = -3t^2$ and

$$\frac{d^2y}{dx^2} = 6t^5$$

Notice that the method we used to find the second derivative was as follows

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\ &= \frac{d}{dt}\left(\frac{dy}{dx}\right) \times \frac{dt}{dx}\\ &= \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}\end{aligned}$$

Now try the questions in exercise 12.

15 min

Exercise 12

An on-line assessment is available at this point, which you might find helpful.

Find $\frac{d^2y}{dx^2}$ in terms of t for the following parametric equations

Q84: $x = t^2$, $y = {}^{4}/_{t}$

Q85: $x = t^2 + 4t$, $y = t^3 - 12t$

Q86: $x = 3 \cos t$, $y = 3 \sin t$

Q87: $x = 3t^2 + 2t$, $y = t - t^3$

Q88: Given that $\frac{d^2y}{dx^2} = 2t^2 - 1$ and $\frac{dy}{dx} = 2t^3 - 3t$ find $\frac{dx}{dt}$

1.4 Related rates

1.4.1 Explicitly defined related rates

Learning Objective

Be able to apply the chain rule to related rates problems

How fast does the water level rise when a cylindrical tank is filled at a rate of 2 litres/second?

This question is asking us to calculate the rate at which the **height** of water in the tank is increasing from the rate at which the **volume** of water in the tank is increasing. This is a related rates problem.

Solution of related rates problems can often be simplified to an application of the chain rule.

Consider the following example.

Example

If $V = {}^{4}/_{3}\ \pi r^3$ and the rate of change of r with respect to time is 2 (ie.$^{dr}/_{dt} = 2$)

then when r = 3 find the rate of change of V

Answer

We are asked to find $^{dV}/_{dt}$

By the chain rule we have $^{dV}/_{dt} = {}^{dV}/_{dr} \times {}^{dr}/_{dt}$

Now $^{dV}/_{dr} = 4\pi r^2$

therefore $^{dV}/_{dt} = {}^{dV}/_{dr} \times {}^{dr}/_{dt}$

$= 4\pi r^2 \times {}^{dr}/_{dt}$

so when $^{dr}/_{dt} = 2$ and r = 3 then $^{dV}/_{dt} = \pi \times 3^2 \times 2 = 72\pi$

Exercise 13

An on-line assessment is available at this point, which you might find helpful.

15 min

Q89: If $V = \pi r^2$ and we are given that $^{dr}/_{dt} = 3$, find $^{dV}/_{dt}$ when r = 2

Q90: If $V = {}^{1}/_{3}\pi r^2 h$ and $^{dh}/_{dt} = 6$, find $^{dV}/_{dt}$ when r = 4

Q91: If $B = t^2 + 4 \sin 3t$, find $^{dB}/_{dx}$ in terms of t, given that $^{dt}/_{dx} = {}^{1}/_{2}$

Q92: If $V = \pi r^2 h$ and we are given that $\frac{dr}{dt} = \frac{3}{\pi r}$, find an expression for $^{dV}/_{dt}$ in terms of h

Q93: Given that $^{dP}/_{dt} = 24h + 9$ and that $P = 4h^2 + 3h$, find $^{dh}/_{dt}$

Q94: If $A = (3t - 2)^4$ and $x = 3t^2 - 4t$, find an expression for $^{dA}/_{dx}$ in terms of t

1.4.2 Implicitly defined related rates

Learning Objective

Be able to apply the chain rule to an implicitly defined equation

We may also encounter related rates problems when dealing with implicitly defined functions.

First read the following example.

Example A point moves on the curve $2x^2 - y^2 = 2$ so that the y-coordinate increases at the constant rate of 12m/sec. That is $^{dy}/_{dt} = 12$

a) At what rate is the x-coordinate changing at the point (3, 4)?

b) What is the slope of the curve at the point (3, 4)?

Answer

a) We are given that $2x^2 - y^2 = 2$ and we are required to find $^{dx}/_{dt}$ at (3, 4)

Notice that x is implicitly defined in terms of y

Differentiate $2x^2 - y^2 = 2$ with respect to t and express $^{dx}/_{dt}$ in terms of $^{dy}/_{dt}$

This gives

$$\begin{aligned} \frac{d}{dt}(2x^2 - y^2) &= \frac{d}{dt}(2) \\ 4x\frac{dx}{dt} - 2y\frac{dy}{dt} &= 0 \\ 4x\frac{dx}{dt} &= 2y\frac{dy}{dt} \\ \frac{dx}{dt} &= \frac{2y}{4x}\frac{dy}{dt} \end{aligned}$$

Now we are given $^{dy}/_{dt} = 12$

$$\text{Therefore at (3, 4) } \frac{dx}{dt} = \frac{2y}{4x}\frac{dy}{dt} = \frac{8}{12} \times 12 = 8$$

b) The slope of the curve is given by $^{dy}/_{dx}$

At (3, 4),
$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{12}{8} = \frac{3}{2}$$

Now try the questions in Exercise 14.

Exercise 14

15 min

An on-line assessment is available at this point, which you might find helpful.

Q95: Given that $x^2 + y^2 = 5^2$, when x and y are functions of t, and ${}^{dx}/_{dt} = 9$, calculate ${}^{dy}/_{dt}$ at the point (4, 3)

Q96: A particle moves on the curve $x^2 - 4y^2 = 32$ with a vertical velocity component ${}^{dy}/_{dt} = 9$

a) Find ${}^{dx}/_{dt}$ at the point (6, -1)
b) What is the instantaneous direction of motion of the particle at (6, -1)? (Give your answer in degrees, relative to the positive direction of the x-axis.)

Q97: Given that $x^2 - 3xy + 4y^2 = 7$ and ${}^{dx}/_{dt} = 4$, calculate ${}^{dy}/_{dt}$ when y = 2

Q98: Given that $x^2 - 5xy - 3y^2 = -7$ and ${}^{dx}/_{dt} = 1$, calculate ${}^{dy}/_{dt}$ at the point (1, 1)

1.4.3 Related rates in practice

Learning Objective

Be able to solve problems involving related rates

Having learned some techniques for dealing with explicitly and implicitly defined related rates, we can now attempt some practical problems.

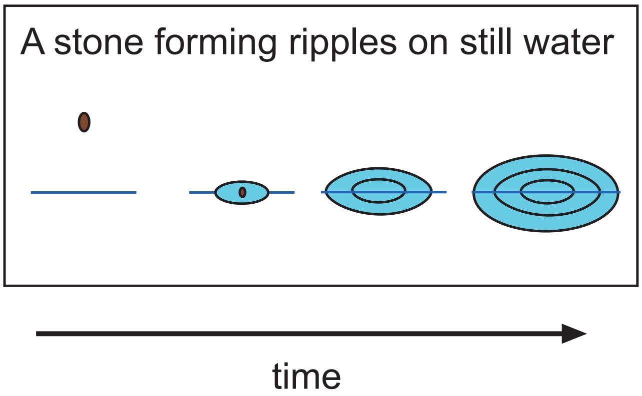
The following strategy may help you with these problems.

Strategy for solving related rates problems

1. Draw a diagram and label the variables and constants. Use *t* for time.
2. Write down any additional numerical information.
3. Write down what you are asked to find. Usually this is a rate and you should write this as a derivative.
4. Write down an equation that describes the relationship between the variables.
5. Differentiate (usually with respect to time).

Read through the following examples carefully .

Example When a pebble is dropped into a still pond, ripples move out from the point where the stone hits, in the form of concentric circles.

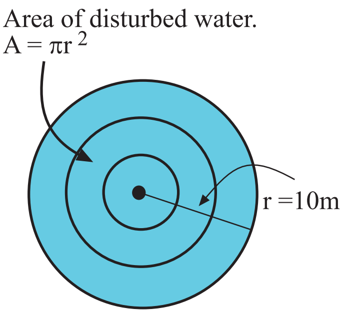


Find the rate at which the **area** of the disturbed water is increasing when the **radius** reaches 10 metres. The radius at this stage is increasing at a rate of 2m/s. i.e. $^{dr}/_{dt} = 2$ when r = 10

Answer

Try to follow the strategy as detailed previously.

1. Make a diagram if it helps.



Note that both the area, A, and the radius, r, change with time. We can think of A and r as differentiable functions of time and use t to represent time. The derivatives $^{dA}/_{dt}$ and $^{dr}/_{dt}$ give the rates at which A and r change.

2. Note that the area of the circle is $A = \pi r^2$ and that when r = 10 then $^{dr}/_{dt} = 2$

3. We are required to find $^{dA}/_{dt}$
4. The equation we can write to connect the variables is $^{dA}/_{dt} = {}^{dA}/_{dr} \times {}^{dr}/_{dt}$
5. Note that when A = πr^2 then $^{dA}/_{dr} = 2\pi r$

$$\begin{aligned} \text{Therefore } \frac{dA}{dt} &= \frac{dA}{dr} \times \frac{dr}{dt} \\ &= 2\pi r \times \frac{dr}{dt} \\ \text{when } r = 1 \quad &= 2\pi 10 \times 2 \\ &= 40\pi \end{aligned}$$

Therefore the area of disturbed water is increasing at a rate of 40π m^2/s when the radius of the outermost ripple reaches 10 metres.

Now try the questions in Exercise 15.

Exercise 15

20 min

The information shown here for various solids may be useful in the following questions.

	VOLUME	CURVED SURFACE AREA
SPHERE	$^4/_3\pi r^3$	$4\pi r^2$
CYLINDER	$\pi r^2 h$	$2\pi rh$
CONE	$^1/_3\pi r^2 h$	πrs

(For the cone, s represents the slant height.)

An on-line assessment is available at this point, which you might find helpful.

Q99: Let V be the volume of a sphere of radius r at time t. Write an equation that relates $^{dV}/_{dt}$ to $^{dr}/_{dt}$

Q100: Let S be the surface area of a cube whose edges have length x at time t. Write an equation that relates $^{dS}/_{dt}$ to $^{dx}/_{dt}$

Q101: If the velocity of a particle is given by $v = 2s^3 + 5s$, where s is the displacement of the particle from the origin at time t, find an expression for the acceleration of the particle in terms of s.

Q102: A snowball, in the shape of a sphere, is rolling down a hill and growing bigger as it gathers more snow. Because of this the radius is increasing at a uniform rate of

0.5 cm/s.

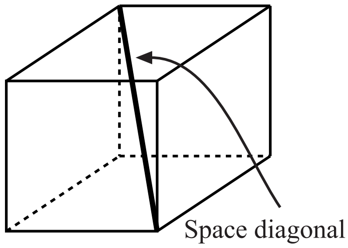
a) How fast is the volume increasing when the radius is 4 cm?

b) How fast is the surface area increasing when the radius is 4 cm?

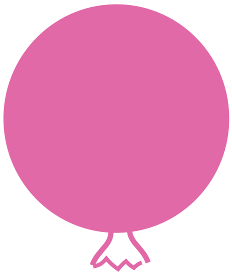
Q103:

A huge block of ice is in the shape of a cube.

At what rate is the space diagonal of the cube decreasing if the edges of the cube are melting at a rate of 2 cm/s?



Q104:

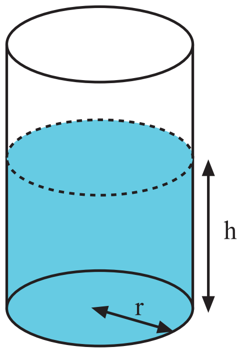


Air is being pumped into a spherical balloon at a rate of 63 cm^3/s. Find the rate at which the radius is increasing when the volume of the balloon is 36π cm^3

(Hint: you will need to calculate the radius of the balloon at the given volume.)

Q105:

How fast does the water level drop when a cylindrical tank is drained at the rate of 2 litres/second?

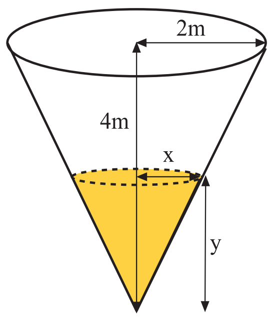


Some problems require a bit more working before a solution can be obtained. Try to understand the following example.

Example

Sand runs into a conical tank at a rate of 5π litres/s. The tank stands vertex down and has a height of 4m and a base radius of 2m.

Find the rate at which the depth of the sand is increasing when the sand is 1 metre deep.



Solution

The variables in the problem are:

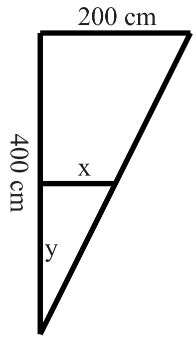
V = the volume of sand in the tank at time t

x = the radius of the surface of the sand at time t

y = the depth of the sand in the tank at time t

The constants are the dimensions of the tank and the rate at which the tank fills with sand = ${}^{dV}/_{dt}$

We are required to find ${}^{dy}/_{dt}$ when y = 1 metre = 100cm and we will be able to calculate this from ${}^{dV}/_{dt} = {}^{dV}/_{dy} \times {}^{dy}/_{dt}$



The volume of sand in the tank is given by the equation $V = {}^{1}/_{3}\pi x^2 y$. However, this last equation involves both the variables x and y and before we can find ${}^{dV}/_{dy}$ we should eliminate x.

We can do this by similar triangles

${}^{x}/_{200} = {}^{y}/_{400}$

$x = {}^{1}/_{2}y$

We can now rewrite V in terms of y alone

$$\begin{aligned} V &= \tfrac{1}{3}\pi\, x^2 y \\ &= \tfrac{1}{3}\pi \left(\tfrac{1}{2}y\right)^2 y \\ &= \tfrac{1}{12}\pi\, y^3 \end{aligned}$$

and we can calculate ${}^{dV}/_{dy} = {}^{1}/_{4}\, \pi y^2$

Now the sand is filling the cone at a rate of ${}^{dV}/_{dt} = 5\pi$ litres/s = $5000\,\pi$ cm^3/s

Therefore $\dfrac{dV}{dt} = \dfrac{dV}{dy} \times \dfrac{dy}{dt}$

becomes $5000\pi = \tfrac{1}{4}\pi y^2 \dfrac{dy}{dt}$

$$\frac{dy}{dt} = \frac{20000}{y^2}$$

and when y = 1 metre = 100 cm then $\dfrac{dy}{dt} = 2$ cm/s

When the sand is 1 metre deep in the tank the level is rising at a rate of 2 cm/s.

Now try the following questions.

Exercise 16

An on-line assessment is available at this point, which you might find helpful.

20 min

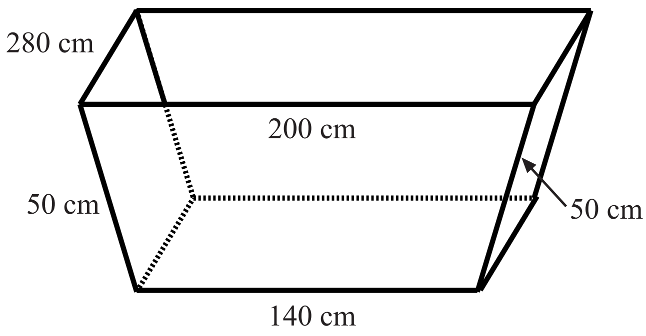
Q106: Water is withdrawn from a conical tank at a constant rate of 4 litres/minute. The tank has base radius 4 metres and height 6 metres and is positioned with its vertex down.

How fast is the water level falling when the depth of water in the tank is 2 metres?

Q107:

Water is pumped into an empty cattle trough at a rate of 77.7 litres/second. The dimensions of the trough are as shown in the diagram.

The cross section of the trough is a trapezium and the front and back have vertical sides.

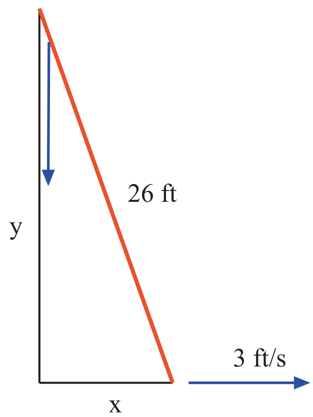


Find the rate at which the depth of the water is increasing when the water reaches a height of 30 cm.

Q108:

A ladder, 26 ft long, leans against a wall. The foot of the ladder is pulled away from the wall at a rate of 3 ft/second.

How fast is the top of the ladder sliding down when the foot is ten feet away from the bottom of the wall?



(Hint: You will need to write down an equation connecting x and y and then use implicit differentiation to solve this problem.)

Q109: A girl is flying a kite and she is trying to keep it at a constant height of 30 metres. The wind is blowing the kite horizontally away from her at a rate of 2.5 m/s. How fast must she let out the string when the kite is 50 metres away from her?

Q110: Another ladder 12 metres long is lying against a building. The top of the ladder begins to slide down the wall so that the angle between the ladder and the ground, θ, is changing at a rate of 3 radians/second. How fast is the distance of the top of the ladder from the ground changing when $\theta = \pi/6$ radians?

Q111: The volume of a cube is decreasing at a rate of 3 cm^3/hour. Find the rate at which the surface area is decreasing at the time when the volume is 512 cm^3.

1.5 Summary

1. $\frac{dy}{dx} = \frac{1}{{}^{dx}/_{dy}}$ and $\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(y)}$
2. $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$

   $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1 - x^2}}$

   $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$
3. When differentiating implicit equations it helps to remember that

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y))\frac{dy}{dx}$$

4. Sometimes, taking the logarithm of an expression makes it easier to differentiate.
5. When a curve is determined by parametric equations in the form x = f (t) and y = g (t), then

$$\frac{dy}{dx} = \frac{{}^{dy}/_{dt}}{{}^{dx}/_{dt}}$$

6. For parametric equations, the method we use to find the second derivative can be written as
   $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right) \Big/ \frac{dx}{dt}$
7. For a particle moving in the plane:
   - the horizontal velocity component is ${}^{dx}/_{dt}$
   - the vertical velocity component is ${}^{dy}/_{dt}$
   - the speed is $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$
   - the instantaneous direction of motion is ${}^{dy}/_{dx}$
8. Solution of related rates problems can often be simplified to an application of the chain rule, such as $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

1.6 Extended Information

There are links on-line to a variety of web sites related to this topic.

The following song might amuse you! Hopefully you will not feel this way after working your way through this unit.

A Calculus Carol: Written by Denis Gannon (1940-1991)

Sung to the tune of 'Oh, Christmas Tree' also known as 'The Red Flag'

Oh, Calculus; Oh, Calculus,
How tough are both your branches.
Oh, Calculus; Oh, Calculus,
To pass what are my chances?
Derivatives I cannot take,
At integrals my fingers shake.
Oh, Calculus; Oh, Calculus,
How tough are both your branches.

Oh, Calculus; Oh, Calculus,
Your theorems I can't master.
Oh, Calculus; Oh, Calculus,
My Proofs are a disaster.
You pull a trick out of the air,
Or find a reason, God knows where.
Oh, Calculus; Oh, Calculus,
Your theorems I can't master.

Oh, Calculus; Oh, Calculus,
Your problems do distress me.
Oh, Calculus; Oh, Calculus,
Related rates depress me.
I walk toward lampposts in my sleep,
And running water makes me weep.
Oh, Calculus; Oh, Calculus,
Your problems do distress me.

Oh, Calculus; Oh,Calculus,
My limit I am reaching.
Oh, Calculus; Oh, Calculus,
For mercy I'm beseeching.
My grades do not approach a B,
They're just an epsilon from D.
Oh, Calculus; Oh,Calculus,
My limit I am reaching.

1.7 Review exercise

Review exercise in further differentiation

An on-line assessment is available at this point, which you might find helpful.

10 min

Q112:

Find the derivative of the function $f(x) = \cos^{-1}(x^3)$, $-1 \leq x \leq 1$

Q113:

Use implicit differentiation to find an expression for $^{dy}/_{dx}$ when $3x^2 - y^2 = 9$

Q114:

A curve is given by the parametric equations $x = 4t^2 + 3$, $y = 2t^3$

Find $^{dy}/_{dx}$ in terms of t

1.8 Advanced review exercise

Advanced review exercise in further differentiation

An on-line assessment is available at this point, which you might find helpful.

20 min

Q115: Let $f(x) = x^2 - 4x - 3$ for $x > 2$

Find the value of $^{d}/_{dx}\ (f^{-1}(x))$ when $f(x) = 2$

Q116: Find the two points where the curve $x^2 + 3xy + y^2 = 9$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

Q117:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = \cos 4t$ and $y = \sin^2 4t$

Q118: A particle moves on the curve $2x^2 - y^2 = 2$ so that its y-coordinate increases at the constant rate of 9 m/s.

a) At what rate is the x-coordinate changing when the particle is at the point (3, 4)?

b) What is the slope of the curve at this point?

1.9 Set review exercise

Set review exercise in further differentiation

An online assessment is provided to help you review this topic.

10 min

An on-line assessment is available at this point, which you will need to attempt to have these answer marked. These questions are not randomised on the web. The questions on the web may be posed in a different manner but you should have the required answers in you working.

Q119: Find the derivative of $f(x) = \tan^{-1}(5x)$

Q120: When $3x^2 + 5xy + 4y^2 = 12$, find ${}^{dy}/_{dx}$ in terms of x and y.

Q121: The parametric equations $x = 3 \cos t$ and $y = 4 \sin t$ describe a curve in the plane. Find ${}^{dy}/_{dx}$

Topic 2

Further Integration

Contents

Learning Objectives

- *Use further integration techniques.*

Minimum Performance Criteria:

- *Integrate a proper rational function where the denominator is a quadratic in a factorised form.*

- *Integrate by parts with one application.*
- *Find a general solution of a first order differential equation (variables separable type).*

2.1 Revision Exercise

Try the following exercise to test your skills.

Some revision may be necessary if you find this difficult.

Revision - Exercise 1

20 min

These questions are intended to practice skills that you should already have. If you have difficulty you could consult your tutor or a classmate.

An on-line assessment is available at this point, which you might find helpful.

Q1: Factorise fully: $x^3 + 11x^2 + 23x - 35$

Q2: Find partial fractions for $\frac{2x + 18}{(x - 3)(x + 5)}$

Q3: Find partial fractions for $\frac{9}{(x + 2)(x - 1)^2}$

Q4: Integrate the following with respect to x: $\int \frac{12}{3x + 2}dx$

Q5: Integrate the following with respect to x: $\int \frac{dx}{(x - 1)^2}$

Q6: Evaluate the following: $\int_1^3 \frac{6x}{x^2 + 1}dx$

2.2 Inverse trigonometric functions

2.2.1 Simple integrals involving inverse trigonometric functions

Learning Objective

Recognise the standard integrals that give inverse sine and tangent functions

The work on further differentiation shows that

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1 + x^2}$$

It therefore follows that

$$\int \frac{1}{\sqrt{1 - x^2}}dx = \sin^{-1}x + C$$

$$\int \frac{1}{1 + x^2}dx = \tan^{-1}x + C$$

We also learned in the previous unit that

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{a^2 - x^2}}$$

$$\frac{dy}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2 + x^2}$$

So again it follows that

$$\int \frac{1}{\sqrt{a^2 - x^2}}dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

You should be aware that often a little algebra may be necessary before applying the above standard formulae. You can see this in the following examples.

Examples

1.

Find $\int \frac{dx}{4x^2 + 1}$

Answer

$$\int \frac{dx}{4x^2 + 1} = \int \frac{dx}{4(x^2 + \frac{1}{4})}$$

$$= \frac{1}{4}\int \frac{dx}{x^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{4}\frac{1}{1/2}\tan^{-1}\left(\frac{x}{1/2}\right) + C$$

$$= \frac{1}{2}\tan^{-1}(2x) + C$$

2. Find $\int \frac{dx}{\sqrt{9 - 25x^2}}$

Answer

$$\int \frac{dx}{\sqrt{9 - 25x^2}} = \int \frac{dx}{\sqrt{25\left(\frac{9}{25} - x^2\right)}}$$

$$= \frac{1}{5}\int \frac{dx}{\sqrt{\left(\frac{3}{5}\right)^2 - x^2}}$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{x}{3/5}\right) + C$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + C$$

Now, try the questions in Exercise 2.

Exercise 2

An on-line assessment is available at this point, which you might find helpful.

20 min

Q7: Find the following indefinite integrals:

a) $\int \frac{dx}{\sqrt{9 - x^2}}$

b) $\int \frac{dx}{16 + x^2}$

c) $\int \frac{5dx}{25 + 4x^2}$

d) $\int \frac{4dx}{\sqrt{16 - 9x^2}}$

e) $\int \frac{6dx}{5x^2 + 4}$

Q8: Evaluate the following integrals (give your answers in terms of π):

a) $\int_0^1 \frac{dx}{\sqrt{1 - x^2}}$

b) $\int_0^{\sqrt{3}} \frac{dx}{1 + x^2}$

c) $\int_3^6 \frac{dx}{\sqrt{36 - x^2}}$

d) $\int_0^4 \frac{dx}{16 + x^2}$

e) $\int_0^{3/4} \frac{dx}{\sqrt{9 - 4x^2}}$

f) $\int_0^4 \frac{9dx}{3x^2 + 16}$

2.2.2 Harder integrals involving inverse trigonometric functions

Learning Objective

Use a combination of techniques to solve integrals that may include an inverse trigonometric function in the answer

First we examine how to find any integral of the form $\int \frac{bx + c}{x^2 + a^2} dx$

In order to do so we make use of the formula

$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

In particular, for example

$\int \frac{2x}{x^2 + 4} dx = \ln(x^2 + 4) + C$

We can now find any integral of the form

$\int \frac{bx + c}{x^2 + a^2} dx$

by writing the integrand

as a multiple of $\frac{2x}{x^2 + a^2}$ + a multiple of $\frac{1}{x^2 + a^2}$

The next example shows how this is done.

Example

Find $\int \frac{x - 3}{x^2 + 4}dx$

Answer

$$\int \frac{x - 3}{x^2 + 4}dx = \int \frac{x}{x^2 + 4}dx - \int \frac{3}{x^2 + 4}dx$$
$$= \frac{1}{2}\int \frac{2x}{x^2 + 4}dx - 3\int \frac{1}{x^2 + 2^2}dx$$
$$= \frac{1}{2}\ln(x^2 + 4) - \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$$

Note that in order to make this integration possible we split the original expression into two separate fractions. This technique will also be useful in the next section when we use partial fractions to find integrals.

Secondly, in this section we see how to find any integral of the form $\int \frac{dx}{ax^2 + bx + c}$ where the quadratic in the denominator does not factorise (ie. where $b^2 - 4ac < 0$)

We do so by 'completing the square' in the denominator and then making an easy substitution to obtain an integral which we can identify as an inverse trigonometric function.

Example

Evaluate $\int_2^7 \frac{dx}{x^2 - 4x + 29}$

Answer

First we complete the square for the denominator.

Then $\int_2^7 \frac{dx}{x^2 - 4x + 29}$ becomes $\int_2^7 \frac{dx}{(x - 2)^2 + 25}$

Now let $u = x - 2$ Then $\frac{du}{dx} = 1$

The limits of the integral will also change.

When x = 7 then u = 5

When x = 2 then u = 0

Therefore $\int_2^7 \frac{dx}{x^2 - 4x + 29} = \int_2^7 \frac{dx}{(x - 2)^2 + 25}$

$$= \int_0^5 \frac{1}{u^2 + 5^2} \frac{dx}{du} du$$
$$= \frac{1}{5} \left[\tan^{-1}\left(\frac{u}{5}\right)\right]_0^5$$
$$= \frac{1}{5} \left[\tan^{-1} 1 - \tan^{-1} 0\right]$$
$$= \frac{1}{5} \left[\frac{\pi}{4} - 0\right]$$
$$= \frac{\pi}{20}$$

Now try the questions in Exercise 3.

Exercise 3

An on-line assessment is available at this point, which you might find helpful.

20 min

Q9: Find the following indefinite integrals:

a) $\int \frac{2x + 1}{x^2 + 9} dx$

b) $\int \frac{3x - 2}{x^2 + 16} dx$

c) $\int \frac{dx}{x^2 + 2x + 10}$

d) $\int \frac{7dx}{x^2 - 6x + 13}$

Q10: Evaluate the following definite integrals:

a) $\int_2^3 \frac{2dx}{x^2 - 4x + 5}$

b) $\int_0^3 \frac{x - 3}{x^2 + 9} dx$

Challenge Question 1

This next question you can regard as a challenge. It involves a combination of the techniques that you have learned in this section.

10 min

Find $\int \frac{x - 2}{x^2 + 2x + 5} dx$

2.3 Partial fractions and integration

We now examine how certain integrals can be found by using partial fractions. In general partial fractions are useful for evaluating integrals of the form

$\int \frac{f(x)}{g(x)} dx$

where f (x) and g (x) are polynomials and g (x) can be factorised into the product of linear and quadratic factors.

In this case expressing $^{f(x)}/_{g(x)}$ in partial fractions will enable us to carry out the integration.

When f (x) and g (x) are polynomials then $^{f(x)}/_{g(x)}$ is called a **rational function**.

You could revise the techniques employed for splitting an algebraic fraction into partial fractions at this stage if you are unsure.

2.3.1 Separate linear factors

Learning Objective

Find the integral of a rational function by expressing the function as partial fractions (separate linear factors in the denominator)

In this section we integrate functions such as $\int \frac{x+7}{(x-1)(x+3)}dx$

The integrand has a denominator which is the product of linear factors.

For $\int f(x)dx = F(x) + C$, f (x) is the **integrand**

In order to do such an integration we must use the standard integral.

$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C$

The following example should make this clear.

Example Find the indefinite integral

$\int \frac{x+7}{(x-1)(x+3)}dx$

Answer

First, express $\frac{x+7}{(x-1)(x+3)}$ as partial fractions.

Then

$$\frac{x+7}{(x-1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+3)}$$
$$= A(x+3) + B(x-1)$$

Let x = -3 then $4 = 0 \times A + (-4) \times B$

B = -1

Let x = 1 then $8 = 4 \times A + 0 \times B$

A = 2

Therefore $\frac{x+7}{(x-1)(x+3)} = \frac{2}{(x-1)} - \frac{1}{(x+3)}$

It is now possible to find the integral

$$\begin{aligned}\int \frac{x + 7}{(x - 1)(x + 3)}dx &= \int \left(\frac{2}{x - 1} - \frac{1}{x + 3}\right)dx \\ &= \int \frac{2}{x - 1}dx - \int \frac{1}{x + 3}dx \\ &= 2 \ln|x - 1| - \ln|x + 3| + C\end{aligned}$$

Note that:

i This method will work for any integral of the form $\int \frac{cx + d}{ax^2 + bx + c}\,dx$, where $ax^2 + bx + c$ can be factorised.

ii The method can be extended to integrals such as $\int \frac{9}{(x - 1)(x + 2)(2x + 1)}\,dx$, where the denominator is a cubic function written as three separate linear factors.

Now try the questions in Exercise 4.

Exercise 4

20 min

An on-line assessment is available at this point, which you might find helpful.

Q11:

Use partial fractions to help find the following indefinite integrals.

a) $\int \frac{10}{(x - 4)(x + 1)}dx$

b) $\int \frac{3x - 8}{(x - 2)(x - 3)}dx$

c) $\int \frac{6 - 2x}{x(x + 3)}dx$

d) $\int \frac{9}{(x - 1)(x + 2)(2x + 1)}dx$

Q12:

a) Factorise the expression2 $x^2 + 7x - 4$

b) Hence find the indefinite integral $\int \frac{9}{2x^2 + 7x - 4}dx$

Q13:

a) Factorise the expression $x^3 - 2x^2 - 9x + 18$

b) Hence find the indefinite integral $\int \frac{-7x + 9}{x^3 - 2x^2 - 9x + 18}dx$

2.3.2 Repeated linear factors

Learning Objective

Find the integral of a rational function by expressing the function as partial fractions (repeated linear factors in the denominator)

We now examine how to integrate rational functions which have repeated linear factors in the denominator, such as $\int \frac{x + 5}{(2x + 1)(x - 1)^2}dx$

In this case we will use the following standard integral

$\int \frac{dx}{(ax + b)^2} = - \frac{1}{a}\frac{1}{(ax + b)} + C$

The next example should make this clear.

Example Find the indefinite integral

$\int \frac{x + 5}{(2x + 1)(x - 1)^2}dx$

Answer

First, express $\frac{x + 5}{(2x + 1)(x - 1)^2}$ as partial fractions.

Since the denominator has a repeated linear factor the working is as follows

$\frac{x + 5}{(2x + 1)(x - 1)^2} = \frac{A}{2x + 1} + \frac{B}{(x - 1)} + \frac{C}{(x - 1)^2}$

$x + 5 = A(x - 1)^2 + B(2x + 1)(x - 1) + C(2x + 1)$

Let $x = 1$ then $6 = 0 \times A + 0 \times B + 3 \times C$

$C = 2$

Let $x = -\frac{1}{2}$ then $\frac{9}{2} = \frac{9}{4}A + 0B + 0C$

$A = 2$

We cannot use the above method to find B.

Therefore we substitute the values we have calculated for A and C back into the above equation

$x + 5 = A(x - 1)^2 + B(2x + 1)(x - 1) + C(2x + 1)$

Then $x + 5 = 2(x - 1)^2 + B(2x + 1)(x - 1) + 2(2x + 1)$

$= (2x^2 - 4x + 2) + B(2x^2 - x - 1) + (4x + 2)$

$= (2 + 2B)x^2 - Bx + (4 - B)$

Now, equating coefficients for the x term gives

$1 = -B \Rightarrow B = -1$

(This value for B could also be obtained by equating the coefficients of the x^2 term or the constant term.)

Therefore $\frac{x + 5}{(2x + 1)(x - 1)^2} = \frac{2}{2x + 1} - \frac{1}{x - 1} + \frac{2}{(x - 1)^2}$

It is now possible to find the integral.

$$\int \frac{x + 5}{(2x + 1)(x - 1)^2}dx = \int \frac{2}{2x + 1}dx - \int \frac{1}{x - 1}dx + \int \frac{2}{(x - 1)^2}dx$$

$$= \frac{2}{2}\ln(2x + 1) - \ln(x - 1) + \frac{2}{-1}(x - 1)^{-1} + C$$

$$= \ln(2x + 1) - \ln(x - 1) - \frac{2}{x - 1} + C$$

Now try the questions in Exercise 5.

Exercise 5

20 min

An on-line assessment is available at this point, which you might find helpful.

Q14: Use partial fractions to help find the following indefinite integrals.

a) $\int \frac{3x^2 + 2}{x(x - 1)^2} dx$

b) $\int \frac{x - 2}{x^2(x - 1)} dx$

c) $\int \frac{3}{(x - 1)(x - 2)^2} dx$

d) $\int \frac{5x^2 - x + 18}{(2x - 1)(x - 3)^2} dx$

2.3.3 Irreducible quadratic factor

Learning Objective

Find the integral of a rational function by expressing the function as partial fractions (irreducible quadratic factor in the denominator)

We now see how to deal with the case where the denominator includes an irreducible quadratic.

We say that a quadratic is **irreducible** when it has no real roots. It cannot be factorised.

factor.

The integral $\int \frac{3x - 1}{(x - 2)(x^2 + 1)} dx$ is an example of this type with $x^2 + 1$ the irreducible quadratic factor in the denominator.

In this case we will use the standard integral

$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

You can see this in the following example.

Example Find the indefinite integral

$\int \frac{3x - 1}{(x - 2)(x^2 + 1)} dx$

Answer

First express $\frac{3x - 1}{(x - 2)(x^2 + 1)}$ as partial fractions.

Since the denominator has an irreducible quadratic factor, the working is as follows

$\frac{3x - 1}{(x - 2)(x^2 + 1)} = \frac{A}{(x - 2)} + \frac{Bx + C}{x^2 + 1}$

$3x - 1 = A(x^2 + 1) + (Bx + C)(x - 2)$

Let $x = 2$ then $5 = 5A \Rightarrow A = 1$

We cannot use the above method to find B and C. Therefore we substitute the value we have found for A back into the above equation (*) and equate coefficients.

Substitute A = 1. Then

$3x - 1 = (x^2 + 1) + (Bx + C)(x - 2)$

$= x^2 + 1 + Bx^2 - 2Bx + Cx - 2C$

$= (1 + B)x^2 + (C - 2B)x + (1 - 2C)$

Now equating cofficients gives,

$1 + B = 0 \Rightarrow B = -1$

$C - 2B = 3 : C + 2 = 3 \Rightarrow C = 1$

Therefore, as partial fractions

$$\frac{3x - 1}{(x - 2)(x^2 + 1)} = \frac{1}{x - 2} + \frac{-x + 1}{x^2 + 1}$$
$$= \frac{1}{x - 2} - \frac{x - 1}{x^2 + 1}$$

It is now possible to find the integral

$$\int \frac{3x - 1}{(x - 2)(x^2 + 1)}dx = \int \frac{dx}{x - 2} - \int \frac{x - 1}{x^2 + 1}dx$$
$$= \int \frac{1}{x - 2}dx - \int \frac{x}{x^2 + 1}dx + \int \frac{1}{x^2 + 1}dx$$
$$= \int \frac{1}{x - 2}dx - \frac{1}{2}\int \frac{2x}{x^2 + 1}dx + \int \frac{1}{x^2 + 1}dx$$
$$= \ln|x - 2| - \frac{1}{2}\ln(x^2 + 1) + \tan^{-1}x + C$$

Now try the questions in Exercise 6.

20 min

Exercise 6

An on-line assessment is available at this point, which you might find helpful.

Q15: Use partial fractions to help find the following indefinite integrals.

a) $\int \frac{9 - x}{(x + 1)(x^2 + 4)}dx$

b) $\int \frac{9x + 8}{(x - 2)(x^2 + 9)}dx$

c) $\int \frac{3x^2 - 4x + 10}{(x + 1)(x^2 + 16)}dx$

d) $\int \frac{2x - 18}{(x - 3)(x^2 + 3)}dx$

30 min

Challenge Question 2

Try to find the following indefinite integral

$\int \frac{dx}{x(x^2 + x + 1)}$

Be warned, this is very nasty.

(You will definitely *not* be asked to do anything as nasty as this in an exam.)

2.3.4 Improper rational functions

Learning Objective

Find the integral of an improper rational function by dividing through and then expressing the function as partial fractions

Let f (x) be a polynomial of degree n and g (x) be a polynomial of degree m.

When $n \geq m$ then ${}^{f(x)}/_{g(x)}$ is an **improper rational function**.

For a polynomial, the **degree** is the value of the highest power.

When we have an improper rational function to integrate we must first divide through and obtain a polynomial plus a proper rational function. We can then integrate using partial fractions as above.

The following example should help make this clear.

Example Find the indefinite integral

$\int \frac{x^3 + 3x^2 + 4}{x^2 + 2x - 3} dx$

Answer

Since the degree of the denominator is greater than the degree of the numerator then the integrand is an improper rational function. So, first of all we should divide by the denominator. This can be done in the following way,

$$\begin{array}{rl} & x + 1 \\ x^2 + 2x - 3 \,\big)\! & \overline{x^3 + 3x^2 + 0x + 4} \\ & \underline{x^3 + 2x^2 - 3x} \\ & x^2 + 3x + 4 \\ & \underline{x^2 + 2x - 3} \\ & x + 7 \end{array}$$

Hence we can now write $\frac{x^3 + 3x^2 + 4}{x^2 + 2x - 3}$ as $x + 1 + \frac{x + 7}{x^2 + 2x - 3}$

$x + 1$ is a polynomial and $\frac{x + 7}{x^2 + 2x - 3}$ is a proper rational function,

and we can integrate both.

Therefore
$$\begin{aligned} \int \frac{x^3 + 3x^2 + 4}{x^2 + 2x - 3} dx &= \int \left(x + 1 + \frac{x + 7}{x^2 + 2x - 3} \right) dx \\ &= \int (x + 1)\, dx + \int \frac{x + 7}{x^2 + 2x - 3} dx \\ &= \int (x + 1)\, dx + \int \frac{x + 7}{(x - 1)(x + 3)} dx \\ &= \int (x + 1)\, dx + \int \frac{2}{x - 1} dx - \int \frac{1}{x + 3} dx \\ &= \frac{1}{2}x^2 + x + 2\ln|x - 1| - \ln|x + 3| + C \end{aligned}$$

(Note that you can see the working for $\int \frac{x + 7}{(x - 1)(x + 3)} dx$ if you look back to section 7.2.1)

Now try the questions in Exercise 7.

Exercise 7

20 min

An on-line assessment is available at this point, which you might find helpful.

Q16: a) Express the **improper rational function** $\frac{x^3 + 3x^2 - x - 8}{x^2 + x - 6}$ as a polynomial plus a proper rational function by dividing the numerator by the denominator.

b) Hence, find the indefinite integral $\int \frac{x^3 + 3x^2 - x - 8}{x^2 + x - 6} dx$

Q17:

a) Divide $x^4 + 2x^3 - 3x^2 - 4x + 13$ by $x^3 + 3x^2 - 4$

b) Factorise $x^3 + 3x^2 - 4$

c) Hence, using partial fractions, find the indefinite integral $\int \frac{x^4 + 2x^3 - 3x^2 - 4x + 13}{x^3 + 3x^2 - 4} dx$

Q18:

a) Divide $3x^3 - 6x^2 + 20x - 53$ by $x^3 - 2x^2 + 9x - 18$

b) Factorise $x^3 - 2x^2 + 9x - 18$

c) Hence, using partial fractions, find the indefinite integral $\int \frac{3x^3 - 6x^2 + 20x - 53}{x^3 - 2x^2 + 9x - 18} dx$

2.4 Integration by parts

2.4.1 One application

Learning Objective

Use integration by parts to find the integral of the product of two functions.

The product rule for differentiation gives rise to a very useful rule for integration known as integration by parts. Remember that the product rule can be stated as

$\frac{d}{dx}(f(x)\, g(x)) = f'(x)\, g(x) + f(x)\, g'(x)$

Integrating both sides of the above equation with respect to x gives the following

$$\int \frac{d}{dx}(f(x)\, g(x))\, dx = \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

$$f(x)\, g(x) = \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

Now rearranging this last equation gives the integration by parts formula

$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$

You might find this easier to remember in a shortened form as

$\int f g' = f g - \int f' g$

This formula gives us a means of finding $\int f(x)\, g'(x)\, dx$ provided we can find $\int f'(x)\, g(x)\, dx$ and so is useful if $\int f'(x)\, g(x)\, dx$ is simpler than $\int f(x)\, g'(x)\, dx$

Alternative notation

If we let u = f (x) and v = g (x)

then $\frac{du}{dx} = f'(x)$ and $\frac{dv}{dx} = g'(x)$

Therefore in terms of u and v we can rewrite the formula for integration by parts as

$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

You can see how the formula for integration by parts is used in the following example.

Example Integrate $\int x \sin 3x dx$

Answer

We can think of this integral as being the product of two functions x and sin 3x.

we choose f (x) = x and g ′ (x) = sin 3x

Then $f'(x) = 1$ and $g(x) = \int g'(x)\, dx = -\frac{1}{3}\cos 3x$

Note that f ′ (x) is simpler than f (x)

The formula for integration by parts $\int f g' = f g - \int f' g$ then gives us

$$\int x \sin 3x dx = -\frac{1}{3}x \cos 3x + \frac{1}{3}\int \cos 3x dx$$
$$= -\frac{1}{3}x \cos 3x + \frac{1}{9}\sin 3x + C$$

Note that by choosing f (x) = x and g ′ (x) = sin 3x then the integral $\int f'(x)\, g(x)dx$ reduces to a simple integral. This is what you should aim to achieve when integrating by parts.

The following strategy may help you with the choices you make for f (x) and g ′ (x).

Strategy for choosing f (x) and g ′ (x)

- For f (x) choose the function that becomes simpler when differentiated. Often this is a polynomial (but not always).
- For g ′ (x) choose the function so that g (x) can be easily determined by integrating. Often this is either a trigonometric or an exponential function.

Another example may make this clearer.

Example Integrate $\int (x + 1)e^{-x}dx$

Answer

Choose f (x) = x + 1 then f ′ (x) = 1 and g ′ (x) = e^{-x} then g (x) = $-e^{-x}$

Now, using the formula for integration by parts $\int f\,g' = f\,g - \int f'\,g$ we have

$$\begin{aligned}\int (x + 1)e^{-x}dx &= -(x + 1)e^{-x} + \int e^{-x}dx \\ &= -(x + 1)e^{-x} - e^{-x} + C \\ &= -e^{-x}(x + 2) + C\end{aligned}$$

Definite integrals using integration by parts

It is also possible to evaluate definite integrals using integration by parts. The integration by parts formula for definite integrals is

$$\int_a^b f(x)\,g'(x)dx = [f(x)\,g(x)]_a^b - \int_a^b f'(x)\,g(x)dx$$

This formula is used in the following example.

Example Evaluate $\int_0^{\pi/2} x \cos x dx$

Answer

Choose $f(x) = x$ then $f'(x) = 1$

and $g'(x) = \cos x$ then $g(x) = \sin x$

Using the formula for integration by parts we have

$$\begin{aligned}\int_0^{\pi/2} x \cos x\, dx &= [x \sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x dx \\ &= [x \sin x]_0^{\pi/2} + [\cos x]_0^{\pi/2} \\ &= \frac{\pi}{2} - 0 + 0 - 1 \\ &= \frac{\pi}{2} - 1\end{aligned}$$

Now try the questions in Exercise 8.

Exercise 8

20 min

An on-line assessment is available at this point, which you might find helpful.

Use integration by parts to find the following indefinite integrals.

Q19: $\int x \sin x\, dx$

Q20: $\int xe^x dx$

Q21: $\int x(2x + 3)^3 dx$

Q22: $\int x \exp(3x)dx$

Q23: $\int x \cos(2x + 1)dx$

Q24: $\int (2x - 3)\sin 2x dx$

Q25: $\int x \ln x \, dx$ (Hint: choose $f(x) = \ln x$ and $g'(x) = x$

Q26: $\int_0^{\pi/4} (x + 4)\sin 2x dx$

Q27: $\int_0^3 x(x - 2)^3 dx$

Q28: $\int_0^{\pi/2} 3x \sin x dx$

Q29: $\int_4^6 x \exp(x - 4) dx$

Q30: $\int_1^2 \frac{\ln x}{x^3} dx$ (Hint: choose $f(x) = \ln x$ and $g'(x) = \frac{1}{x^3}$)

Use integration by parts to evaluate the following definite integrals.

Q31: $\int_0^{\pi/4} (x + 4)\sin 2x dx$

Q32: $\int_0^3 x(x - 2)^3 dx$

Q33: $\int_0^{\pi/2} 3x \sin x dx$

Q34: $\int_4^6 x \exp(x - 4) dx$

Q35: $\int_1^2 \frac{\ln x}{x^3} dx$ (Hint: choose $f(x) = \ln x$ and $g'(x) = \frac{1}{x^3}$)

2.4.2 Repeated applications

Learning Objective

Use integration by parts more than once in order to find the integral of the product of two functions.

Sometimes we may have to use integration by parts more that once to obtain an answer. Here is an example.

Example Find the indefinite integral $\int x^2 \cos x \, dx$

Answer

Choose $f(x) = x^2$ then $f'(x) = 2x$

and $g'(x) = \cos x$ then $g(x) = \sin x$

The formula for integration by parts $\int f g' = f g - \int f' g$ then gives us

$\int x^2\cos x\,dx = x^2\sin x - \int 2x\sin x\,dx$

We now find $\int 2x\sin x\,dx$ by using integration by parts once more.

Choose $f(x) = 2x$ then $f'(x) = 2$

and $g'(x) = \sin x$ then $g(x) = -\cos x$

The formula for integration by parts $\int f g' = f g - \int f' g$ then gives us

$$\begin{aligned}\int 2x\sin x\,dx &= -2x\cos x + \int 2\cos x\,dx \\ &= -2x\cos x + 2\sin x + C\end{aligned}$$

Finally we have

$$\begin{aligned}\int x^2\cos x\,dx &= x^2\sin x - \int 2x\sin x\,dx \\ &= x^2\sin x + 2x\cos x - 2\sin x + C \\ &= \left(x^2 - 2\right)\sin x + 2x\cos x + C\end{aligned}$$

Now try the questions in Exercise 9.

Exercise 9

20 min

An on-line assessment is available at this point, which you might find helpful.

Find the following indefinite integrals.These questions may require repeated application of the formula for integration by parts.

Q36: $\int x^2 e^x\,dx$

Q37: $\int x^2\cos(3x)\,dx$

Q38: $\int x^2 e^{-4x}\,dx$

Q39: $\int x^3 e^x\,dx$

Evaluate the following definite integrals.

Q40: $\int_0^2 x^2 e^x\,dx$

Q41: $\int_0^{\pi/6} x^2\sin 3x\,dx$

2.4.3 Further examples

Learning Objective

Use integration by parts to find the integral of $\ln(x)$ and $\tan^{-1}(x)$

Next we see how to find an integral of the form $I = \int e^{ax}\sin(bx)dx$ or similar. Sometimes an integration by parts can take us round in a circle back to where we started. We integrate by parts twice to obtain a simple equation for I which is then easy to solve.

Example Find the indefinite integral $I = \int e^x \sin x \, dx$

Answer

Let $f(x) = e^x$ then $f'(x) = e^x$

Let $g'(x) = \sin x$ then $g(x) = -\cos x$

The formula for integration by parts $\int f g' = f g - \int f' g$ then gives us

$I = \int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

Now apply integration by parts to $\int e^x \cos x \, dx$

Let $f = e^x$ then $f' = e^x$

Let $g' = \cos x$ then $g = \sin x$

This time the formula for integration by parts gives us

$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

You should notice that the integral on the right hand side of the above equation takes us back to the integral we started with, I. Thus

$$\begin{aligned} \int e^x \sin x \, dx &= -e^x \cos x + \int e^x \cos x \, dx \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx \\ I &= -e^x \cos x + e^x \sin x - I \end{aligned}$$

Rearranging the above equation gives

$2I = e^x (\sin x - \cos x) + C$

$I = {}^1/_2 \, e^x (\sin x - \cos x) + C$

Hence

$\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + C$

(Note that since C is an arbitrary constant we leave ${}^1/_2 C$ as C in the working.)

At first sight the following integration does not appear to be a suitable integral to integrate by parts as there is only one function, ln x

Example Integrate $\int \ln x \, dx$

Solution

If we rewrite the integrand as $1 \times \ln x$ we can then find the integral by using integration by parts as follows.

Choose $f(x) = \ln x$ and $g'(x) = 1$

then $f'(x) = {}^1/_x$ and $g(x) = x$

The formula for integration by parts $\int f g' = f g - \int f' g$ then gives us,

$$\int (1 \times \ln x)\,dx = x\ln x - \int \frac{1}{x} x\,dx$$
$$= x\ln x - \int 1dx$$
$$= x\ln x - x + C$$

Now try the questions in Exercise 10.

Exercise 10

20 min

An on-line assessment is available at this point, which you might find helpful.

Q42: $\int e^{2x}\cos x\,dx$

Q43: $\int e^{3x}\sin 2x dx$

Q44: $\int \frac{\ln x}{x}dx$

Q45: $\int e^{x}\sin (1 - 2x)dx$

Q46: $\int \tan^{-1}x\,dx$

Q47: $\int \tan^{-1}(3x)dx$

Q48: $\int \ln (x^2)dx$

Q49: $\int (\ln x)^2dx$

Challenge Question 3

15 min

Use integration by parts to find the indefinite integral

$\int \sin^{-1}x\,dx$

2.5 First order differential equations

2.5.1 Simple differential equations

Learning Objective

Be able to identify and know the associated terminology for differential equations

Formulate and solve simple differential equations

A **differential equation** is an equation involving an unknown function and its derivatives.

The **order of a differential equation** is that of the highest-order derivative appearing in the equation.

Examples

1. $^{dy}/_{dx} = x^2y$ is a first order differential equation.

2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^3 = \sin x$ is a second order differential equation.

A first order differential equation is **linear** if it can be written in the form

$^{dy}/_{dx} + k(x)\,y = h(x)$

The equation is termed linear as it involves only first order terms in $^{dy}/_{dx}$ and y but not terms such as y^2, $y\frac{dy}{dx}$, y^3, cos(y), etc.

Examples

1.

$(4 - x^2)^{dy}/_{dx} + 3y = (4 - x^2)^2$ is a linear first order differential equation

2.

$x^{dy}/_{dx} + y^2 = 0$ is a first order differential equation which is not linear. It contains a y term of degree 2.

Differential equations arise in many situations and they are usually solved by integration. A first order equation is usually solved by a single integration and the solution obtained contains a constant of integration. A solution containing a constant of integration is known as a **general solution** of the equation. Since the constant can assume any value, the differential equation actually has infinitely many solutions.

In applications of differential equations the solution often has to satisfy an additional condition - usually called the initial condition. The initial condition can be used to find the value of the constant in the general solution. The solution so obtained is called a **particular solution** of the equation. This is illustrated in the following examples.

An **arbitrary constant** is a constant that occurs in the general solution of a differential equation.

The **general solution** of a differential equation contains an arbitrary constant and gives infinitely many solutions that all satisfy the differential equation.

For a differential equation an **initial condition** is additional information required to determine a particular solution. This could be a coordinate on a curve, a velocity at $t = 0$, the amount of money in a bank account on 1st January 2000, etc.

The **particular solution** of a differential equation is a solution which is often obtained from the general solution when an initial condition is known.

Examples

1. The gradient of the tangent to a curve is given by the **differential equation**

$^{dy}/_{dx} = 2x - 4$

Find the equation of the curve.

Answer

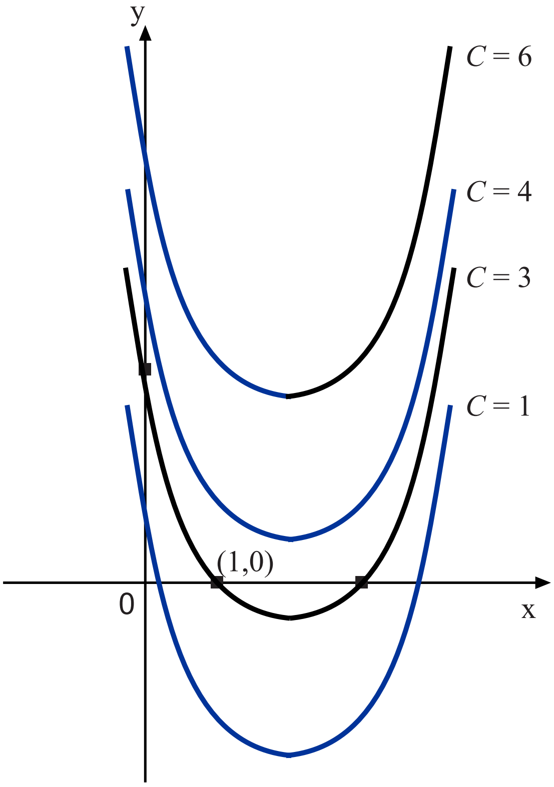
We can solve this problem by integrating

$$\int \frac{dy}{dx}dx = \int (2x - 4)dx$$

$$y = x^2 - 4x + C$$

where C is an arbitrary constant.

$y = x^2 - 4x + C$ is a **general solution** which gives a family of curves each of which satisfies the differential equation $^{dy}/_{dx} = 2x - 4$



The graphs shown here are some of the curves of the form

$y = x^2 - 4x + C$

The curves all have the same gradient given by the differential equation

$^{dy}/_{dx} = 2x - 4$

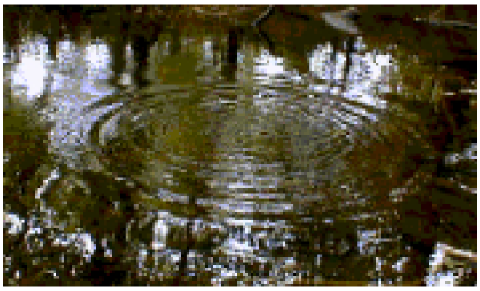
If we are given more information then we can identify one curve in particular. So, for example, if we know that the point (1, 0) lies on the curve then we know that y = 0 when x = 1

Hence substituting into $y = x^2 - 4x + C$ we obtain $0 = 1 - 4 + C$ and so $C = 3$

In this case $y(1) = 0$ is the **initial condition** for the differential equation.

Using this initial condition we obtain the **particular solution** $y = x^2 - 4x + 3$

2.



A stone is dropped into a still pond, ripples move out from the the point where the stone hits in the form of concentric circles. The area, Am^2, of disturbed water increases at a rate of 10t m/s, where t is the time in seconds.

Calculate the area of disturbed water after 2 seconds.

Answer

The rate of change of area is given by the differential equation $^{dA}/_{dt} = 10t$

Integrating this with respect to t gives

$$\int \frac{dA}{dt}dt = \int 10t\, dt$$

$$A = 5t^2 + C$$

We have now obtained the **general solution** of the equation.

To find the appropriate **particular solution** we must use a suitable **initial condition**. In this case we use the fact that A = 0 when t = 0 (this corresponds to the fact that the area = 0 at the instant that the stone is dropped).

Substituting these values into $A = 5t^2 + C$ gives C = 0

Therefore the particular solution to the differential equation is $A = 5t^2$

We can now use this particular solution to calculate the area of disturbed water after 2 seconds.

When t = 2 then $A = 5t^2 = 20m^2$

The following questions in Exercise 11 revise these techniques.

Exercise 11

An on-line assessment is available at this point, which you might find helpful.

15 min

Q50: The gradient of a tangent to a curve is given by the differential equation

$\frac{dy}{dx} = 4x - \frac{3}{x^2}$. The curve passes throught the point (1,5). Find the equation of the curve.

Q51: The number of bacteria in an experiment is growing at a rate of $9e^{0.5t}$ bacteria per hour, where t is the time in hours. Given that there were initially 18 bacteria in the experiment calculate how many are present after 12 hours.

Q52: A particle is moving with velocity (metres/second) given by the formula

$v = \frac{dx}{dt} = 3\cos\left(3t - \frac{\pi}{6}\right)$

Given that the particle started from the origin at t = 0, find how far it has travelled after π seconds. (x denotes the distance of the particle from the origin.)

Q53: On the moon, the acceleration due to gravity is given by $\frac{d^2x}{dt^2} = 1.6$ m/sec^2.

If a rock is dropped into a crevasse how fast is it travelling just before it hits the bottom 20 seconds later?

2.5.2 First order differential equations with separable variables

Learning Objective

Solve first order differential equations with separable variables

Not all equations can be solved by a straightforward integration as in the previous section. In this section we look at a slightly more complicated case - first order separable differential equations. These equations can be written in the form $^{dy}/_{dx} = f(x)\,g(y)$

We can rewrite $\frac{dy}{dx} = f(x)\,g(y)$ as $\frac{1}{g(y)}dy = f(x)dx$

Then integrating both sides with respect to x gives

$$\int \frac{1}{g(y)} \frac{dy}{dx} dx = \int f(x) dx$$
$$\int \frac{1}{g(y)} dy = \int f(x) dx$$

If both integrations can be carried out in this equation then we can find a general solution.

In practice it is easier to obtain the above formula by initially separating the variables (i.e. we get all the terms involving x, including dx, on one side of the equation and all the terms involving y, including dy, on the other side).

ie. we can rewrite ${}^{dy}/_{dx} = f(x)\, g(y)$ as

$\frac{1}{g(y)} dy = f(x) dx$

and integrating, we obtain

$\int \frac{1}{g(y)} dy = \int f(x)\, dx$

exactly as before.

The following examples should make this clearer.

Examples

1. Find the general solution for the differential equation

$\frac{dy}{dx} = \frac{3x^2}{y}$

Answer

Separating variables we have $y\, dy = 3x^2\, dx$

Now integrating both sides we obtain the following

$$\int y\, dy = \int 3x^2 dx$$
$$\tfrac{1}{2} y^2 = x^3 + C$$
$$y^2 = 2x^3 + C$$
$$y = \pm \sqrt{2x^3 + C}$$

(Note that in the third line of the working above 2C was rewritten as C. We can do this as C is just an arbitrary constant.)

2. Find the general solution for the differential equation ${}^{dy}/_{dx} = x^3 y$

Answer

Separating the variables we have ${}^{1}/_{y}\, dy = x^3\, dx$

Now integrating both sides we obtain the following

$$\int \frac{1}{y}dy = \int x^3 dx$$
$$\ln y = \tfrac{1}{4}x^4 + C$$
$$y = \exp\left(\tfrac{1}{4}x^4 + C\right)$$
$$= \exp\left(\tfrac{1}{4}x^4\right) \times \exp(C)$$
$$= A\exp\left(\tfrac{1}{4}x^4\right), \quad \text{where } A = \exp(C)$$

3. Given that y = -1 when x = 0, find the particular solution for the differential equation $x\ {}^{dy}/_{dx} = y^2 - {}^{dy}/_{dx}$

Answer

First of all we need to rearrange the equation so that we can separate the variables

$(x + 1){}^{dy}/_{dx} = y^2$

Then, separating variables

$\frac{1}{y^2}dy = \frac{1}{x+1}dx$

Hence

$$\int \frac{1}{y^2}dy = \int \frac{1}{x+1}dx$$
$$-\frac{1}{y} = \ln|x + 1| + C$$

Substitute in x = 0 and y = -1

1 = ln 1 + C

C = 1

Therefore the particular solution is

$$-\frac{1}{y} = \ln|x + 1| + 1$$
$$y = -\frac{1}{\ln|x + 1| + 1}$$

Now try the questions in Exercise 12.

Exercise 12

An on-line assessment is available at this point, which you might find helpful.

30 min

Q54: Find the **general solution** for the following differential equations, giving y in terms of x.

a) $\frac{dy}{dx} = \frac{e^x}{y}$

b) ${}^{dy}/_{dx} = 3y$

c) ${}^{dy}/_{dx} = {}^{\sin x}/_{2y}$

d) ${}^{dy}/_{dx} = 4x(1 + y^2)$

e) ${}^{dy}/_{dx} + y = xy$

f) ${}^{dy}/_{dx} = x^2 \sec y$

g) $x^{dy}/_{dx} + {}^{dy}/_{dx} + 1 = y$

h) $\frac{dy}{dx} = \frac{xy}{1 + x^2}$

Q55: Find the **particular solution** for the following differential equations with the given initial conditions.

a) ${}^{dy}/_{dx} = 2xy$, x = 0, y = 1

b) ${}^{dy}/_{dx} = \exp\ ({}^{1}/_{2}\ x - y)$, x = 0, y = 0

c) $2x^{dy}/_{dx} + y^2 = {}^{dy}/_{dx}$, x = 1, y = -2

d) $\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{x}$, $x = 1, y = \frac{1}{2}$

e) $x^{dy}/_{dx} - 2 = 2y - 3^{dy}/_{dx}$, x = 1, y = 3

Q56: The following questions will require some of the techniques that you learned earlier in this topic such as integration by parts and partial fractions.

a) Find the general solution of $2^{dy}/_{dx} = x\ (y^2 - 1)$

b) Find the particular solution of $\frac{1}{x}\frac{dy}{dx} = y\ \sin x$, when $x = \frac{\pi}{2}, y = 1$

c) Find the particular solution of $2x^{dy}/_{dx} + 1 = y^2 - 2^{dy}/_{dx}$ when x = 2, y = -2

Extra Help: Differential equations - variables separable

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

2.5.3 Growth and decay

Learning Objective

Be able to formulate a simple statement involving rate of change and know the laws of growth and decay

Many real-life situations can be modelled by differential equations. In this section we practise forming and solving these equations.

Consider the following situations.

Bacterial growth

In an experiment the number of bacteria B (t) present in a culture after t days is found to be increasing at a rate proportional to the number of bacteria present.

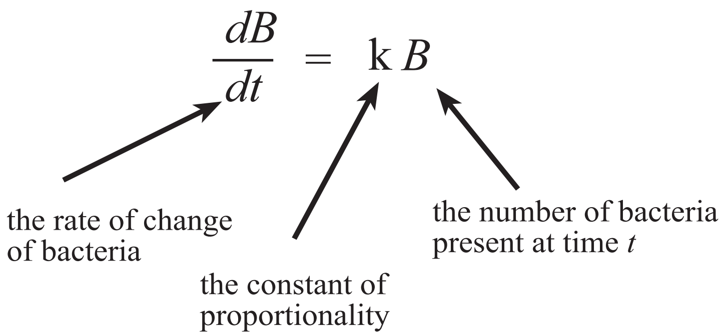
From the above statement we can write down a differential equation for B (t) and solve it to find B in terms of t. This is done in the following way:

The rate of change of B = ${}^{dB}/_{dt}$

But as the rate of change of B is proportional to B:

The rate of change of B = kB

Hence we obtain the differential equation



Separating the variables gives ${}^{dB}/_{B}$ = k dt Hence

$$\int \frac{1}{B} dB = \int k dt$$

$\ln B = kt + C$

$B = e^{kt + C}$

$B = Ae^{kt}$, where $A = e^C$

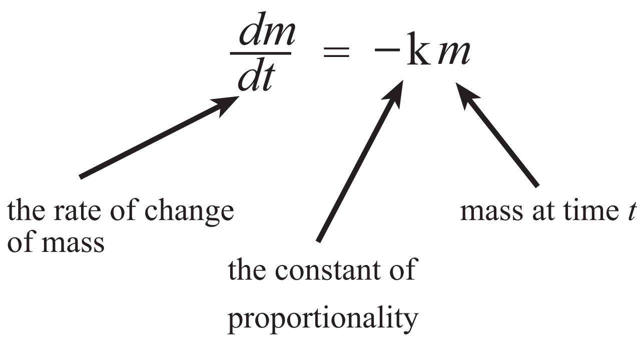
We have found the general solution of the equation. If it is known that there are B_0 bacteria present at the start of the experiment i.e. $B = B_0$ when t = 0 then we must have $B_0 = Ae^{0k}$ i.e. $B_0 = A$ and so we have the particular solution $B(t) = B_0e^{kt}$

Radioactive decay

When a radioactive atom emits some of its mass as radiation, the remainder of the atom reforms to make an atom of some new substance. This process is called radioactive decay. So for example, radium decays into lead and carbon-14 decays into nitrogen-14.

For a radioactive substance, the mass m (grams) at time t (years) decreases at a rate proportional to the mass at that time.

In a similar way to the above example for growth we can obtain an equation for decay. This time the constant of proportionality is negative because the mass is decreasing.



Also, similar to the example for bacterial growth, we can solve this to obtain $m = m_0e^{-kt}$ where m_0 is the original mass.

Check this for yourself.

The two examples above illustrating exponential growth and decay are of fundamental importance.

Exponential growth occurs when the rate of growth of a population is proportional to the size of the population, and we can write

$^{dP}/_{dt} = kP$

The general solution is $P(t) = A e^{kt}$

Exponential decay occurs when the population decreases at a rate proportional to the size of the population, and we can write

$^{dP}/_{dt} = -kP$

The general solution is $P(t) = Ae^{-kt}$

Examples

1. Bacterial growth

In an experiment the number of bacteria B (t) present in a culture after t days is increasing at a rate proportional to the number of bacteria present.

Suppose there are B_0 bacteria present at the start of the experiment, i.e. $B(0) = B_0$ If it is found that the bacteria population doubles in 10 hours i.e. $B = 2B_0$ when t = 10 then calculate k, the constant of proportionality and hence find a formula for B (t) in terms of t

Solution

The population growth is described by the differential equation $^{dB}/_{dt} = kB$

Hence

$$\int \frac{dB}{B} = \int k\,dt$$

$$\ln B = kt + C$$

$$B = e^{kt + C}$$

Since $B(0) = B_0$ then $B = B_0e^{kt}$

Also, since $B(10) = 2B_0$ then substituting 10 for t we have

$$2B_0 = B_0e^{10k}$$

$$e^{10k} = 2$$

$$10k = \ln 2$$

$$k = \frac{\ln 2}{10} = 0.069 \text{ (3 d.p.)}$$

Hence the number of bacteria present in the culture at time t is given by $B(t) = B_0e^{0.069t}$

2. Radioactive decay

For a radioactive substance, the mass m (grams) at time t (years) decreases at a rate proportional to the mass at that time.

The half-life of any radioactive material is the time taken for half of the mass to decay. Given that the original mass of a particular radioactive substance is 800 grams and after 10 years it has decayed to 600 grams find the half-life for this substance.

Answer

The radioactive decay is described by the differential equation $^{dm}/_{dt} = -km$

Hence

$$\int \frac{dm}{m} = \int - kdt$$

$\ln m = - kt + C$

$m = e^{- kt + C}$

Since m (0) = 800 then $m = 800e^{-kt}$

Now we are also given that m = 600 when t = 10

Hence $600 = 800e^{-10k}$

Thus

$e^{- 10k} = 0.75$

$- 10k = \ln 0.75$

$$k = \frac{\ln 0.75}{- 10} = 0.0288 \text{ (to 3 s.f.)}$$

Therefore the mass of the radioactive substance at time t is given by $m(t) = 800e^{-0.0288t}$

To find the half life for this substance we let $m(t) = {}^{1}/_{2} \times 800 = 400$ and solve the above equation for t

$400 = 800e^{-0.0288t}$

$e^{-0.0288t} = 0.5$

$-0.0288t = \ln (0.5)$

t = 24.1 years approx

So it will take approximately 24.1 years for half of the mass to decay.

3. 'Flu virus

In a town with population 40 000 a 'flu virus spread rapidly last winter. The **percentage** P of the population infected in t days after the initial outbreak satisfies the differential equation ${}^{dP}/_{dt} = kP$ where k is a constant

a) If 100 people are infected initially, find, in terms of k, the percentage infected t days later.

b) Given that 500 people have 'flu after 7 days, how many more are likely to have contracted the virus after 10 days ?

This example comes from CSYS Mathematics 1998, Paper 1.

Answer

Since ${}^{dP}/_{dt} = kP$ then **a)**

$$\int \frac{1}{P}dP = \int kdt$$

$\ln P = kt + C$

$P = e^{kt + C}$

$= Ae^{kt}$ where $A = e^{C}$

Since 100 people were infected initially when t = 0 then

P = the percentage of the population infected = $100 \times 100 \div 40000 = 0.25$We can substitute these values into the equation for P

Therefore $P = Ae^{kt}$ becomes

$0.25 = Ae^0 : A = 0.25$

So in terms of k, the percentage infected t days later is given by $P = 0.25e^{kt}$

b) Since 500 people have 'flu after 7 days then when t = 7

$P = \frac{500}{40000} \times 100 = 1.25$

Hence

$$\begin{aligned} 1.25 &= 0.25e^{7k} \\ e^{7k} &= 5 \\ 7k &= \ln 5 \\ k &= \tfrac{1}{7}\ln 5 \approx 0.230 \text{ (to 3 s.f.)} \end{aligned}$$

To calculate the total number of people with 'flu after 10 days we substitute t = 10 and k = 0.230 into the formula for P. Hence $P = 0.25e^{230} = 2.49$ (to 3 sf.)

Therefore there will be approximately 2.49% of 40000 = 997 people with 'flu. This is an increase of about 497 people.

Half-life of a radioactive element

20 min

There is an interactive exercise on the web for you to try at this point if you wish.

Exercise 13

20 min

An on-line assessment is available at this point, which you might find helpful.

Q57: The number of strands of bacteria B (t) present in a culture after t days of growth is assumed to be increasing at a rate proportional to the number of strands present. Write down a differential equation for B and solve it to find B in terms of t.

Given that the number of strands observed after 1 day is 502 and after 4 days is 1833, find the number of strands initially present.

CSYS Mathematics 1993, Paper 1.

Q58: A capacitor is a device for storing electrical charge, Q. As a result of leakage, an electrical capacitor discharges the charge at a rate proportional to the charge present. If the charge has the value Q_0 at time t = 0, find Q as a function of t and the constant of proportionality, k

Q59: Atmospheric pressure P decreases as altitude H above sea level increases. In fact the rate that atmospheric pressure changes with altitude is proportional to P. Write down a differential equation for P and solve it to find P in terms of H.

Given that the pressure at sea level is 1000 millibars and that the pressure at an altitude of 10 km is 223 millibars determine the pressure at an altitude of 20 km. Give your answer to three significant figures.

Q60: The half-life for polonium is 140 days. In an experiment the polonium is no longer useful after 80% of the radioactive nuclei originally present have disintegrated. To the nearest day, for how many days can you use the polonium in the experiment.

2.5.4 Further applications of differential equations

Learning Objective

Construct and solve differential equations with separable variables in scientific contexts

Newton's law of cooling

When coffee is left standing in a cup, it cools until its temperature drops to that of the surrounding air. When a hot metal ingot is dropped into water its temperature drops to that of the surrounding water. These are examples that obey Newton's law of cooling.

Newton's law states that the rate of cooling of a previously heated body is proportional to the **difference** between its temperature T after time t, and the temperature T_s of the surrounding environment.

We can represent this statement by a differential equation in the following way

The rate of change of temperature T is $^{dT}/_{dt}$

But as the rate of change of T is proportional to the **difference** between T and T_s then: the rate of change of temperature T = $-k(T - T_s)$

Hence we obtain the differential equation $^{dT}/_{dt} = -k(T - T_s)$

Example a)

A hot liquid is cooling in a surrounding environment whose temperature remains at T_s. If T denotes the temperature of the liquid at time t, it follows from Newton's Law of Cooling that $^{dT}/_{dt} = -k\ (T - T_s)$ where k is a constant of proportionality.

If the temperature of the liquid is T_0 at time t = 0 show that $T = T_s + (T_0 - T_s)e^{-kt}$

b)

A cup of tea cools from 90°C to 60°C after 10 minutes in a room whose temperature was 20°C. Use Newton's law of cooling to calculate how much longer it would take the tea to cool to 40°C.

Answer

a) Since $^{dT}/_{dt} = -k(T - T_s)$ by separating the variables and integrating we have

$$\frac{dT}{dt} = -k(T - T_S)$$

$$\int \frac{dT}{(T - T_S)} = \int -kdt$$

$$\ln(T - T_S) = -kt + C$$

$$T - T_S = e^{-kt + C}$$

$$T - T_S = Ae^{-kt}, \quad \text{where } A = e^C$$

At $t = 0$, $T = T_0$ therefore

$T_0 - T_s = A\,e^0$

$A = (T_0 - T_s)$

Hence

$T - T_s = (T_0 - T_s)e^{-kt}$

$T = T_s + (T_0 - T_s)e^{-kt}$

b) For the cup of tea we have $T_0 = 90$ and $T_s = 20$

Hence$T = 20 + 70e^{-kt}$

Since $T = 60$ when $t = 10$ we have

$60 = 20 + 70e^{-10k}$

$40 = 70e^{-10k}$

$e^{-10k} = {}^{40}/_{70}$

$-10k = \ln({}^{4}/_{7})$

$k = -0.056$ (3 sf.)

Using this value for k we are now able to calculate how long it will take for the tea to cool to 40°C. This time we are required to find t. We have

$$40 = 20 + 70e^{-0.0560t}$$

$$20 = 70e^{-0.0560t}$$

$$e^{-0.0560t} = \frac{20}{70} = \frac{2}{7}$$

$$-0.0560t = \ln\left(\frac{2}{7}\right)$$

$$t = -\frac{1}{-0.0560}\ln\left(\frac{2}{7}\right) = 22.4 \text{ minutes}$$

Therefore it takes an extra 12.4 minutes for the tea to cool down to 40°C

Finally we solve an old CSYS question which is rather difficult.

Example : Water pollution

An accident at a factory on a river results in the release of a polluting chemical. Immediately after the accident, the concentration of the chemical in the river becomes k g/m^3. The river flows at a constant rate of w m^3/hour into a loch of volume V m^3. Water flows over the dam at the other end of the loch at the same rate. The level, x g/m^3, of the pollutant in the loch t hours after the accident satisfies the differential equation

$V^{dx}/_{dt} = w\,(k - x)$

a) Find the general solution for x in terms of t

b) In this particular case, the values of the constants are V = 16 000 000, w = 8000 and k = 1000

i Before the accident, the level of chemical in the loch was zero. Fish in the loch will be poisoned if the level of pollutant in the loch reaches 10 g/m^3. How long do the authorities have to stop the leak before this level is reached?

ii In fact, when the level of pollutant in the loch has reached 5 g/m^3, the leak is located and plugged. The level in the river then drops to zero and the level in the loch falls according to the differential equation $V^{dx}/_{dt} = -wx$

According to European Union standards, $1g/m^3$ is a safe level for the chemical. How much longer will it be before the level in the loch drops to this value?

This example comes from CSYS Mathematics 1997, Paper 1.

Answer

a) Since $V^{dx}/_{dt} = w(k - x)$

separating the variables and integrating gives

$$\int \frac{V}{k - x}dx = \int w dt$$
$$-V\ln(k - x) = wt + C$$
$$\ln(k - x) = -\frac{wt}{V} + C$$
$$k - x = \exp\left(-\frac{wt}{V} + C\right)$$
$$x = k - A\exp\left(-\frac{wt}{V}\right), \quad A = \exp(C)$$

b)

i

Since V = 16 000 000, w = 8000 and k = 1000 we have

$$x = 1000 - A\exp\left(-\frac{8000t}{16\,000\,000}\right)$$
$$= 1000 - A\exp\left(-\frac{t}{2000}\right)$$

Since x = 0 when t = 0, $0 = 1000 - A\exp(0) \Rightarrow A = 1000$

Hence

$x = 1\,000\left(1 - \exp\left(-\frac{t}{2\,000}\right)\right)$

To find how long it will take the level of pollutant in the loch to reach $10g/m^3$ we now substitute x = 10 into our formula.

$$10 = 1000\left(1 - \exp\left(-\frac{t}{2000}\right)\right)$$

$$1 - \exp\left(-\frac{t}{2000}\right) = 0.01$$

$$\exp\left(-\frac{t}{2000}\right) = 1 - 0.01 = 0.99$$

$$-\frac{t}{2000} = \ln(0.99)$$

$$t = -2000\ln(0.99) = 20.1 \text{ hours}$$

ii After the leak has been plugged the level of pollutant in the loch is now given by $V^{dx}/_{dt} = -wx$

Separating the variables and integrating we have

$$\int \frac{dx}{x} = -\int \frac{w}{V}dt$$

$$\ln x = -\frac{wt}{V} + C$$

$$x = \exp\left(-\frac{wt}{V} + C\right)$$

$$= B\exp\left(-\frac{wt}{V}\right), \quad \text{where } B = \exp(C)$$

Now we are given that when the leak is plugged the level of pollutant in the loch is 5 g/m^3

Hence x = 5 when t = 0 and so

5 = B exp (0)

B = 5

Using the values that we were given for the variables above we now have

$x = 5\exp\left(-\frac{t}{2000}\right)$

Hence, we can calculate how long it takes for the level of pollution to reach 1 g/m^3 by substituting x = 1 into the previous formula.

$$1 = 5\exp\left(-\frac{t}{2000}\right)$$

$$\exp\left(-\frac{t}{2000}\right) = 0.2$$

$$-\frac{t}{2000} = \ln(0.2)$$

$$t = -2000\ln(0.2)$$

$$= 3219 \text{ hours}$$

Now try the questions in Exercise 14.

Exercise 14

An on-line assessment is available at this point, which you might find helpful.

30 min

Q61: After 15 minutes in a room that is at a temperature of 18°C a cup of hot chocolate cools from 90°C to 50°C. Use Newton's Law of Cooling to estimate how much longer it will take to cool to 30°C.

Q62: A hard-boiled egg is put in a sink of water at 16°C to cool. After 5 minutes the egg's temperature is found to be 40°C and then after a further 15 minutes it has cooled to 20°C. Assuming that the water has not warmed significantly, use Newton's Law of Cooling to estimate the original temperature of the hard-boiled egg when it was put in the water. (Give your answer to the nearest degree.)

Q63: Suppose the salmon population, P (t), in a loch is attacked by disease at time t = 0, with the result that ${}^{dP}/_{dt} = -k\sqrt{P}$

If there were initially 1600 salmon in the loch and 576 were left after 8 weeks, how long would it take all the salmon in the loch to die?

Q64: When a valve is opened, the rate at which the water drains from a pool is proportional to the square root of the depth of the water.

This can be represented by the differential equation

$\frac{dh}{dt} = -\frac{\sqrt{h}}{10}, \quad h \geqslant 0$

Where h is the depth (in metres) of the water and t is the time (in minutes) elapsed since the valve was opened.

a) Express h as a function of t.

b) Find the solution of the equation, given that the pool was initially 4 metres deep.

c) The next time the pool had to be drained, the water was initially 9 metres deep. How long did it take to drain the pool on this occasion?

CSYS Mathematics 1996, Paper 1.

Q65: A large population of N individuals contains, at time t = 0, just one individual with a contagious disease. Assume that the spread of the disease is governed by the equation ${}^{dN}/_{dt} = k(N - n)n$

where n (t) is the number of infected individuals after a time t days and k is a constant.

a) Find n explicitly as a function of t.

(You might find this quite tricky. It will help if you are able to rewrite ${}^{1}/_{[(N-n)n]}$ as the partial fractions $\frac{1}{N}\left(\frac{1}{N-n} + \frac{1}{n}\right)$. Try this for yourself.)

b) Given that measurements reveal that half of the population is infected after 100 days, show that

$k = \{\ln(N - 1)\}/100N$

c) What value does n (t) approach as t tends to infinity?

CSYS Mathematics 1992, Paper 1.

2.6 Summary

1. $\int \frac{1}{\sqrt{1 - x^2}} dx = \sin^{-1} x + C$

   $\int \frac{1}{1 + x^2} dx = \tan^{-1} x + C$

2. $\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

   $\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

3. When g (x) is a polynomial that can be factorised into the product of linear and quadratic factors then expressing the integral $\int \frac{f(x)}{g(x)} dx$ in partial fractions enables us to carry out the integration.

4. Integration by parts allows us to integrate the product of two functions.

   $\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$

5. A separable differential equation can be written in the form $dy/dx = f(x)\, g(y)$

   It is solved in the following way

   $\int \frac{1}{g(y)} dy = \int f(x) dx$

2.7 Extended Information

There are links on-line to a variety of web sites related to this topic

2.8 Review exercise

Review exercise in further integration

10 min

An on-line assessment is available at this point, which you might find helpful.

Q66: Find $\int \frac{3x + 4}{x(x - 4)} dx$

Q67: Use the method of integration by parts to evaluate $\int_0^{\pi} x \sin x dx$

Q68: Find the general solution of the differential equation $dy/dx = x^2 y$

2.9 Advanced review exercise

Advanced review exercise in further integration

An on-line assessment is available at this point, which you might find helpful.

40 min

Q69:

a) Find partial fractions for
$\frac{4}{x^2 - 4}$
b) By using a) obtain
$\int \frac{x^2}{x^2 - 4} dx$

CSYS Mathematics 1997, Paper 1.

Q70: Use integration by parts to obtain

$\int_0^3 x\sqrt{x + 1}\, dx$

CSYS Mathematics 1999, Paper 1.

Q71:

a) Find a real root of the cubic polynomial
$c(x) = x^3 - x^2 - x - 2$
and hence factorise it as a product of a linear term l (x) and a quadratic term q (x)
b) Show that c (x) cannot be written as a product of three linear factors.
c) Use your factorisation to find values of A, B and C such that

$\frac{5x + 4}{x^3 - x^2 - x - 2} = \frac{A}{l(x)} + \frac{Bx + C}{q(x)}$

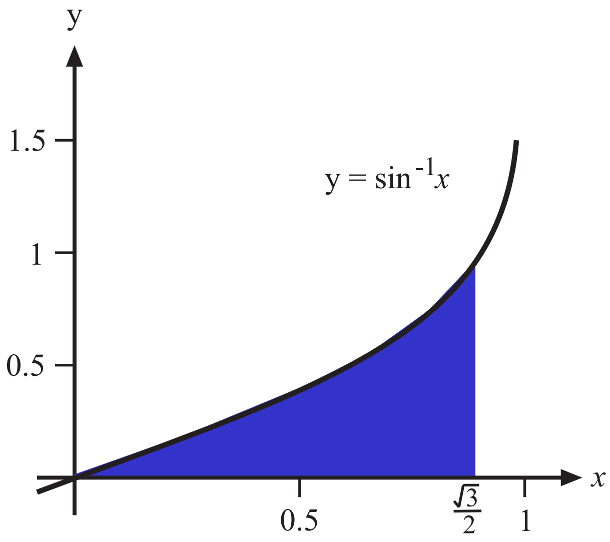
Hence obtain the indefinite integral

$\int \frac{5x + 4}{x^3 - x^2 - x - 2} dx.$

CSYS Mathematics 1996, Paper 1.

Q72:

a) Find the derivative of the function $g(x) = \sqrt{(1 - x^2)}$ and hence, or otherwise, obtain the indefinite integral
$\int \frac{x}{\sqrt{1 - x^2}} dx$
b) Use integration by parts to show that
$\int f(x)dx = x f(x) - \int x f'(x)dx$
where f is a differentiable function with derivative f'
c) The diagram below represents the graph of $y = \sin^{-1}x$. Use the previous results to calculate the area of the shaded region which lies between $x = 0$ and $x = \frac{\sqrt{3}}{2}$



CSYS Mathematics 1998, Paper 1.

Q73: In a chemical reaction, two substances X and Y combine to form a third substance Z. Let Q (t) denote the number of grams of Z formed t minutes after the reaction begins. The rate at which Q (t) varies is governed by the differential equation

$$\frac{dQ}{dt} = \frac{(30 - Q)(15 - Q)}{900}$$

a) Express $\frac{900}{(30 - Q)(15 - Q)}$ in partial fractions.

b) Use your answer to a) to show that the general solution of the differential equation can be written in the form

$$A \ln\left(\frac{30 - Q}{15 - Q}\right) = t + C$$

where A and C are constants.

State the value of A and, given that Q (0) = 0, find the value of C.

Find, correct to two decimal places:

i the time taken to form 5 grams of Z;

ii the number of grams of Z formed 45 minutes after the reaction begins.

CSYS Mathematics 1999, Paper 1.

2.10 Set review exercise

Set review exercise in further integration

10 min

An on-line assessment is available at this point, which you will need to attempt to have these answers marked. These questions are not randomised on the web. The questions on the web may be posed in a different manner but you should have the required answers in you working.

Q74: Find partial fractions for the expression $f(x) = \frac{2x + 5}{(x - 2)(x + 1)}$ Hence determine $\int f(x)\, dx$

Q75: Use integration by parts to evaluate $\int_{0}^{\pi/4} x \sin(-2x)dx$

Q76: Find the general solution of the differential equation $\frac{dy}{dx} = \frac{\cos\left(\frac{x}{2}\right)}{4y}$

Topic 3

Complex numbers

Contents

Learning Objectives

- *Understand and use complex numbers.*

Minimum Performance Criteria:

- *Perform a simple arithmetic operation on two complex numbers of the form a + bi*

- *Evaluate the modulus and argument of a complex number.*
- *Convert from Cartesian to polar form.*
- *Plot a complex number on an Argand diagram.*

3.1 Revision exercise

Learning Objective

Identify areas which need revision

This exercise should to help identify any areas of weakness in techniques which are required for the study of this unit. Some revision may be necessary if any of the questions seem difficult. There is a web exercise if you prefer it.

Revision exercise

15 min

Q1: Solve the quadratic $3x^2 - 13x + 4 = 0$

Q2: Expand $(2x + 3)(x - 2)$

Q3: Expand $(2x - y)^4$ using the Binomial theorem.

Q4: Show that $\cos(a + \pi) = -\cos a$

Q5: Simplify the expression $\frac{(3\sqrt{48} - \sqrt{27})}{(2\sqrt{12} + \sqrt{75})}$

3.2 Introduction

Learning Objective

State why the complex numbers set is required

Usually in mathematics the first number system that is encountered is the set of natural numbers ($\mathbb{N}$), i.e. 1, 2, 3, ...

This set is subsequently enlarged to cope with various difficulties:

- The set of integers ($\mathbb{Z}$) allows subtractions such as $3 - 5 = -2$
  ($-2 \notin \mathbb{N}$ but $-2 \in \mathbb{Z}$).
- The set of rationals ($\mathbb{Q}$) enables divisions such as $4 \div 6 = {}^2/_3$
  (${}^2/_3 \notin \mathbb{Z}$ but ${}^2/_3 \in \mathbb{Q}$).
- The set of reals ($\mathbb{R}$) provides for solutions to certain quadratic equations.
  For example, $x^2 - 2 = 0$. The solutions are $x = \pm\sqrt{2}$
  ($\pm\sqrt{2} \notin \mathbb{Q}$ but $\pm\sqrt{2} \in \mathbb{R}$).

However the set $\mathbb{R}$ will not allow solutions to all quadratic equations.
For example, $x^2 + 1 = 0$. This implies that $x^2 = -1$, which has no real solution since x^2 is always positive for real numbers.

This difficulty can be overcome by enlarging the set $\mathbb{R}$ to include a new 'number' denoted i with the property that $i^2 = -1$

This number i leads to a new and extensive branch of Mathematics.

3.3 Complex numbers and the complex plane

Learning Objective

Demonstrate the spatial representation of complex numbers and perform arithmetic on them

The introduction of this new number i overcomes the problem of finding the square roots of a negative number.

Example : Finding the square roots of a negative number

Find the square roots of -25

Answer:

$-25 = 25 \times -1 = 25 \times i^2 = (5 \times i)^2$

So this gives $\sqrt{-25} = \pm 5i$

Q6: Find the square roots of -16

Q7: Find the square roots of -64

Q8: Find the square roots of -8

Q9: Write down three quadratics which have no solution in the real numbers.

With the existence of the square roots of a negative number, it is possible to find the solutions of any quadratic equation of the form $ax^2 + bx + c = 0$ using the quadratic formula.

Examples

1. Solve the quadratic equation $x^2 - 6x + 25 = 0$

Answer:

The quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives

$x = \frac{6 \pm \sqrt{36 - 100}}{2} = \frac{6 \pm \sqrt{-64}}{2}$

i.e. $x = 3 + 4i$ and $x = 3 - 4i$

(here $a = 1$, $b = -6$, and $c = 25$)

2. Find the solutions of the quadratic $x^2 - 2x + 3 = 0$

Answer:

The solutions are found by using $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1$, $b = -2$ and $c = 3$

Thus the solutions are

$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm \sqrt{-8}}{2} = 1 \pm i\sqrt{2}$

That is, the solutions are $x = 1 \pm i2\sqrt{2}$

Note the way that these solutions are written. It does not matter whether the i comes before or after the rest of the term. Here by putting it first, confusion with $\sqrt{2i}$ is avoided.

Q10: Solve the quadratic equation $x^2 - 2x + 5 = 0$ using the quadratic formula.
(Your answer will involve the new number i)

Q11: Solve the quadratic equation $x^2 + 2x + 6 = 0$ using the quadratic formula.

Q12: Solve the quadratic equation $x^2 + x + 1 = 0$ using the quadratic formula.

The solutions to these problems are in the form a + bi (or a + ib). Such expressions are called complex numbers.

A **complex number** is a number of the form a + bi where a and b are real numbers and $i = \sqrt{-1}$

The complex number may also be written as a + ib

The set of complex numbers is written as $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

Note that it is customary to denote a complex number by the letter z.

If z = a + ib then:

- The number a is called the real part of z
- The number b is called the imaginary part of z

These are sometimes denoted Re(z) for a and Im(z) for b

That is, Re (a + ib) = a and Im (a + ib) = b

Exercise on finding real and imaginary points

You might find it helpful to try the web exercise, which is similar.

5 min

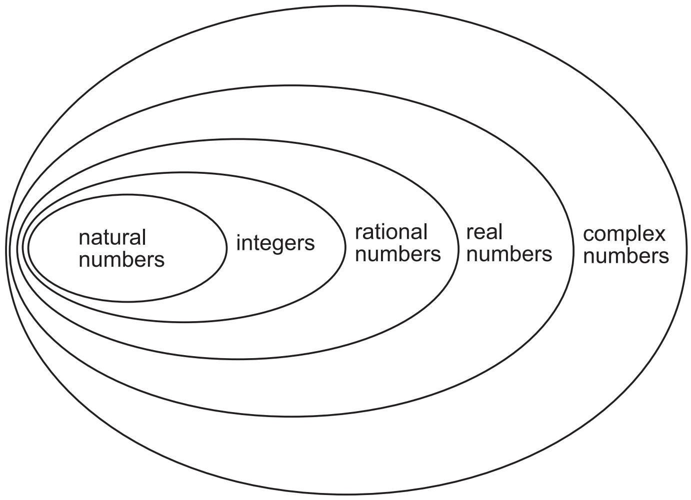
Q13: What is the imaginary part of the complex number 5 - 7i?

Q14: What is the real part of the complex number 3i?

If the real part of a complex number is zero, the number is said to be *purely imaginary*.

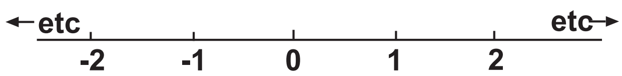
If the imaginary part of a complex number is zero, the number is just a real number. So the real numbers are a subset of the complex numbers.

Here is how the complex numbers fit in with the standard number sets already known.



The real numbers can be represented on the number line.

The number line



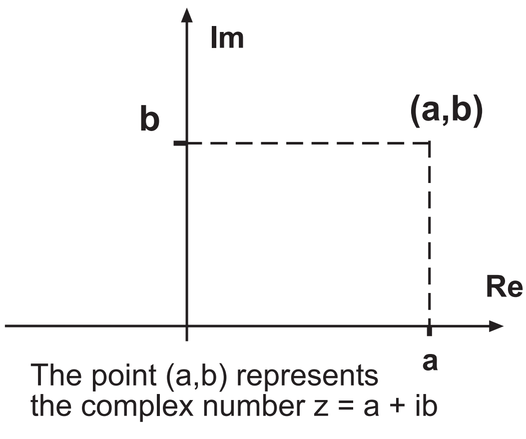
Is there a similar representation for the complex numbers?

The definition of a complex number involves two real numbers. Two real numbers give a point on a plane.

So complex numbers can be plotted in a plane by using the x-axis for the real part and the y-axis for the imaginary part.

This plane is called *The Complex Plane* or *Argand Diagram*.

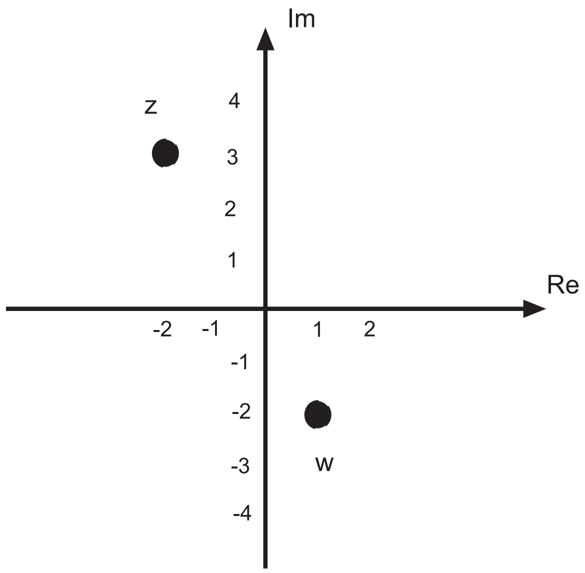
An Argand Diagram



Example Show the complex numbers z = -2 + 3i and w = 1 - 2i on an Argand diagram.

Answer:

The complex numbers z and w are shown on the following diagram.



Plotting complex numbers exercise

There are questions on identifying points in the complex plane on the web.

5 min

Q15: Draw an Argand diagram and plot the points which represent the complex numbers:

- a = 3 + 2i
- b = 4 - 3i
- c = -2 + 3i
- d = -3 - 2i

3.4 The arithmetic of complex numbers

Learning Objective

Perform basic arithmetic operations on complex numbers

3.4.1 Addition and subtraction of complex numbers

Take two complex numbers, 3 + 4i and 2 - i

The rule for adding these numbers is straightforward:

- Add the real parts together to give the real part 5 (i.e. 3 + 2 = 5)
- Add the imaginary parts together to give the imaginary part 3 (i.e. 4 + (-1) = 3)

So (3 + 4i) + (2 - i) = (3 + 2) + (4i - i) = (5 + 3i) (a + ib) + (c + id) = (a + c) + i(b + d)

- Add the real parts.
- Add the imaginary parts.

Example : Adding complex numbers

Add $\frac{1}{2} - i\frac{\sqrt{3}}{4}$ and $3 + i\frac{\sqrt{3}}{2}$

Answer:

$$\left(\frac{1}{2} - i\frac{\sqrt{3}}{4}\right) + \left(3 + i\frac{\sqrt{3}}{2}\right) = \left(\frac{1}{2} + 3\right) + i\left(\frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) = \frac{7}{2} + i\frac{\sqrt{3}}{4}$$

[Or alternatively

Adding the real parts gives $\left(\frac{1}{2} + 3\right) = \frac{7}{2}$

Adding the imaginary parts gives $\left(\frac{-\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{4}$]

So the sum is $\frac{7}{2} + i\frac{\sqrt{3}}{4}$

5 min

Adding complex numbers exercise

There are some randomised questions on adding complex numbers on the web if you prefer them.

Q16: Add the complex numbers 3 - i and 2 + 5i

Q17: Add the complex numbers 4 - 2i and 1 + 2i

Q18: Add the complex numbers 2 - i and -2

Q19: Add the complex numbers 3i and 3 + 2i

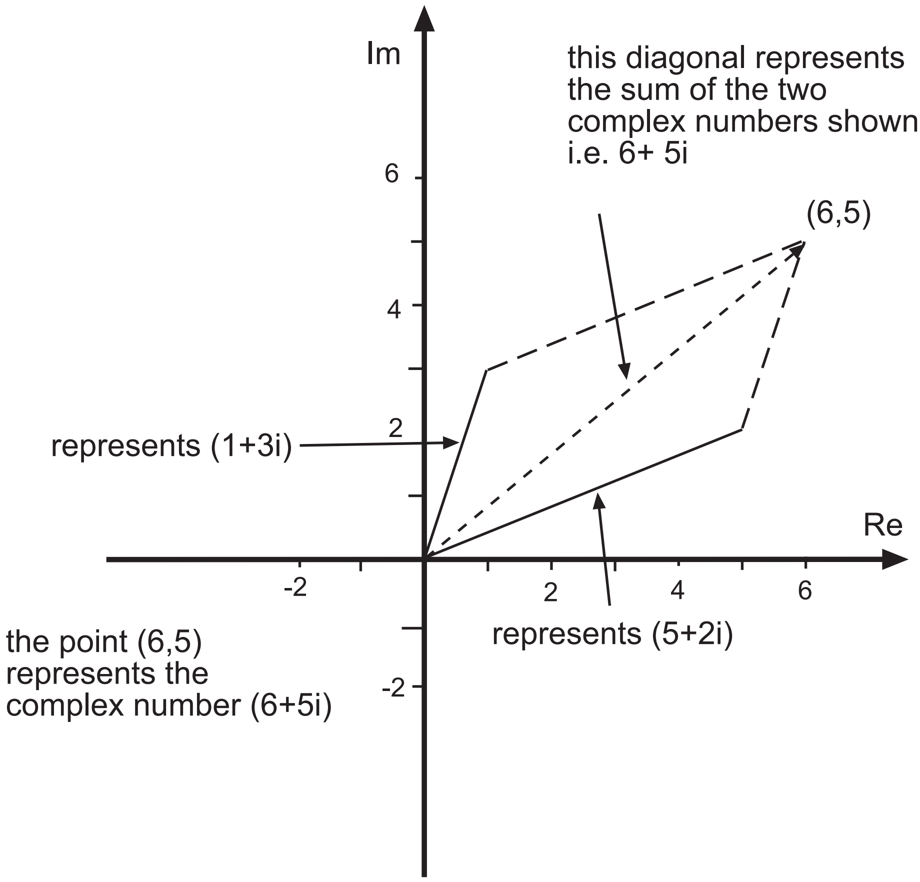
5 min

Demonstration of adding two complex numbers

There is a visual demonstration of the adding of two complex numbers on the web.

In the complex plane this addition can be interpreted geometrically as follows.

Example: (1 + 3i) + (5 + 2i) = 6 + 5i



Note the construction of the diagram:

- The points representing the complex numbers are plotted.
- Lines are drawn joining each of them to the origin.
- Two further lines are drawn to complete a parallelogram.
- The diagonal of this parallelogram is drawn starting from the origin.

This diagonal corresponds to the sum.

Subtraction of complex numbers is just as straightforward as addition.

(a + ib) - (c + id) = (a - c) + i(b - d)

- Subtract the real parts.
- Subtract the imaginary parts.

Example : Subtracting two complex numbers

Subtract 3 - i from 7 + i

Answer:

(7 + i) - (3 - i) = (7 - 3) + i(1 - (-1)) = 4 + 2i

5 min

Subtracting complex numbers exercise

You might like to try the web questions on subtracting complex numbers.

Q20: Subtract 3 from 2 - 4i

Q21: Subtract 3i from 4 - i

5 min

Demonstration of subtracting complex numbers

There is a visual demonstration of subtracting complex numbers on the web.

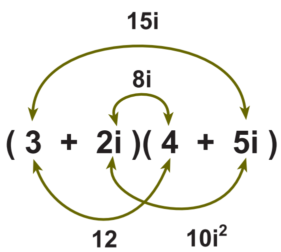
3.4.2 Multiplication of complex numbers

The technique for multiplying two complex numbers is similar to that used when multiplying out two brackets.

For example, $(x + 2)(x + 3) = x^2 + 2x + 3x + 6 = x^2 + 5x + 6$

Now consider the two complex numbers (3 + 2i) and (4 + 5i)

To multiply these together take the same approach.



This will give $(3 \times 4) + (3 \times 5i) + (2i \times 4) + (2i \times 5i)$

$= 12 + 15i + 8i + 10i^2$

$= 12 + 23i - 10$ (Remember that $10i^2 = 10(-1) = -10$)

$= 2 + 23i$

Thus (3 + 2i)(4 + 5i) = (2 + 23i)

- Use the technique for multiplying out two brackets.

  (a + ib)(c + id) = (ac - bd) + i(bc + ad)

Examples

1. Multiplying two complex numbers

Multiply (1 + 2i) and (2 - 3i)

Answer:

$(1 + 2i)(2 - 3i) = (1 \times 2) + (1\times (-3i)) + (2i \times 2) + (2i \times (-3i))$

$= 2 - 3i + 4i + 6 = 8 + i$

2. Multiply (4 - 2i) and (2 + 3i)

Answer:

$(4 - 2i)(2 + 3i) = 8 +12i - 4i + 6 = 14 + 8i$

Multiplying complex numbers exercise

There are further questions on multiplying two complex numbers on the web to give you extra practice.

10 min

Q22: Multiply (1 - i) and (2 + i)

Q23: Multiply (3 - i) and (-2 - i)

Q24: Multiply (-4 + 4i) and (3 + 3i)

Square roots of a complex number

Earlier examples showed that the square roots of a negative real number could be found in terms of i in the set of complex numbers. It is also possible to find the square roots of any complex number.

Example : Finding the square roots of a complex number

Find the square roots of the complex number 5 + 12i

Answer:

Let the square root of 5 + 12i be the complex number (a + ib) so $(a + ib)^2 = 5 + 12i$

Note: the trick is to equate the real parts and the imaginary parts to give two equations, which can be solved simultaneously.

First of all multiply out the brackets.

$(a + ib)(a + ib) = a^2 - b^2 + 2abi$

So $(a^2 - b^2) + 2abi = 5 +12i$

Equate the real parts to give $a^2 - b^2 = 5$ (call this equation 1).

Equate the imaginary parts to give 2ab = 12 (call this equation 2).

Rearranging equation 2 to make b the subject gives $b = \frac{6}{a}$

Substitute $b = \frac{6}{a}$ in equation 1 to give $5 = a^2 - \frac{36}{a^2}$

So $a^4 - 5a^2 - 36 = 0$

Hence $(a^2 - 9)(a^2 + 4) = 0 \Rightarrow a^2 = -4$ or 9

Since $a \in \mathbb{R}$ then $a^2 \geq 0$ and $a^2 = -4$ is impossible.

This leaves $a^2 = 9$ which gives $a = \pm 3$

Substitute a = 3 in equation 2 to give b = 2

Substitute a = -3 in equation 2 to give b = -2

Hence the square roots are 3 + 2i and -3 - 2i

Square root exercise

10 min

There are more questions on finding the square roots of a complex number on the web if you wish to try them.

Q25: Find the square roots of the complex number -3 - 4i

Q26: Find the square roots of the complex number 5 - 12i

Q27: Find the square roots of the complex number 3 - 4i

Multiplication by i

Multiplication by i has an interesting geometric interpretation.

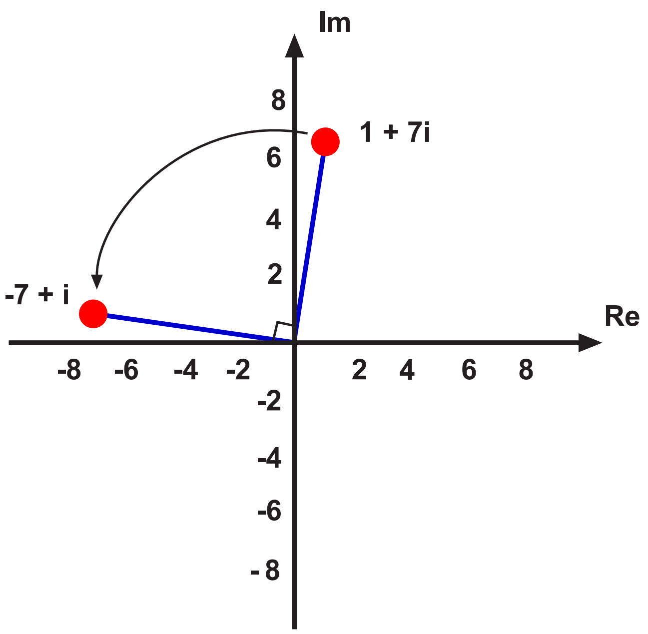
The following examples should demonstrate what happens.

Examples

1. Take the complex number 1 + 7i and multiply it by i

Answer:

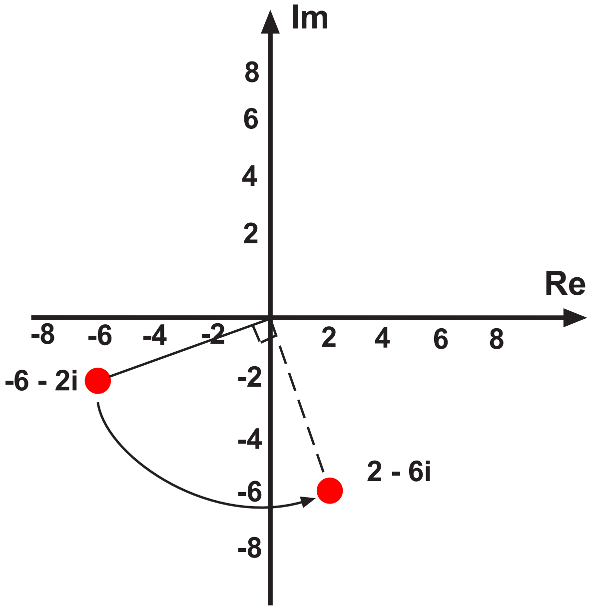
$(1 + 7i)i = i + 7i^2 = -7 + i$



2. Take the complex number -6 - 2i and multiply it by i

Answer:

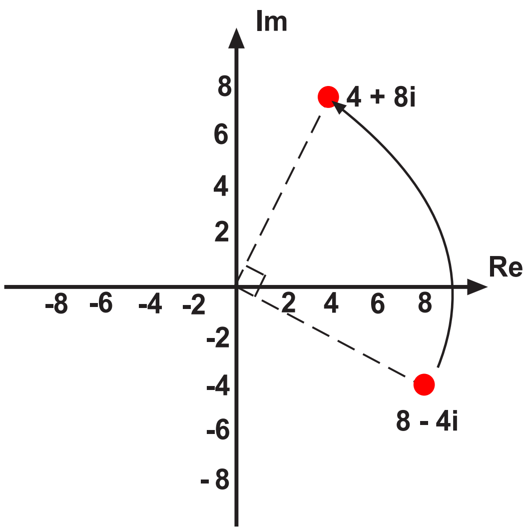
$(-6 - 2i)i = -6i - 2i^2 = 2 - 6i$



3. Take the complex number 8 - 4i and multiply it by i

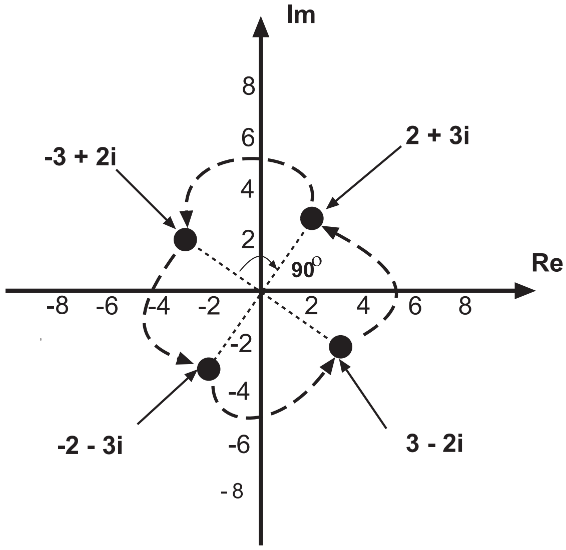
Answer:

$(8 - 4i)i = 8i - 4i^2 = 4 + 8i$



In each of the examples it can be seen that the effect of multiplying a complex number by i is a rotation of the point on the Argand diagram through 90° or $\frac{\pi}{2}$

Repeated multiplication by i can be summed up by the following diagram.



The diagram shows that if z is multiplied by i four times the answer is z; this is not surprising as $i^4 = i^2 \times i^2 = (-1)(-1) = 1$

Activity

Take the complex number 2 + 3i. Plot the results on an Argand diagram when this number is repeatedly multiplied by the complex number i.

Hence give a geometric interpretation of multiplication by i and check this interpretation with other complex numbers.

3.4.3 Conjugates of complex numbers

Learning Objective

Identify the conjugate of a complex number

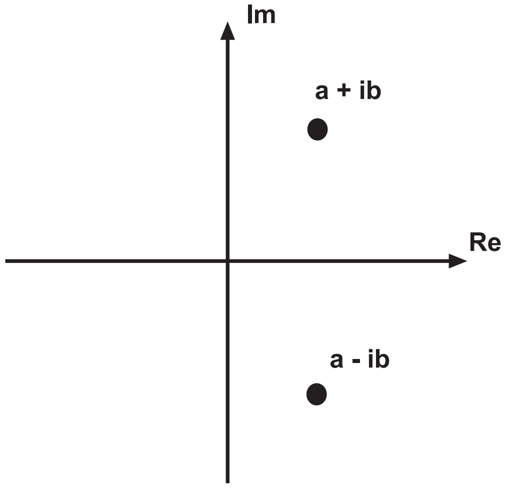
The basic operations of adding, subtracting and multiplying complex numbers are straightforward. Division is also possible but requires the use of the conjugate of a complex number.

The **conjugate** of the complex number z = a + ib is denoted by $\bar{z}$

and defined by $\bar{z} = a - ib$

The conjugate is sometimes denoted by z^*

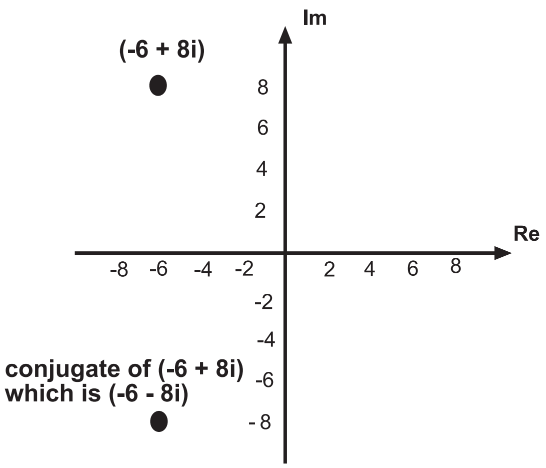
Geometrically the conjugate of a complex number can be shown in an Argand diagram as the reflection of the point in the x-axis.



Example Give the conjugate of the complex number z = -6 + 8i and show z and $\bar{z}$ on an Argand diagram.

Answer:

The conjugate is $\bar{z}$ = -6 - 8i



Find the conjugate demonstration


5 min

There is an interactive exercise to demonstrate the conjugate in an Argand diagram on the web.

Find the conjugate exercise


5 min

There are randomised conjugate questions on the web if you would like to use them instead.

Q28: Give the conjugate of 3 - 4i

Q29: Give the conjugate of -2 - i

Q30: What is the conjugate of 5i?

Q31: What is the conjugate of 3?

Q32: If z = 1 + 5i find its conjugate and multiply z by $\bar{z}$

3.4.4 Division of complex numbers

The technique used to divide complex numbers is similar to that used to simplify surd expressions such as $\frac{3}{\sqrt{2}}$

The trick is to multiply both top and bottom by a suitable factor, in this case $\sqrt{2}$

Here $\frac{3}{\sqrt{2}}$ is multiplied by $\frac{\sqrt{2}}{\sqrt{2}}$ to give $\frac{3}{2}\sqrt{2}$

Now consider (5 + i) divided by (2 - 3i) i.e. $\frac{5 + i}{2 - 3i}$

In this case a suitable factor by which to multiply top and bottom is (2 + 3i)

(the complex conjugate of 2 - 3i).

(2 - 3i)(2 + 3i) simplifies to give the real number 13 on the denominator.

Hence

$$\frac{5 + i}{2 - 3i} = \frac{(5 + i)(2 + 3i)}{(2 - 3i)(2 + 3i)} = \frac{10 + 17i - 3}{13} = \frac{(7 + 17i)}{13} = \frac{7}{13} + \frac{17}{13}i$$

Example : Dividing one complex number by another

If z_1 = 2 - 3i and z_2 = 3 - 4i, find $\frac{z_1}{z_2}$

Answer:

Multiply top and bottom by $\bar{z}$ = (3 + 4i).

Thus

$$\frac{z_1}{z_2} = \frac{2 - 3i}{3 - 4i} = \frac{(2 - 3i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{(18 - i)}{25} = \frac{18}{25} - \frac{1}{25}i$$

- Find the conjugate of the denominator.
- Multiply the complex fraction, both top and bottom, by this conjugate to give an integer on the denominator.
- Express the answer in the form a + bi

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + i(bc - ad)}{(c^2 + d^2)}$$
$$= \frac{(ac + bd)}{(c^2 + d^2)} + \frac{i(bc - ad)}{(c^2 + d^2)}$$

Example Divide 5 - 2i by 4 + 3i

Answer:

Multiply top and bottom by $\overline{4 + 3i} = 4 - 3i$

Thus $\frac{5 - 2i}{4 + 3i} = \frac{(5 - 2i)(4 - 3i)}{(4 + 3i)(4 - 3i)} = \frac{14 - 23i}{25} = \frac{14}{25} - \frac{23}{25}i$

Dividing two complex numbers exercise

You might like to try the questions on division of complex numbers on the web.

10 min

Q33: If $z_1 = 7 + i$ and $z_2 = 3 - 4i$, find $\frac{z_1}{z_2}$

Q34: If $z_1 = 7 + i$ and $z_2 = 3 - 4i$, find $\frac{z_2}{z_1}$

Q35: Find the real and imaginary parts of the fraction $\frac{2 - 3i}{- 1 + 5i}$

Q36: Find the real and imaginary parts of the complex fraction $\frac{3 + i}{4 + 2i}$

Q37: Find the real and imaginary parts of the complex fraction $\frac{1}{i}$

Activity

Take the complex number 2 + 3i. Plot the results on an Argand diagram when this number is repeatedly divided by the complex number i.

Hence give a geometric interpretation of division by i and check this interpretation with other complex numbers.

Multiplying and dividing activity

There are interactive exercises on the web to show the geometric effects of multiplying and dividing two complex numbers.

10 min

3.5 The modulus, argument and polar form of a complex number

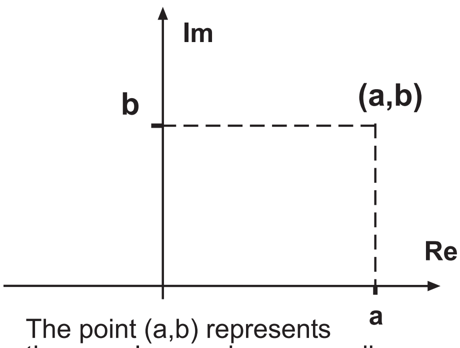
Learning Objective

Find and use the modulus and argument to obtain the polar form

Remember the Argand diagram in which the point (a, b) corresponds to the complex number z = a + ib

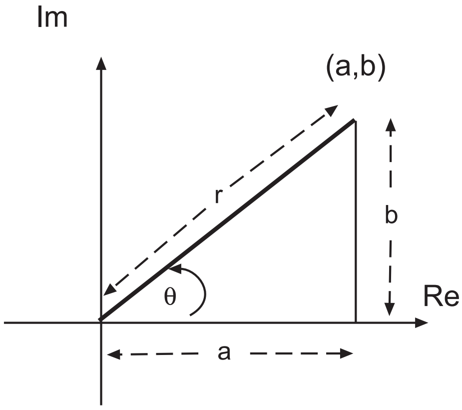
When the complex number is written as a + ib where a and b are real numbers, this is known as the **Cartesian form**.

An Argand Diagram



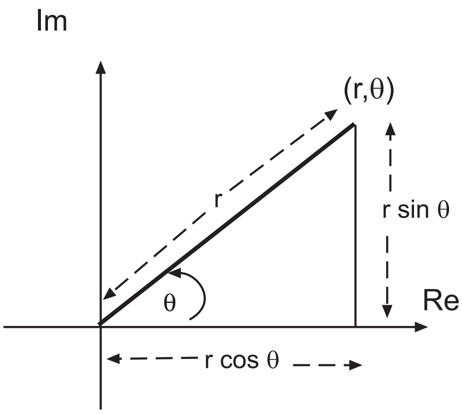
The point (a,b) represents
the complex number z = a + ib

This point (a,b) can also be specified by giving the distance, r, of the point from the origin and the angle, θ, between the line joining the point to the origin and the positive x-axis.



The point (a,b) represents
the complex number z = a + ib

By some simple trigonometry it follows that a = r cos θ and b = r sin θ



The point (r, θ) represents
the complex number z = r cosθ + i r sinθ

Thus the complex number z can be written as r cos θ + i r sin θ

This is known as the **polar form** of a complex number.

r is called the **modulus** of z and θ is the **argument** of z.

The definitions of these terms are:

The modulus r of a complex number z = a + ib is written $|z|$ and defined by $|z| = \sqrt{a^2 + b^2}$

Geometrically the modulus is the distance, on an Argand diagram, of the point representing z from the origin.

The argument θ of a complex number is the angle between the positive x-axis and the line representing the complex number on an Argand diagram. It is denoted arg (z).

The polar form of a complex number is $z = r(\cos\theta + i\sin\theta)$ where r is the modulus and θ is the argument.

In some cases calculations in polar form are much simpler so it is important to be able to work with complex numbers in both forms.

There will be times when conversion between these forms is necessary.

Given a modulus (r) and argument (θ) of a complex number it is easy to find the number in **Cartesian form**.

Use the following steps to do this:

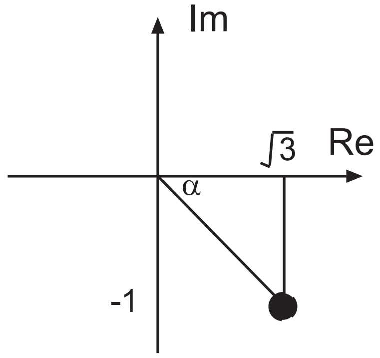
- Evaluate 'a' = $r\cos\theta$ and 'b' = $r\sin\theta$
- Write down the number in the form a + ib

Examples

1. If a complex number z has modulus of 2 and argument of $\frac{-\pi}{6}$, express z in the form a + ib and plot the point which represents the number in an Argand diagram.

Answer:

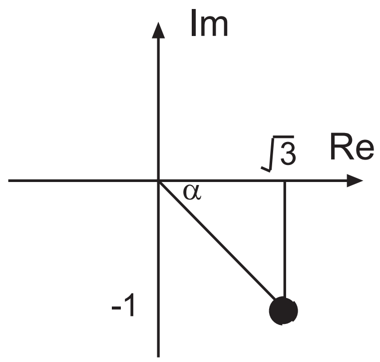
- $a = 2\cos\left(\frac{-\pi}{6}\right) = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$
- $b = 2\sin\left(\frac{-\pi}{6}\right) = 2 \times \frac{-1}{2} = -1$
- So $a + ib = \sqrt{3} - i$



Check: 3 - i lies in the fourth quadrant and $\frac{-\pi}{6}$ is also in the fourth quadrant.

2. Plot the complex number with modulus 2 and argument $\frac{11\pi}{6}$ in an Argand diagram.

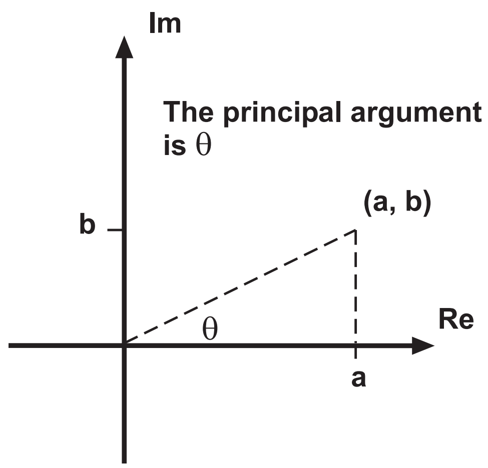
Answer:



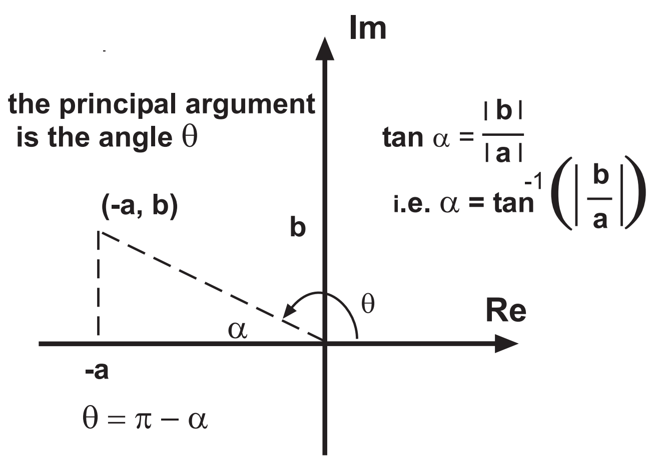
NB: notice that this is exactly the same point as the previous example. This duplication demonstrates that different arguments can give the same complex number. To prevent confusion one argument is referred to as the principal value of the argument.

The principal value of an argument is the value which lies between $-\pi$ and π

The values of the principal argument for a complex number in each quadrant are shown on the following diagrams.



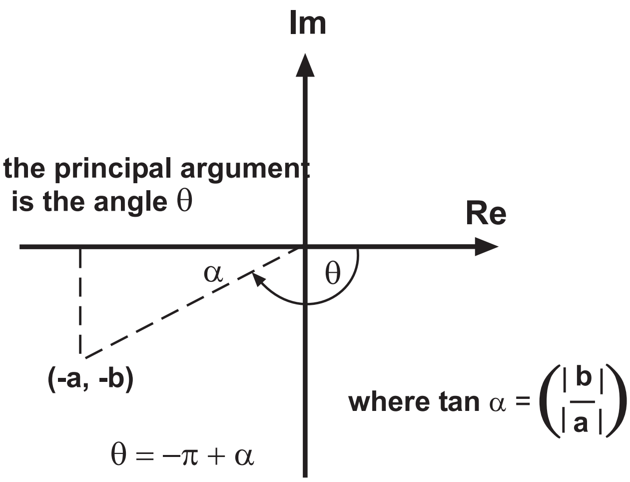
arg z = θ where tan θ = $\left|\frac{b}{a}\right|$, i.e. $\theta = \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$



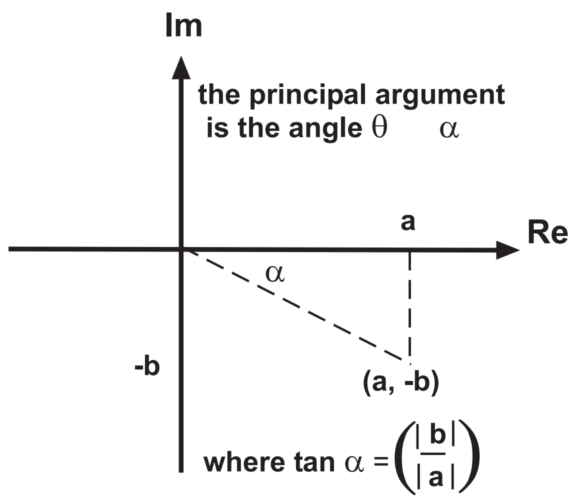
arg z $= \theta$ where $\theta = \pi - \alpha$ and tan $\alpha = \left|\frac{b}{a}\right|$, i.e. $\alpha = \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$

In the previous two examples the complex number is represented by a point in either the first or second quadrant and the principle argument is positive (lying between 0 and π).

In the next two examples when the complex number is represented in the third and fourth quadrants the principal argument is negative (lying between 0 and -π).



arg z $= \theta$ where $\theta =$ - π + α and tan $\alpha = \left|\frac{b}{a}\right|$, i.e. $\alpha =$ tan $^{-1}\left(\left|\frac{b}{a}\right|\right)$



arg z $= \theta$ where $\theta =$ - α and tan $\alpha = \left|\frac{b}{a}\right|$, i.e. $\alpha =$ tan $^{-1}\left(\left|\frac{b}{a}\right|\right)$

NB: Taking tan $^{-1}\left(\frac{b}{a}\right)$ without the modulus sign on a calculator may give a different value than required.

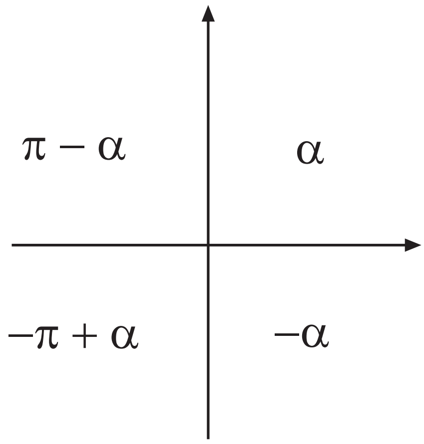
Strategy for finding the modulus and argument

For the complex number z = a + ib:

- Plot the point (a, b) in an Argand diagram.
- Find the modulus r = $\sqrt{a^2 + b^2}$

- Evaluate α =tan$^{-1}\left(\left|\frac{b}{a}\right|\right)$
- Use the Argand diagram to find the value of the argument between -π and +π

In carrying out the final step the following quadrant diagram may be useful:



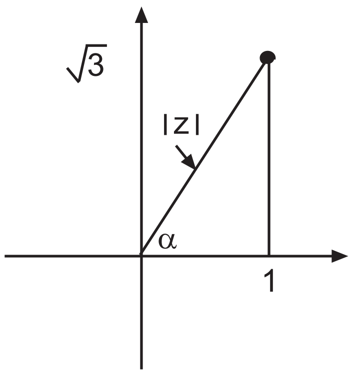
Once the modulus and argument of z are known the **polar form** of a complex number can be written down.

Examples

1. Find the modulus and principal argument of the complex number z = 1 + i$\sqrt{3}$

Answer:

First plot the point 1 + i$\sqrt{3}$ in the Argand diagram.



Then $|\,z\,|$ = $\sqrt{a^2 + b^2} = \sqrt{(1)^2 + (\sqrt{3})^2} = 2.$ The modulus is 2

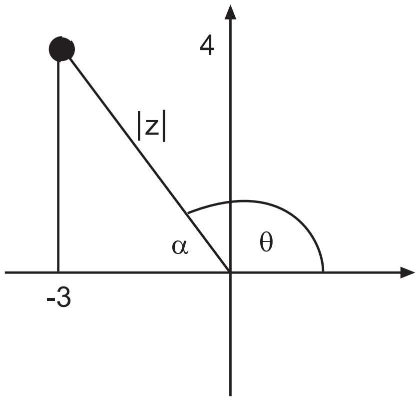
Since z is in the first quadrant arg z = α where tan α = $\frac{\sqrt{3}}{1}$

Hence α = tan$^{-1}\sqrt{3}$ = 60° and so arg (z) = 60°

2. Find the modulus and principal argument of the complex number z = -3 + 4i

Answer:

Plot the point corresponding to the complex number in the Argand diagram.



$|z| = \sqrt{a^2 + b^2} = \sqrt{(-3)^2 + (4)^2} = 5$. The modulus is 5

As z is in the second quadrant arg (z) = π - α where tan α = $^4/_3$

Hence, arg z = π - $\tan^{-1}\left(\frac{4}{3}\right) = 2.214$ to 3 decimal places.

The argument is 2.214 (in radians).

Alternatively it is probably easier to work in degrees and use 180° instead of π

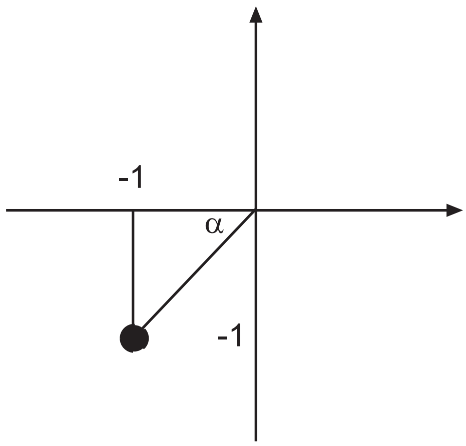
So arg z = 180° - α where tan α = $^4/_3$

So arg z = 180° - $\tan^{-1}\left(\left|\frac{4}{3}\right|\right)$ = 126.87°

3. Find the modulus and principal argument of the complex number z = -1 - i

Answer:

Plot the point corresponding to the complex number in the Argand diagram.



$|z| = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$

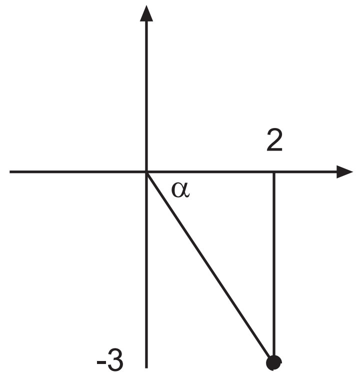
As z is in the third quadrant arg z = -180° + α where tan α = $^1/_1$ = 1

Hence α = $\tan^{-1}(1)$ = 45° and so arg (z) = -180° + α = -135°

4. Find the modulus and principal argument of the complex number z = 2 - 3i

Answer:

Draw the point corresponding to the complex number in the Argand diagram.



$|z| = \sqrt{a^2 + b^2} = \sqrt{(2)^2 + (-3)^2} = \sqrt{13}$

Since z lies in the fourth quadrant arg (z) = - α where tan α = $^3/_2$

Hence $\alpha = \tan^{-1}\left(\frac{3}{2}\right)$ = 0.983 and so arg z = -0.983 (in radians).

15 min

Modulus, argument and polar form exercise

There are more questions on finding modulus and argument on the web in two exercises if you want to try them.

For the following questions the argument can be given in either radians or degrees, whichever is most suitable.

Q38: Find the modulus and the principal argument of the complex number 3 + 5i

Q39: Find the polar form of the complex number 1 - i

Q40: Find the modulus and the principal argument of the complex number -2 + 4i

Q41: Find the modulus and the principal argument of the complex number 4 - i

Q42: Find the modulus and the principal argument of the complex number -3 - 6i

Q43: Find the modulus and the principal argument for the complex number 12 + 5i and hence state the number in polar form.

Q44: Convert $-1 + \sqrt{3}\,i$ into polar form.

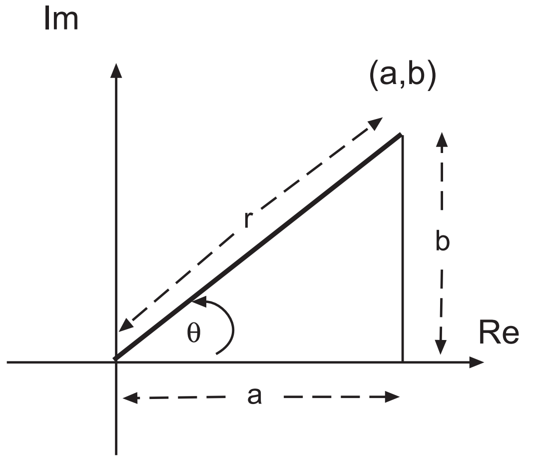
Q45: Find the modulus and the principal argument of the complex number $-\sqrt{3} - i$

3.6 Geometric interpretations

Learning Objective

Interpret particular equations in the complex plane

As shown earlier complex numbers can be represented in an Argand diagram.



The point (a,b) represents
the complex number z = a + ib

Sometimes equations involving complex numbers have interesting geometric interpretations. For example, the family of all solutions of an equation lie on a circle or on a straight line.

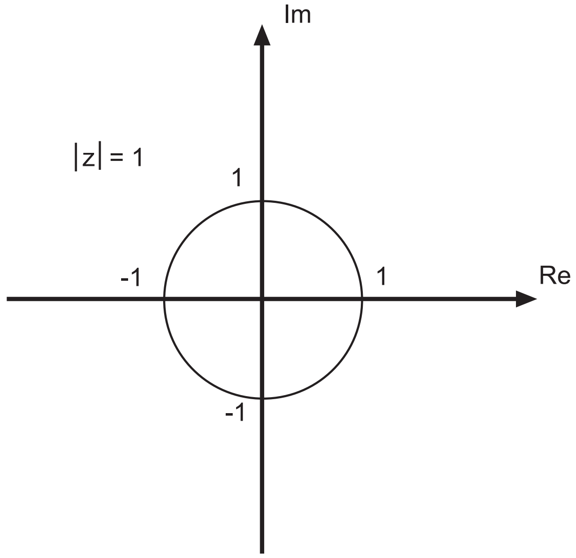
Circle representations

Consider the equation $|z| = 1$

What are the solutions to this equation? That is, which complex numbers z satisfy $|z| = 1$

Geometrically, since $|z|$ is the distance between the point representing z and the origin, $|z| = 1$ if and only if the distance between z and the origin is 1

This can only occur if z lies on the circle centre (0, 0) and radius 1



Algebraically, the same result can be established using x and y coordinates.

Suppose z = x + iy

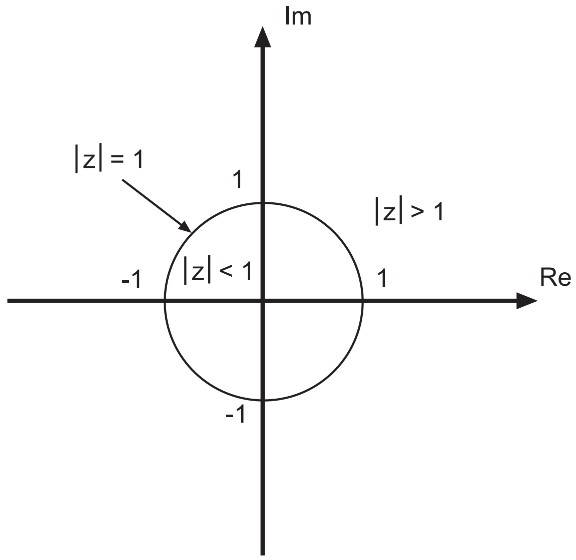
Then $|z| = 1 \Leftrightarrow \sqrt{x^2 + y^2} = 1 \Leftrightarrow x^2 + y^2 = 1$ which is the equation of the circle with centre (0, 0) and radius 1

Consider now the inequality $|\, z \,| > 1$

Clearly $|\, z \,| > 1 \Leftrightarrow$ the distance from z to the origin is greater than 1

Hence the solution set is the set of all points which lie strictly outside the circle, centre (0, 0) and radius 1

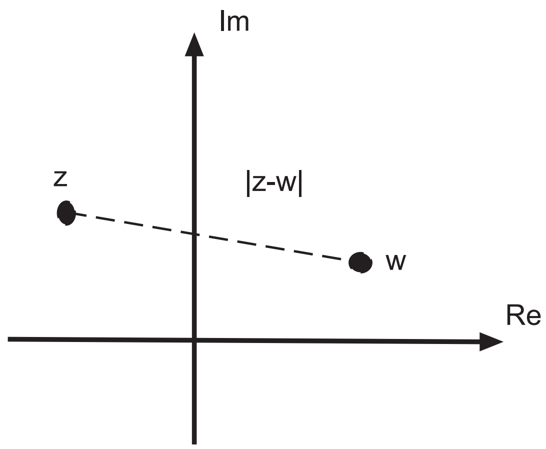
Similarly, $|\, z \,| < 1$ has a solution set of all the points which lie strictly inside the unit circle.



There are also equations which correspond to circles whose centres do not lie at the origin.

If z and w are two complex numbers then

$|\, z - w \,|$ = the distance between the points representing z and w in the Argand diagram.



Activity

Calculate the distance between the two complex numbers z = 3 - 2i and w = 3 and verify the previous diagram.

Now consider the equation $|\, z - 3 \,| = 2$

Remember that the complex number 3 is represented by the point (3, 0) in the Argand diagram.

Hence $|z - 3|$ = the distance between the point z and (3, 0) in the Argand diagram.

So $|z - 3| = 2$ says that the distance between the point z and (3, 0) is equal to 2

Thus the solution set is the set of all points on a circle of radius 2 with centre (3, 0)

Q46: Give a geometrical interpretation of the equation $|z - 3| > 2$

Straight line representations

Finally, look at equations of the form $|z - a| = |z - b|$

The following example shows that the solution set of such equations is a straight line.

Example Find a geometrical interpretation for the equation $|z - 3| = |z - 4i|$

Answer:

The complex numbers 3 and 4i are represented by the points (3, 0) and (0, 4) in the Argand diagram.

Hence $|z - 3| = |z - 4i| \Leftrightarrow$ distance from (3, 0) to z = distance from (0, 4) to z

$\Leftrightarrow$ z is the same distance from (3, 0) and (0, 4)

$\Leftrightarrow$ z lies on the perpendicular bisector of the line joining (3, 0) and (0, 4)

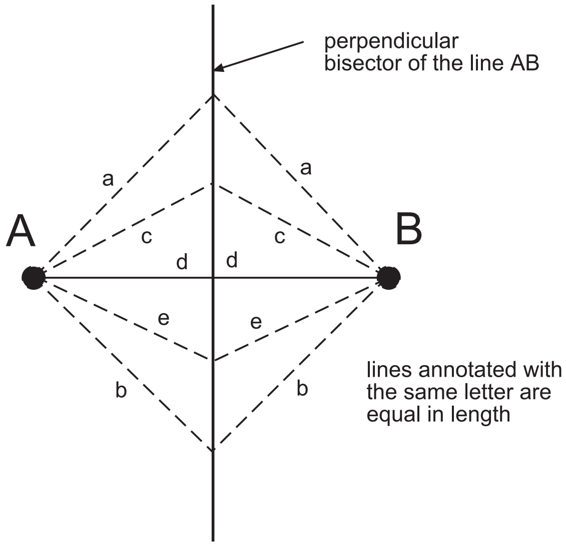
The next activity will demonstrate the fact used above that the perpendicular bisector of a straight line consists of all the points that are equidistant from the ends of the line.

10 min

Geometric activity

- Plot the points (3, 0) and (0, 4), which represent the complex numbers 3 and 4i on an Argand diagram.
- Plot four more points that are, in turn, equidistant from these.
- Join these new points, which will lie in a straight line.

Here is a diagram which shows the construction of a perpendicular bisector of the line joining two points.



The equation of the perpendicular bisector in terms of x and y can be found as follows:

Let z = x + iy

$$|z - 3| = |z - 4i|$$
$$|x + iy - 3| = |x + iy - 4i|$$
$$|(x - 3) + i\,y| = |x + i(y - 4)|$$
$$(x - 3)^2 + y^2 = x^2 + (y - 4)^2$$
$$x^2 - 6x + 9 + y^2 = x^2 + y^2 - 8y + 16$$
$$8y = 6x + 7$$

These examples give some general rules that can be used to find a geometric interpretation in the complex plane of an equation.

Equations of the form $|z| = r$

The solutions of the equation form a circle with centre (0, 0) and radius r.

Equations of the form $|z - a| = r$

The solutions of the equation form a circle of radius r units with centre a, where a is a point in the complex plane.

Equations of the form $|z - a| > r$

The solutions of the equation are the set of all points that lie outside the circle with radius r units with centre a, where a is a point in the complex plane.

Equations of the form $|z - a| < r$

The solutions of the equation are the set of all points that lie inside the circle with radius r units with centre a, where a is a point in the complex plane.

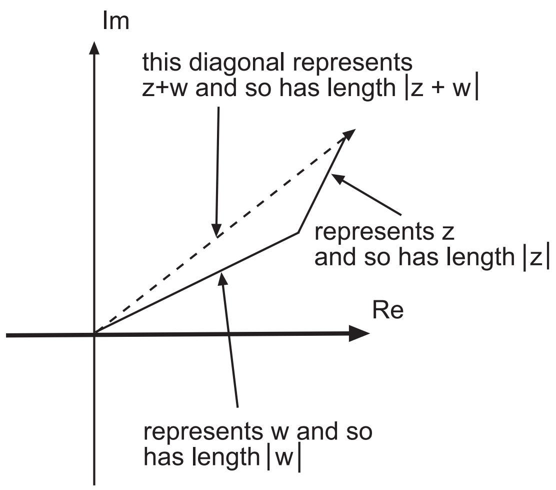
Equations of the form $|z - a| = |z - b|$

The solutions of the equation are the set of points that lie on the perpendicular bisector of the line joining a and b.

Finally it is worth mentioning another property of complex numbers involving the modulus, which has an interesting geometric interpretation.

If z and w are complex numbers then $|z + w| \leqslant |z| + |w|$

This is seen clearly in the following diagram.



The geometric interpretation of this is that there are no occasions where it is possible in a triangle for one side of a triangle to be larger than the sum of the lengths of the other two sides.

Activity

Take a long ruler, a pen and a small pencil. Verify the triangle inequality.

The proof of the triangle inequality is only included for interest.

Geometric representation exercise

There are some questions on the geometrical interpretation of complex equations on the web.

10 min

Q47: Explain the geometric representation of $|z + 2| = 5$

Q48: What is the solution set for the equation $|z - 2i| = 4$

Q49: Give the geometric interpretation of $|z - 1| = 4$

Q50: Give the equation whose solution set is a circle centre -3i and radius 4 in the complex plane.

Q51: Give a geometric interpretation of $|z + 2i| = |z + 3|$ and find its equation using x, y coordinates.

Q52: Give the equation whose solution set is the set of all points which lie within a circle of radius 2 and centre 2i in the complex plane.

3.7 Fundamental theorem of algebra

Learning Objective

Remember the fundamental theorem of algebra

If the equation $x^2 + ax + b = 0$ has solutions $x = \alpha$ and $x = \beta$

then $x^2 + ax + b = (x - \alpha)(x - \beta)$

Thus, the factors of the polynomial can be found from the roots and vice versa.

By using the set of complex numbers, every quadratic equation can be solved and so every quadratic equation can be factorised.

Examples

1. Consider the equation $x^2 + 1 = 0$

The equation has solutions x = i and x = -i

It follows that $x^2 + 1 = (x + i)(x - i)$ and this can be easily checked.

2. Consider $x^2 + x + 1 = 0$

By the quadratic formula the equation has solutions $x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

So $x^2 + x + 1 = \left(x + \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left(x + \frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

Similar results, for solving polynomial equations and factorising polynomial expressions, hold for polynomials of higher order.

Mathematicians find this a very striking result - hence the name of the theorem.

Let $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a polynomial of degree n (with real or complex coefficients).

The **fundamental theorem of algebra** states

that $P(z) = 0$ has n solutions $\alpha_1, ..., \alpha_n$ in the complex numbers

and $P(z) = (z - \alpha_1)(z - \alpha_2)...(z - \alpha_n)$

3.8 Solving equations

Learning Objective

Apply the fundamental theorem to finding roots of polynomials

Quadratic equations can always be solved by using the quadratic formula and hence the factors can be found.

For higher order equations the solutions are much harder to find.

In some cases, however, it is possible to find easy solutions by inspection -

simply by trying x = $\pm$1, $\pm$2, $\pm$3, ...

Example Find a solution of $x^3 + x - 2 = 0$

Answer:

Try x = 1 then $x^3 + x - 2 = 1^3 + 1 - 2 = 0$

Thus x = 1 is a solution.

Having found one solution it is sometimes possible to find other solutions.

Example Find all the solutions of the equation $x^3 - x^2 - x - 2 = 0$

Answer:

Let $f(x) = x^3 - x^2 - x - 2$

Then f (1) = -3, f (-1) = -3, f (2) = 0

Since f (z) = 0, x - 2 is a factor.

Using long division gives $x^3 - x^2 - x - 2 = (x - 2)(x^2 + x + 1) = 0$

But $(x^2 + x + 1) = 0 \Rightarrow x = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

Hence the equation has solutions x = 2, $x = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $x = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Given one complex root, another can be found by considering the conjugate.

It is easy to see from the quadratic formula that if $z = \alpha$ is a solution of a quadratic equation

then so is $z = \overline{\alpha}$

In fact this is true in general.

Suppose P (x) is a polynomial with real coefficients. If $z = \alpha$ is a solution of P (x) = 0 then so is $z = \overline{\alpha}$

The roots $z = \alpha$ and $z = \overline{\alpha}$ are said to be *conjugate pairs*.

The proof of the conjugate roots property can be found as proof (10) in the section headed Proofs near the end of this topic. It is only included for interest.

With this property it is possible to find conjugate roots and hence quadratic factors.

If z = α and z = $\overline{\alpha}$ are complex roots of P (x)

then P (x) has factors (x - α) and (x - $\overline{\alpha}$)

P (x) then factorises to (x - α)(x - $\overline{\alpha}$)P_1 (x) for some other polynomial P_1 (x)

Q (x) = (x - α)(x - $\overline{\alpha}$) will always have real coefficients and so provides a quadratic factor of P (x) with real coefficients.

Example Suppose x = i is one root of a polynomial.

By the conjugate roots property x = -i is also a root.

Then (x - i) and (x + i) are factors.

Hence (x - i)(x + i) = x^2 + 1 is a quadratic factor with real coefficients.

Strategy for solving a cubic:

- Find one solution α by inspection.
- Using long division, obtain a factorisation of the form P (x) = (x - α) Q (x) where Q (x) is a quadratic.
- Solve Q (x) = 0

Strategy for solving a quartic given a complex solution z = α, where $\alpha \notin \mathbb{R}$:

- Take z = $\overline{\alpha}$ also as a solution.
- Find a quadratic factor Q_1(x) = (x - α)(x - $\overline{\alpha}$) of P (x)
- Obtain a factorisation of P (x) of the form P (x) = Q_1 (x) Q_2 (x) where Q_2 (x) is another quadratic, by using long division.
- Solve Q_2 (x) = 0

Examples

1. Find the roots of the polynomial $2x^3 + x^2 + x + 2$

Answer:

By the fundamental theorem of algebra there will be 3 roots.

Try to find a real root by inspection. There is one, namely x = -1

Use long division to divide $2x^3 + x^2 + x + 2$ by (x +1) to give ($2x^2$ - x + 2)

Thus, the polynomial factorises to (x + 1)($2x^2$ - x + 2)

The task now is to find the roots of $2x^2$ - x + 2 and this does not readily factorise.

The quadratic formula, however, gives two complex roots, x = $\frac{1}{4} \pm i\frac{\sqrt{15}}{4}$

The three roots of this polynomial are x = -1, $\frac{1}{4} + i\frac{\sqrt{15}}{4}$, and $\frac{1}{4} - i\frac{\sqrt{15}}{4}$

2. Find the other roots of the quartic $x^4 + x^3 + 2x^2 + x + 1$ given that one root is x = i

Answer:

By the fundamental theorem of algebra there will be 4 roots.

By the conjugate roots property, complex roots occur in conjugate pairs.

Thus x = -i must also be a root since x = i is given as a root.

Hence x - i and x + i are factors and so $(x - i)(x + i) = x^2 + 1$ is a factor.

Long division gives $x^4 + x^3 + 2x^2 + x + 1 = (x^2 + 1)(x^2 + x + 1)$

Since $x^2 + x + 1$ does not factorise readily use the quadratic formula to find the other two roots, which are $x = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $x = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$

The four roots are x = i, -i, $\frac{-1}{2} + \frac{\sqrt{3}}{2}i$ and $\frac{-1}{2} - \frac{\sqrt{3}}{2}i$

3. Given that (-2 - i) and (3 + 2i) are the roots of a quadratic $x^2 - bx - c$ where $b, c \in \mathbb{C}$, find the complex numbers b and c

Answer:

The factors of the quadratic are (x - (-2 - i)) = (x + 2 + i) and (x - (3 + 2i)) = (x - 3 - 2i)

Thus the quadratic is

$(x + 2 + i)(x - 3 - 2i) = x^2 + (-1 - i)x + (-4 - 7i) = x^2 - (1 + i)x - (4 + 7i)$

Equate the coefficients of the same powers of x to give b = (1 + i) and c = (4 + 7i)

Solving complex equations exercise

15 min

There are more questions on finding the solutions of complex equations on the web which you might like to try.

Q53: Find the solutions of the equation $x^3 + x^2 + 3x - 5 = 0$

Q54: Find the roots of the polynomial $x^3 + 2x^2 + 4x + 3$

Q55: The roots of a quadratic $x^2 + bx + c$ are x = 2 - 3i and x = 1 + 2i
where b and $c \in \mathbb{C}$. Find the complex numbers b and c

Q56: Find the roots of the quartic $x^4 - 2x^3 + 8x^2 - 2x + 7$ given that one of them is
$x = 1 + i\sqrt{6}$

Q57: Find the roots of the quartic $x^4 - 2x^3 - x^2 + 2x + 10$ given that one of the roots is
x = -1 + i

3.9 De Moivre's theorem

Learning Objective

Understand and use De Moivre's theorem for expansions

3.9.1 Multiplication using polar form

The multiplication of two complex numbers becomes much easier using the polar form.

Take two complex numbers

$z = r_1(\cos\theta + i\sin\theta)$ and $w = r_2(\cos\phi + i\sin\phi)$ and multiply them together.

Thus $zw = r_1r_2(\cos\theta\cos\phi - \sin\theta\sin\phi) + i\,r_1r_2(\sin\theta\cos\phi + \sin\phi\cos\theta)$

Using trig. formulae this simplifies to $r_1r_2\{\cos(\theta + \phi) + i\sin(\theta + \phi)\}$

So the modulus of the product, r_1r_2, is the product of the moduli of z and w, namely r_1 and r_2

The argument of the product, $\theta + \phi$, is the sum of the arguments of z and w, namely θ and ϕ

This gives a simple rule for multiplying complex numbers in polar form.

- Multiply the moduli.
- Add the arguments.

$(r_1\cos\theta_1 + i\,r_1\sin\theta_1)(r_2\cos\theta_2 + i\,r_2\sin\theta_2) = r_1r_2\{\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\}$

Q58: If $z = 3(\cos 30° + i\sin 30°)$ and $w = 4(\cos 60° + i\sin 60°)$, find zw.

In a similar way division can be discussed using polar form.

If $z = r_1(\cos\theta + i\sin\theta)$ and $w = r_2(\cos\phi + i\sin\phi)$ then

$$\frac{z}{w} = \frac{r_1(\cos\theta + i\sin\theta)}{r_2(\cos\phi + i\sin\phi)}$$

$$= \frac{r_1}{r_2} \times \frac{(\cos\theta + i\sin\theta)(\cos\phi - i\sin\phi)}{(\cos\phi + i\sin\phi)(\cos\phi - i\sin\phi)}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta\cos\phi - \sin\theta\sin\phi + i(\sin\theta\cos\phi - \cos\theta\sin\phi)}{(\cos^2\phi + \sin^2\phi)}$$

$$= \frac{r_1}{r_2}\{\cos(\theta - \phi) + i\sin(\theta - \phi)\}$$

This gives another simple rule, this time for dividing two complex numbers in polar form.

- Divide the moduli.
- Subtract the arguments.

$\frac{r_1 \cos \theta_1 + i\, r_1 \sin \theta_1}{r_2 \cos \theta_2 + i\, r_2 \sin \theta_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)\}$

3.9.2 De Moivre's theorem

Learning Objective

Understand and apply De Moivre's Theorem to expansions

Multiplying two complex numbers in polar form is easily undertaken by using the rules set out in the last section.

This leads to a very simple formula for calculating powers of complex numbers - known as De Moivre's theorem.

Consider the product of z with itself (z^2) if $z = r \cos \theta + i\, r \sin \theta$

The rules of multiplying the moduli and adding the arguments gives

$z^2 = r\,.\,r\{\cos(\theta + \theta) + i \sin(\theta + \theta)\} = r^2(\cos 2\theta + i \sin 2\theta)$

Now consider z^3

Take $z^2 = r^2(\cos 2\theta + i \sin 2\theta)$ and multiply this by $z = r \cos \theta + i\, r \sin \theta$

This gives $z^3 = (r^2(\cos 2\theta + i \sin 2\theta))(r(\cos \theta + i \sin \theta)) = r^3(\cos 3\theta + i \sin 3\theta)$

A pattern has emerged.

This result can be extended to the nth power and is known as De Moivre's Theorem.

If $z = r \cos \theta + i\, r \sin \theta$, then $z^n = r^n(\cos n\theta + i \sin n\theta)$ for all $n \in \mathbb{N}$

The proof of De Moivre's theorem can be found as proof (11) in the section headed Proofs near the end of this topic. Again it is only included for interest.

This is a very useful result as it makes it simple to find z^n once z is expressed in polar form.

Example Calculate $\left\{2\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)\right\}^5$

Answer:

Using De Moivre's theorem

$$\begin{aligned}\left\{2\left(\cos \frac{\pi}{5} + i \sin \frac{\pi}{5}\right)\right\}^5 &= 2^5\left\{\cos\left(5 \times \frac{\pi}{5}\right) + i \sin\left(5 \times \frac{\pi}{5}\right)\right\} \\ &= 32(\cos \pi + i \sin \pi) \\ &= -32\end{aligned}$$

[To reach this result multiply the moduli to give 2^5 and add the arguments to give $\frac{5\pi}{5} = \pi$]

15 min

De Moivre's expansion exercise

There are further questions on expansions using De Moivre's theorem on the web which you may wish to try.

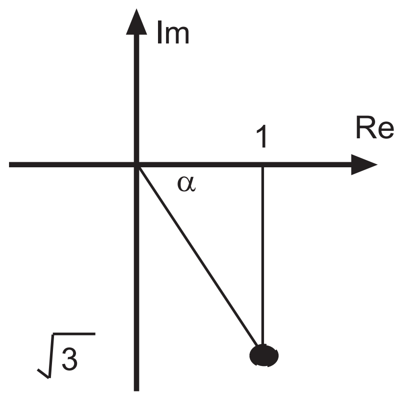
Q59: Calculate $\left\{3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right\}^3$ using De Moivre's theorem.

Q60: Calculate $\left\{2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right\}^4$ using De Moivre's theorem.

Example Calculate $(1 - i\sqrt{3})^6$

Answer:

First express $z = 1 - i\sqrt{3}$ in polar form.



$|z| = \sqrt{a^2 + b^2} = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

As z is in the fourth quadrant arg $z = -\alpha$ where $\tan\alpha = \sqrt{3}$, i.e. $\alpha = \frac{\pi}{3}$

So arg $z = \frac{-\pi}{3}$ and $z = 2\left\{\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right\}$

Using De Moivre's theorem

$$z^6 = 2^6\left\{\cos\left(\frac{-6\pi}{3}\right) + i\sin\left(\frac{-6\pi}{3}\right)\right\} = 64\ \{\cos(-2\pi) + i\sin(-2\pi)\} = 64$$

5 min

Further De Moivre expansion exercise

There are more questions on the web if you prefer them.

Q61: Find the real and imaginary parts of $(-1 - i)^4$ using De Moivre's theorem.

Q62: Calculate $(-\sqrt{3} + i)^3$ using De Moivre's theorem.

De Moivre's theorem and the rule for dividing complex numbers in polar form can be used to simplify fractions involving powers.

Example Simplify $\frac{(1+i)^6}{(1-i\sqrt{3})^4}$

Answer:

First express 1 - i$\sqrt{3}$ and 1 + i in polar form.

From the previous example 1 - i$\sqrt{3}$ = 2 $\left\{\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right\}$

From an Argand diagram 1 + i = $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

By De Moivre's theorem

$$\begin{aligned}(1-i\sqrt{3})^4 &= 2^4\left\{\cos\left(4\times\frac{-\pi}{3}\right) + i\sin\left(4\times\frac{-\pi}{3}\right)\right\}\\ &= 16\left\{\cos\left(\frac{-4\pi}{3}\right) + i\sin\left(\frac{-4\pi}{3}\right)\right\}\end{aligned}$$

and

$$\begin{aligned}(1+i)^6 &= \sqrt{2}^6\left\{\cos\left(6\times\frac{\pi}{4}\right) + i\sin\left(6\times\frac{\pi}{4}\right)\right\}\\ &= 8\left\{\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right\}\end{aligned}$$

Hence

$$\begin{aligned}\frac{(1+i)^6}{(1-i\sqrt{3})^4} &= \frac{8}{16}\left\{\cos\left(\frac{3\pi}{2}+\frac{4\pi}{3}\right) + i\sin\left(\frac{3\pi}{2}+\frac{4\pi}{3}\right)\right\}\\ &= \frac{1}{2}\left\{\cos\left(\frac{17\pi}{6}\right) + i\sin\left(\frac{17\pi}{6}\right)\right\}\end{aligned}$$

[To divide complex numbers in polar form, divide the moduli and subtract the arguments.]

Q63: Simplify $\frac{(3-3i)^4}{(\sqrt{3}+i)^3}$

Q64: Simplify $\frac{(\sqrt{3}+i)^4}{(1-i)^3}$

3.9.3 De Moivre's theorem and multiple angle formulae

Learning Objective

Use De Moivre to solve trigonometric problems

De Moivre's theorem is extremely useful in deriving trigonometric formulae.

The examples which follow demonstrate the strategy to obtain these.

Examples

1. Find a formula for cos 3θ in terms of powers of cos θ by using De Moivre's theorem.

Hence express $\cos^3\theta$ in terms of cos θ and cos 3θ

Answer:

Use De Moivre

$\cos 3\theta + i \sin 3\theta = (\cos \theta + i \sin \theta)^3$

By the Binomial theorem

$(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$

Hence

$\cos 3\theta + i \sin 3\theta = \cos^3 \theta + 3i \cos^2 \theta \sin \theta - 3 \cos \theta \sin^2 \theta - i \sin^3 \theta$

Equating real parts

$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$

Hence

$$\begin{aligned}\cos 3\theta &= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)\\ &= \cos^3\theta - 3\cos\theta + 3\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta\end{aligned}$$

Rearranging gives $\cos^3\theta = \frac{\cos 3\theta + 3\cos\theta}{4}$

2. Using De Moivre's and the Binomial theorem express $\cos 4\theta$ as a polynomial in $\cos \theta$

Answer:

By De Moivre

$(\cos \theta + i \sin \theta)^4 = \cos 4\theta + i \sin 4\theta$

By the Binomial theorem

$(\cos \theta + i \sin \theta)^4 = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta$

Hence

$\cos 4\theta + i \sin 4\theta = \cos^4 \theta + 4i \cos^3 \theta \sin \theta - 6 \cos^2 \theta \sin^2 \theta - 4i \cos \theta \sin^3 \theta + \sin^4 \theta$

Equating real parts

$\cos 4\theta = \cos^4 \theta - 6 \cos^2 \theta \sin^2 \theta + \sin^4 \theta$

Hence

$\cos 4\theta = (1 - \cos^2 \theta)^2 - 6 \cos^2 \theta(1 - \cos^2 \theta) + \cos^4 \theta$

and so

$\cos 4\theta = 8 \cos^4 \theta - 8 \cos^2 \theta + 1$

Q65: Express $\frac{\sin 5\theta}{\sin \theta}$ as a polynomial in $\sin \theta$

Q66: Express $\sin 3\theta$ as a polynomial in $\sin \theta$ and hence express $\sin^3\theta$ in terms of $\sin \theta$ and $\sin 3\theta$.

Q67: Express $\cos 5\theta$ as a polynomial in $\cos \theta$

3.9.4 De Moivre's theorem and nth roots

Learning Objective

Find the nth roots of unity using De Moivre's theorem

De Moivre's theorem is not only true for the integers but can be extended to fractions.

$$\{r(\cos\theta + i\sin\theta)\}^{p/q} = r^{p/q}\left\{\cos\left(\tfrac{p}{q}\theta\right) + i\sin\left(\tfrac{p}{q}\theta\right)\right\}$$

Example Calculate $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3}$

By De Moivre's theorem for fractional powers

$\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3} = \left\{\cos\left(\frac{1}{3}\times\frac{\pi}{4}\right) + i\sin\left(\frac{1}{3}\times\frac{\pi}{4}\right)\right\} = \left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

Q68: Calculate $\left\{27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right\}^{1/3}$

Q69: Using De Moivre's theorem calculate $\left\{8\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{2/3}$

The previous worked example showed that $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{1/3} = \left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

That is, $\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}$ is a cube root of $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$

This cube root is obtained by dividing the argument of the original number by 3

However, the cube roots of $\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}$ are complex numbers z which satisfy $z^3 = 1$ and so by the Fundamental theorem of algebra, since this equation is of degree 3, there should be 3 roots. That is, in general, a complex number should have 3 cube roots.

Given a complex number these 3 cube roots can always be found

Strategy for finding the cube roots of a complex number

- Write the complex number in polar form $z = r(\cos\theta + i\sin\theta)$
- Write z in two more equivalent alternative ways by adding 2π to the argument.

  $z = r\{\cos(\theta + 2\pi) + i\sin(\theta + 2\pi)\}$

  $z = r\{\cos(\theta + 4\pi) + i\sin(\theta + 4\pi)\}$
- Write down the cube roots of z by taking the cube root of r and dividing each of the arguments by 3

NB: the previous strategy gives the three cube roots as

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)\right\}$$

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\right\}$$

$$r^{1/3}\left\{\cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{4\pi}{3}\right)\right\}$$

If $z = r(\cos\theta + i\sin\theta)$ is written in any further alternative ways such as

$z = r\{\cos(\theta + 6\pi) + i\sin(\theta + 6\pi)\}$, this gives a cube root of

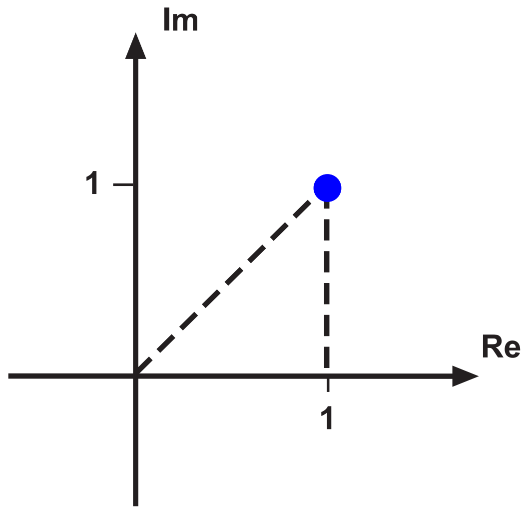
$r^{1/3}\left\{\cos\left(\frac{\theta}{3} + \frac{6\pi}{3}\right) + i\sin\left(\frac{\theta}{3} + \frac{6\pi}{3}\right)\right\} = r^{1/3}\left\{\cos\left(\frac{\theta}{3}\right) + i\sin\left(\frac{\theta}{3}\right)\right\}$

which is the same as one of the previously mentioned roots.

It is impossible to find any more.

Example Find the cube roots of 1 + i

First express 1 + i in polar form



$|1 + i| = \sqrt{2}$ and $\arg(1 + i) = \frac{\pi}{4}$

Hence1 + i can be expressed as $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

But 1 + i can also be expressed as

$$1 + i = \sqrt{2}\left\{\cos\left(\frac{\pi}{4} + 2\pi\right) + i\sin\left(\frac{\pi}{4} + 2\pi\right)\right\} = \sqrt{2}\left\{\cos\left(\frac{9\pi}{4}\right) + i\sin\left(\frac{9\pi}{4}\right)\right\}$$

and

$$1 + i = \sqrt{2}\left\{\cos\left(\frac{\pi}{4} + 4\pi\right) + i\sin\left(\frac{\pi}{4} + 4\pi\right)\right\} = \sqrt{2}\left\{\cos\left(\frac{17\pi}{4}\right) + i\sin\left(\frac{17\pi}{4}\right)\right\}$$

Hence, taking the cube root of the modulus and dividing the argument by 3, the cube roots of 1 + i are

$$z = (2^{1/2})^{1/3}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) = (2^{1/6})\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$
$$z = (2^{1/6})\left\{\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right\}$$
$$z = (2^{1/6})\left\{\cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)\right\}$$

In this way the nth roots of any complex number can be found.

Example Find the cube roots of z = 64(cos 30° + i sin 30°)

Answer:

This is in polar form. Use 2π = 360° and 4π = 720°

z = 64(cos 30° + i sin 30°)

z can also be written as

z = 64{cos (30 + 360)° + i sin (30 + 360)°}

and

z = 64{cos (30 + 720)° + i sin (30 + 720)°}

Since $64^{1/3} = \sqrt[3]{64} = 4$, the cube roots of z are

4(cos 10° + i sin 10°), 4(cos 130° + i sin 130°), 4(cos 250° + i sin 250°)

Q70: Find the fourth roots of 81i, that is of 81 $\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Q71: Find the sixth roots of $\sqrt{3} + i$

It is easy and important to find the nth roots of 1

i.e. complex numbers such that $z^n = 1$

Such numbers are often referred to as the nth roots of unity.

The nth roots of unity are those numbers that satisfy the equation $z^n = 1$

Since 1 = cos 2π + i sin 2π, it follows that $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ is an nth root of unity.

But 1 can be written using different arguments as follows:

$$\begin{aligned}1 &= \cos 2\pi + i\sin 2\pi\\ &= \cos 4\pi + i\sin 4\pi\\ &= \cos 6\pi + i\sin 6\pi\\ &= \ldots\ldots\ldots\ldots\\ &= \cos 2n\pi + i\sin 2n\pi\end{aligned}$$

Hence dividing the argument in each case by n gives the following nth roots of unity.

$z = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$

$z = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n}$

$z = \cos\frac{6\pi}{n} + i\sin\frac{6\pi}{n}$ and so on.

Note that arguments increase by $\frac{2\pi}{n}$ each time. The roots of unity are regularly spaced in an Argand diagram.

Example Find the cube roots of unity and plot them on an Argand diagram.

Answer:

Since 1 can be written in polar form as

$1 = \cos 2\pi + i\sin 2\pi$

$1 = \cos 4\pi + i\sin 4\pi$

$1 = \cos 6\pi + i\sin 6\pi$

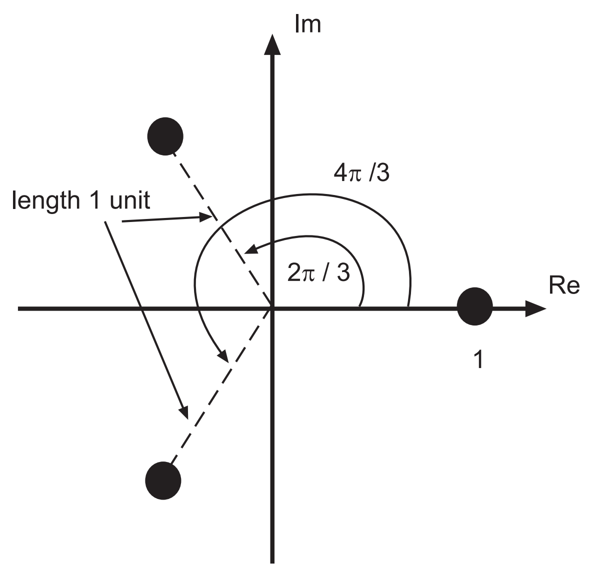
the cube roots of unity are

$z = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{1}{2}(-1 + i\sqrt{3})$

$z = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{1}{2}(-1 - i\sqrt{3})$

$z = \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} = 1$

On the Argand diagram



Q72: Find the fourth roots of unity and plot them on an Argand diagram.

Q73: Find the solutions of the equation $z^6 - 1 = 0$. Plot the answers on an Argand diagram.

Graphical display of the roots of unity

5 min

There is an on-line activity to show the roots of unity.

Extension activities

15 min

Try these two activities if you would like a challenge.

- Find the solutions of the equation $z^4 = i$ and plot them on an Argand diagram.
- Consider the solutions of the equation $z^4 = 16$
- Compare the solutions of $z^4 = 16$ and the solutions of $z^4 = -16$

- Take $z = r(\cos\theta + i\sin\theta)$ and calculate $\frac{1}{z}$ by division methods.
- Conclude whether or not De Moivre's theorem also is true for the integer -1

3.10 Conjugate properties

Learning Objective

Identify and use conjugate properties where required

There are special identities which apply to complex numbers and their conjugates. These are listed below.

Let the two complex numbers be z and w with conjugates $\bar{z}$ and $\bar{w}$ respectively.

1. $\bar{\bar{z}} = z$
2. $z = \bar{z}$ where $z \in \mathbb{R}$
3. $z + \bar{z} = 2\text{Re}(z)$
4. $|\bar{z}| = |z|$
5. $z\bar{z} = |z|^2$
6. $\overline{(z + w)} = \bar{z} + \bar{w}$
7. $\overline{(zw)} = \bar{z}\ \bar{w}$

These properties of conjugate lead easily to the following properties of modulus.

Let z and w be complex numbers. Then:

1. $|zw| = |z|\ |w|$
2. $|z + w| \leqslant |z| + |w|$ This is commonly known as the triangle inequality.

Activity

Try to confirm these properties using the Argand diagram or algebraic techniques on the complex number z = a + ib.

Now check out the proofs or explanations of these identities shown as proofs (1) to (9) in the section headed Proofs near the end of this topic. They are only included for interest.

The next exercise should be done using these properties where appropriate.

Modulus properties exercise

10 min

There is a different exercise on the web on modulus properties.

Q74: If z = -3 + 4i and $w = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ evaluate the following expressions:

1. zw
2. $\bar{z}\,\bar{w}$
3. $|z|$
4. $z\bar{z}$

3.11 Proofs

Learning Objective

Know the structure of the proofs

Proof (1): $\bar{\bar{z}} = z$

Let z = a + bi for a and b $\in \mathbb{R}$

By the definition of a conjugate $\bar{z}$ = a - bi = a + (-b)i

Again using the conjugate definition $\bar{\bar{z}} = \overline{a + (-b)i}$ = a - (-b)i = a + bi = z

Proof (2): **z =** $\bar{z}$ **if z** $\in \mathbb{R}$

If z $\in \mathbb{R}$ then z = a + 0i = a (since 0i = 0) for some a $\in \mathbb{R}$

By the definition of a conjugate $\bar{z}$ = a - 0i, but 0i = 0

so $\bar{z}$ = a = z

Proof (3): **z +** $\bar{z}$ **= 2Re (z)**

Let z = a + bi then $\bar{z}$ = a - bi

z + $\bar{z}$ = a + bi + a - bi = 2a = 2Re (z)

Proof (4): $|\bar{z}| = |z|$

Let z = a + bi

$|z| = \sqrt{a^2 + b^2}$ and $\bar{z}$ = a - bi so

$|\bar{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$

Proof (5): $z\bar{z}$ **=** $|z|^2$

Let z = a + bi then $\bar{z}$ = a - bi

Hence $z\bar{z} = (a + bi)(a - bi) = a^2 + b^2$

But $|z| = \sqrt{a^2 + b^2}$ so $|z|^2 = a^2 + b^2$

Thus $z\bar{z} = a^2 + b^2 = |z|^2$

Proof (6): $(\overline{z + w})$ **=** $\bar{z}$ **+** $\bar{w}$

Let z = a + bi and w = c + di

Then z + w = (a + c) + i(b + d) so $(\overline{z + w})$ = (a + c) - i(b + d)

Also $\bar{z}$ = a - bi and $\bar{w}$ = c - di and so

$\bar{z} + \bar{w} = (a - bi) + (c - di) = (a + c) - i(b + d) = (\overline{z + w})$

Proof (7): $(\overline{zw})$ **=** $\bar{z}\ \bar{w}$

Let z = a + bi and w = c + di

Then

zw = (a + bi)(c+ di) = (ac - bd) + i(bc + ad) and so $(\overline{zw})$ = (ac - bd) - i(bc + ad)

Also

$\bar{z}$ = a - bi and $\bar{w}$ = c - di and so $\bar{z}\ \bar{w}$ = (a- bi)(c - di) = (ac - bd) - i(bc + ad) = $(\overline{zw})$

Proof (8): $|zw| = |z||w|$

$|zw|^2 = zw(\overline{zw})$ by proof 5

$= zw\bar{z}\bar{w}$ by proof 7

$= z\bar{z}w\bar{w}$

$= |z|^2\ |w|^2$ by proof 5

Hence taking the square root of each side

$|zw| = |z|\ |w|$

Proof (9): Triangle inequality

This proof uses another result: Re (z) $\leq |z|$ for all $z \in \mathbb{C}$.

(If time permits, try this proof)

The triangle inequality states that if z and w $\in \mathbb{C}$ then $|z + w| \leqslant |z| + |w|$

So let z and w $\in \mathbb{C}$

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} && \text{from proof 5}\\
&= (z + w)(\overline{z} + \overline{w}) && \text{from proof 6}\\
&= z\overline{z} + w\overline{w} + z\overline{w} + w\overline{z}\\
&= |z|^2 + |w|^2 + z\overline{w} + \overline{(z\overline{w})} && \text{from proof 5}\\
&= |z|^2 + |w|^2 + 2\text{Re}\,(z\overline{w}) && \text{from proof 3}\\
&\leqslant |z|^2 + |w|^2 + 2|z\overline{w}| && \text{from result given at start}\\
&\leqslant |z|^2 + |w|^2 + 2|z||\overline{w}| && \text{from proof 8}\\
&\leqslant |z|^2 + |w|^2 + 2|z||w| && \text{from proof 4}\\
&\leqslant (|z| + |w|)^2
\end{aligned}$$

Since $|z + w| \geqslant 0$ and $|z| + |w| \geqslant 0$

Taking the square root of each side gives $|z + w| \leqslant |z| + |w|$

Proof (10): Conjugate roots property

Suppose that α is a root of the polynomial

$P(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_2 z^2 + a_1 z + a_0$

Then $P(\alpha) = 0$

i.e. $a_n \alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_2\alpha^2 + a_1\alpha + a_0 = 0$

Hence

$$\begin{aligned}
\overline{P(\alpha)} &= \overline{a_n\alpha^n + a_{n-1}\alpha^{n-1} + \ldots + a_2\alpha^2 + a_1\alpha + a_0}\\
&= \overline{a_n\alpha^n} + \overline{a_{n-1}\alpha^{n-1}} + \ldots + \overline{a_2\alpha^2} + \overline{a_1\alpha} + \overline{a_0}\\
&= \overline{a_n}\left(\overline{\alpha^n}\right) + \overline{a_{n-1}}\left(\overline{\alpha^{n-1}}\right) + \ldots + \overline{a_2}\,\overline{\alpha^2} + \overline{a_1}\,\overline{\alpha} + \overline{a_0}\\
&= a_n(\overline{\alpha})^n + a_{n-1}(\overline{\alpha})^{n-1} + \ldots + a_2(\overline{\alpha})^2 + a_1\overline{\alpha} + a_0\\
&= P(\overline{\alpha})
\end{aligned}$$

However $\overline{P(\alpha)} = \overline{0} = 0$ and so $P(\overline{\alpha}) = 0$

Therefore $\overline{\alpha}$ is a root of the polynomial.

Proof (11): Proof of De Moivre's theorem

The theorem states that if $z = r(\cos\theta + i\sin\theta)$ then

$z^n = r^n(\cos n\theta + i\sin n\theta)$ for all $n \in \mathbb{N}$.

This is a proof by induction. The technique will be explained in greater detail in the topic on Number Theory.

For n = 1

$\{r(\cos\theta + i\sin\theta)\}^1 = r(\cos\theta + i\sin\theta) = r^1(\cos 1\theta + i\sin 1\theta)$.

And so it is true for n = 1

Now suppose that the result is true for n = k then

$\{r(\cos\theta + i\sin\theta)\}^k = r^k(\cos k\theta + i\sin k\theta)$

Consider n = k + 1 then

$$\{r(\cos\theta + i\sin\theta)\}^{k+1} = \{r(\cos\theta + i\sin\theta)\}\{r(\cos\theta + i\sin\theta)\}^{k}$$
$$= \{r(\cos\theta + i\sin\theta)\}\{r^k(\cos k\theta + i\sin k\theta)\}$$
$$= r^{k+1}\{(\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\sin\theta\cos k\theta + \cos\theta\sin k\theta)\}$$
$$= r^{k+1}\{\cos(\theta + k\theta) + i\sin(\theta + k\theta)\}$$
$$= r^{k+1}\{\cos((k + 1)\theta) + i\sin((k + 1)\theta)\}$$

So the result is true for n = k + 1 if it is true for n = k. Since it is also true for n = 1, then it is true for all $n \in \mathbb{N}$

3.12 Summary

At this stage the following ideas and techniques covered in this topic should be known:

- Basic arithmetic operations on complex numbers.
- The spatial representation of complex equations.
- Fundamental theorem of algebra and its applications.
- Polar and cartesian forms and the conversion from one to the other.
- De Moivre's theorem and its applications.

3.13 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web which give a selection of interesting web sites to visit. These sites can lead to an advanced study of the topic but there are many areas which will be of passing interest.

CARDANO

Cardano in 1545 published the 'Ars magna' which includes the use of negative numbers. He described them as 'ficticious' but said that imaginary numbers were 'sophistic'.

BOMBELLI

He was probably the first mathematician to have a clear idea of a complex number. He discussed imaginary and complex numbers in a treatise written in 1572 called 'L'Algebra'.

DESCARTES

He first called numbers involving the square root of a negative number 'imaginary' but assumed from this that the problem was insoluble.

EULER

In 1748 he found the formula $e^{i\theta} = \cos\theta + i\sin\theta$ (now known as Euler's formula) where $i = \sqrt{-1}$. This is the exponential form of a complex number.

ARGAND

Although the idea of representing a complex number by a point in a plane had been suggested by several mathematicians earlier, it was Argand's proposal that was accepted.

GAUSS

He was the first mathematician to mention a complex number plane. He also gave the first proof of the Fundamental Theorem of Algebra in his Ph.D. thesis.

HAMILTON

Hamilton, an Irish mathematician, in 1833 introduced the complex number notation a + bi and made the connection with the point (a, b) in the plane although many mathematicians argued that they had found this earlier. He is well known for his work on graph theory and networks.

DE MOIVRE

A French mathematician and statistician who discovered an easy way to take the power of a complex number. He is probably best known for his work on statistics.

3.14 Review exercise

Review exercise

15 min

There is a web version of this review exercise available if you wish extra tests.

Q75: If z and w are complex numbers such that z = -1 - i and w = 2 - 3i, express zw in cartesian form and plot zw on an Argand diagram.

Q76: Find the modulus and argument of the complex number $u = 1 - i\sqrt{3}$ and hence write u in polar form.

Q77: Find the solutions of the equation $x^3 - x^2 + 5x + 7 = 0$

Q78: Using De Moivre's theorem expand and simplify $\left\{2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)\right\}^5$

Q79: Find the complex equation which is represented by a circle with centre -2i and radius 4

3.15 Advanced review exercise

Advanced review exercise

15 min

There is another version of this test on the web.

Q80: Find the fifth roots of unity and plot them on an Argand diagram.

Q81: Express cos 3θ in terms of cos θ and sin θ

Q82: Let $z = \frac{(1 + i)^9}{(1 - i\sqrt{3})^5}$

Find, by using De Moivre's theorem, the modulus and argument of z. Find a positive integer n such that z^n is purely imaginary. (CSYS 1986, Paper 1.)

Q83: If $1 - i\sqrt{2}$ is one root of the quartic $x^4 - 2x^3 + 4x^2 - 2x + 3$, find the other roots.

3.16 Set review exercise

Set review exercise

15 min

The answers to these questions are available in a similar web exercise. These questions are not randomised on the web and may be posed in a different manner but you should have the required answers in your working.

Q84: Find the solutions of the equation $x^3 - x^2 + 2x + 4 = 0$

Q85: Let z = 2 + i and w = 4 - 3i be two complex numbers.

a) Express zw in Cartesian form and plot zw on an Argand diagram.

b) Find the modulus and argument of w and hence write w in polar form.

Q86: Expand $3(\cos(t) + i\sin(t))^5$ using De Moivre's theorem.

Topic 4

Sequences and Series

Contents

Learning Objectives

- *Understand and use sequences and series.*

Minimum Performance Criteria:

- *Find the nth term and the sum of the first n terms of an arithmetic sequence.*
- *Find the nth term and the sum of the first n terms of a geometric sequence.*

4.1 Revision exercise

Learning Objective

Identify areas which need revision

Revision exercise

15 min

If any question in this revision exercise causes difficulty then it may be necessary to revise the techniques before starting the unit. Ask a tutor for advice. There is a web version of this exercise of you prefer it.

Q1: Find the first four terms in the expansion of $(2x - 3)^6$

Q2: 1000 pounds is invested at a rate of 7% per annum. Form a recurrence relation equation and use it to determine the value of the investment after 5 years.

Q3: For the recurrence relation $u_{n+1} = 0.3u_n + 4$ with $u_1 = 5$, find u_4

Q4: Simplify by identifying a common denominator

$\frac{-1}{(n+1)} - \frac{1}{(n-1)} + \frac{2}{n} + \frac{3}{(n^3 - n)}$

4.2 Introduction

The terms ‘sequence’ and ‘series’ are used in a wide variety of contexts. There are, for example, film sequences, television series, a sequence of events and so on. The two terms are often interchangeable and in fact most dictionaries will give similar definitions of both terms.

There are however mathematical sequences and series and these have distinctly different meanings although there is a strong relationship between them.

In particular, this topic will investigate two important types of sequence and series. Other different types of mathematical sequences and series will also be described. The concepts of mathematical limits, infinity, convergence and divergence will be discussed.

4.3 What is a sequence?

Learning Objective

Recognise sequences and apply the techniques and laws related to them

This section of the topic will discuss particular properties associated with some sets of numbers, why these sets possess such properties and how they can be identified.

So, what is a sequence?

A **sequence** is an ordered list of terms.

Example

The following are sequences:

- 25, 2, 9, 13, 7, 10, 37
- x, x^2, x^3, x^4
- $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$
- 1, 2, 3, 4, 5, ..., 50, ..., 99, 100

In the last example the dots between the terms in the sequence represent missing terms. This sequence is the list of natural numbers between 1 and 100

These examples demonstrate that not all sequences have an obvious pattern. In this topic the emphasis will be on numeric sequences that have a clear relationship between the terms.

Each number in a sequence is called a **term** or an element. The nth term (or general term) is often denoted by u_n

Example Find the third term in the sequence 2, 3, 5, 8, 12

Answer:

Count from the left. The third term is 5

Q5: What is the fifth term of the sequence 2, 4, 6, 8, 10, 12

All the sequences mentioned previously are finite sequences. Each sequence specifically states the first and last terms.

The general example of a finite sequence is u_1, u_2, u_3, ..., u_n where u_1 is the first term and u_n is the last term.

A **finite sequence** is one which has a last term.

The general example of a sequence which does not have a last term is

u_1, u_2, u_3, ... where the three dots at the end indicate that the sequence continues indefinitely.

Such a sequence is called an infinite sequence.

An **infinite sequence** is one which continues indefinitely.

Example

These are all infinite sequences:

- 1, 2, 3, 4, 5, ..., 50, ..., 99, 100, ... (the natural numbers listed in order).
- 2, 3, 5, 7, 11, 13, 17, ... (the prime numbers listed in ascending order).
- $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$ (the reciprocals of the natural numbers listed in order).

If a sequence can be defined by a general term such as u_n (or a_n, x_n, etc.) then it is common for the sequence to be written as $\{u_n\}$ where $n \in \mathbb{N}$.

Example

Here are some sequences defined by a general term:

- $\left\{\frac{1}{n}\right\}$ denotes the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, ...$
- $\{2n + 1\}$ denotes the sequence 3, 5, 7, 9, 11, 13, 15, ...
- $\{n^2 + n - 2\}$ denotes the sequence 0, 4, 10, 18, ...

These examples show that there are different ways to obtain sequences. Some of these ways this will be explored in the sections to come.

Q6: Write out the first five terms of the sequence $\{(n - 2)^3 + 2\}$

4.4 Sequences and recurrence relations

Learning Objective

Identify the link between a recurrence relation and a sequence

Recall that a linear recurrence relation can be defined by

$u_{n+1} = au_n + b$ where $a \neq 0$ and the starting value u_1 is specified.

It is a linear recurrence relation since the equation is of the form $y = mx + c$

The sequence defined by $\{10n + 2\}$ where $n \in \mathbb{N}$ (i.e. 12, 22, 32, 42, 52) can also be defined in this way.

The starting value is $u_1 = 12$ and the equation is $u_{n+1} = u_n + 10$

It is also possible for a sequence to be defined by a second order recurrence relation in which the first two terms are specified.

An example of this is the sequence 3, 6, 15, 33, 78, ...

This can be defined by the recurrence relation $u_{n+2} = u_{n+1} + 3u_n$
where $u_1 = 3$ and $u_2 = 6$

Note carefully that, unless otherwise stated, in this course, the first term of the sequence is actually described as u_1

for example, u_1, u_2, u_3, u_4, u_5, ... or a_1, a_2, a_3, a_4, a_5, ...

The letter is unimportant but other textbooks may adopt u_0 as the first term of the sequence.

This leads back to the general term of a sequence. As already stated, another way of describing it is the nth term of a sequence. Thus if the general term or nth term has a formula it is possible to find any specific term in the sequence.

Examples

1. Finding the sequence given a formula for the nth term

A sequence is defined by $u_n = \frac{2n}{n+1}$. Find the first four terms.

Answer:

Substitute n = 1 to give the first term of $u_1 = 1$

Continue to substitute with n = 2, n = 3 and n = 4 to give the next three terms which are $\frac{4}{3}, \frac{3}{2}, \frac{8}{5}$

2. Using the recurrence relation to find a named term

Find the 4th term of the sequence defined by the recurrence relation

$u_{n+1} = 0.5u_n + 3$ where $u_1 = 2$

Answer:

Here it is easy enough to take the recurrence relation and substitute upwards to find the 4th term. The sequence is $u_1 = 2$, $u_2 = 4$, $u_3 = 5$ and $u_4 = 5.5$ and so the fourth term is 5.5

Finding the nth term of a given sequence

You may want to try the randomised questions on finding terms of a sequence on the web.

10 min

Q7: Find the 3rd term of the sequence defined by $\{2^{n-1}\}$ where $n \in \mathbb{N}$.

Q8: Find the 5th term of the sequence specified by the recurrence relation

$u_{n+1} = \frac{1}{u_n} + 2$ where $u_1 = 1$

Q9: The nth term of a sequence is given by $3n^2$ where $n \in \mathbb{N}$. Find the first four terms of the sequence.

4.5 Arithmetic sequences

Learning Objective

Calculate the nth term of an arithmetic sequence

The sequences mentioned in the previous sections have been constructed in a variety of ways. This section will now examine a sequence formed in one particular way. This sequence is called an arithmetic sequence.

It is defined as follows:

An arithmetic sequence is one which takes the form

a, a + d, a + 2d, a + 3d, ... where a is the first term and d is the common difference.

The common difference in an arithmetic sequence is the difference between any two consecutive terms in the sequence.

The link with recurrence relations still remains.

Consider the general linear recurrence relation

$u_{n+1} = bu_n + c$ where $b, c \in \mathbb{R}$ and $n \in \mathbb{N}$.

Let b = 1 and the equation becomes $u_{n+1} = u_n + c$

Define $c \neq 0$ and the equation represents an arithmetic sequence where the first term is $a = u_1$ and the sequence has a **common difference** of d = c.

Note that 'b' and 'c' are used here to prevent confusion with the first term 'a' of the arithmetic sequence.

Q10: Consider the arithmetic sequence defined by the recurrence relation $u_{n+1} = u_n + c$ with c = -2 and $u_1 = 3$.

State a, d and hence the first 4 terms of the sequence.

This idea of finding the terms one by one until the required term is reached is fine for terms near the beginning of a sequence. However it would be rather laborious to do this to find, say, the 40th term in a sequence.

There is a way round this which comes directly from the definition.

Since the sequence is defined as a, a + d, a + 2d, ... then

$u_1 = a$

$u_2 = a + 1d$

$u_3 = a + 2d$ and so on.

So $u_n = a + (n - 1)d$ and the sequence can be written as $\{a + (n - 1)d\}$ where $n \in \mathbb{N}$.

The nth term of an arithmetic sequence is given by a + (n - 1)d where a is the first term of the sequence, d represents the common difference and n is the number of the term ($n \in \mathbb{N}$).

Note that if the values of the terms of an arithmetic sequence are plotted on a graph against the term number the relationship can clearly be seen as a linear one.

Arithmetic sequence graphs

10 min

Work out a variety of arithmetic sequences and plot the values of the terms against the term number using a graphics calculator. Check that this linear relationship exists. There is a web animation for this activity if you wish to view it.

Q11: Using the definition, which of the following are arithmetic sequences?

a) -12, -9, -6, -3
b) 5, 15, 45 ,55, 85, 95, 125, ...
c) 9, 14, 20, 27, ..., 64, 76, ...
d) 0, 1, 2, 3, 4, 5, ..., 45, ..., 90, 91
e) 0.01, 0.001, 0.0001, 0.00001, 0.000001, ...
f) 1, 8, 27, 64, 125, ...

Examples

1. Finding the nth term of an arithmetic sequence

Find the 20th term of the arithmetic sequence 4, 11, 18, 25, 32, ...

Answer:

The formula is

$u_{20} = a + (20 - 1)d = 4 + 19 \times 7 = 137$

2. Defining the sequence from two terms

If the sixth term of an arithmetic sequence is -22 and the third term is -10 define the sequence.

Answer:

For the sixth term -22 = a + 5d and for the third term -10 = a + 2d.

Solving these simultaneously gives d = -4.

So a = -2 and the sequence is defined by {-2 - 4(n - 1)} = {-4n + 2}

Arithmetic sequence exercise

10 min

There are randomised web questions on defining the sequence which may be of interest.

Q12: Find the 16th term of the arithmetic sequence 1, $\frac{1}{2}$, 0, - $\frac{1}{2}$, - 1, ...

Q13: If the fifth term of an arithmetic sequence is 15 and the seventh term is 21 define the sequence.

Q14: If the third term of an arithmetic sequence is -13 and the sum of the first two terms is -38, define the sequence and give the tenth term.

Q15: The sum of the second and third terms of an arithmetic sequence is 2 and the sum of the sixth and seventh terms is -14. Define the sequence and find u_{40}

4.6 Geometric sequences

Learning Objective

Calculate the nth term of a geometric sequence

Another type of sequence is the geometric sequence.

A geometric sequence is one which has the form a, ar, ar^2, ar^3, ... where a is the first term and r is the common ratio.

The common ratio in a geometric sequence is the ratio $r = \frac{u_{n+1}}{u_n}$ of two consecutive terms.

Example 5, 15, 45, 135, 405, ... is a geometric sequence with a = 5 and r = 3

This sequence can be linked to the general linear recurrence relation

$u_{n+1} = bu_n + c$ where $b, c \in \mathbb{R}$ and $n \in \mathbb{N}$.

In this equation consider the outcome when $b \neq 0$ with $c = 0$

This gives $u_{n+1} = bu_n$

This represents a geometric sequence.

The first term is $a = u_1$ and the sequence has a common ratio of r = b

Note that 'b' and 'c' are used here to prevent confusion with the first term 'a' of the geometric sequence.

Q16: Consider the geometric sequence defined by the recurrence relation

$u_{n+1} = bu_n$ with b = -3 and $u_1 = -1$. State a, r and hence the first 4 terms of the sequence.

As with arithmetic sequences it would be rather laborious to use this approach to find, say, the 20th term in a sequence.

Again there is an alternative approach which comes directly from the definition.

Since the sequence is defined as a, ar, ar^2, ... then

$u_1 = a$

$u_2 = ar^1$

$u_3 = ar^2$ and so on.

So $u_n = ar^{n-1}$ and the sequence can be written as $\{ar^{n-1}\}$ where $n \in \mathbb{N}$.

The nth term of a geometric sequence is given by ar^{n-1} where a is the first term of the sequence, r represents the common ratio and n is the number of the term ($n \in \mathbb{N}$).

The terms in a geometric sequence demonstrate an exponential relationship with the equation in the form $u_n = ab^n$

Geometric sequence graphs

10 min

Work out a variety of simple geometric sequences and using a graphics calculator check that the exponential relationship holds by plotting log u_n against n. If it does, the graph should show a linear relationship.
There is a web animation for this activity if you wish to view it.

Q17: Using the definition, which of the following are geometric sequences?

a) -1, 1, -1, 1, -1, 1, -1, 1, -1, ...
b) 5, -10, 20, -40, 80, ...
c) 1, 4, 9, 16, 25, 36, 49, ...
d) 2.2, 3.3, 4.4, 5.5, ..., 9.9
e) 0.01, 0.001, ..., 0.0000001, 0.00000001, ...
f) 64, 32, 16, 8, 4

Examples

1. Finding the nth term of a geometric sequence

Find the 12th term of the geometric sequence 4, 1.2, 0.36, 0.108, 0.0324, ...

Answer:

The formula is

$u_{12} = ar^{11} = 4 \times 0.3^{11} = 7.08588 \times 10^{-6}$

2. Defining the sequence from two terms

If the sixth term of a geometric sequence is -486 and the third term is 18 define the sequence.

Answer:

For the sixth term $-486 = ar^5$ and for the third term $18 = ar^2$

Dividing one equation by the other gives $\frac{-486}{18} = -27 = \frac{ar^5}{ar^2} = r^3$

So $-27 = r^3 \Rightarrow r = -3$

Substituting gives a = 2 and the sequence is defined by $\{2 \times (-3)^{(n-1)}\}$

Geometric sequence exercise

10 min

There are further questions on defining a geometric sequence on the web which may be of use.

Q18: Find the 16th term of the geometric sequence 0.1, 0.4, 1.6, 6.4, 25.6, ...

Q19: If the fifth term of a geometric sequence with all terms positive is 16 and the seventh term is 64 define the sequence.

Q20: A deposit of \$500 earns interest at the rate of 5.5% per annum (compound). Calculate what the balance will be at the end of the sixth year to the nearest cent.

4.7 Fibonacci and other sequences

Learning Objective

Show an awareness of the different types of sequences

There are many interesting sequences. Here are a few.

An alternating sequence is any sequence which has alternate positive and negative terms.

This type of sequence was mentioned in the geometric sequence section.

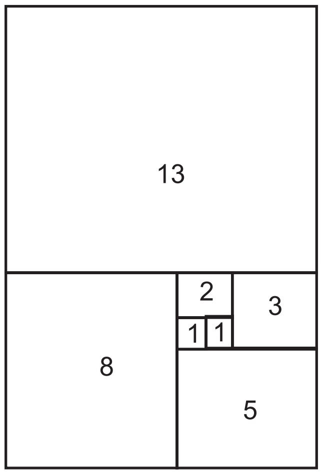
The example given was -1, 1, -1, 1, -1, ...

The Fibonacci sequence is formed by taking a first and second term equal to 1. Each subsequent term is formed by adding the two terms immediately before it.

This sequence is defined by a second order recurrence relation of the form

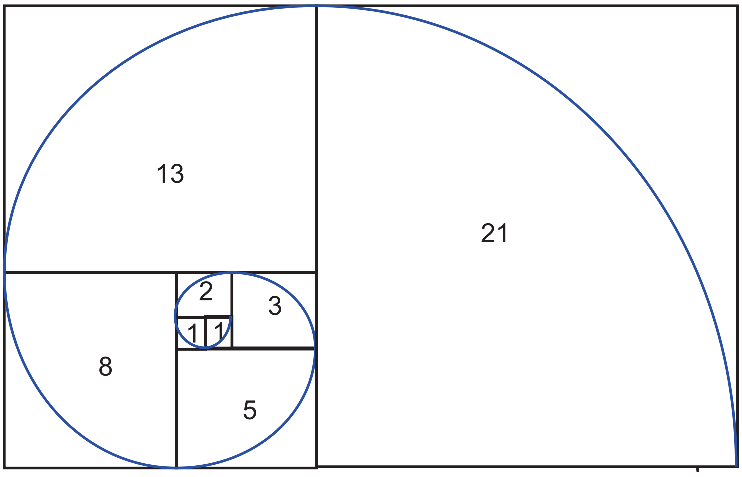
$u_{n+2} = u_{n+1} + u_n$

The Fibonacci sequence can be shown using squares as follows where the length of each side represents a term in the Fibonacci sequence.



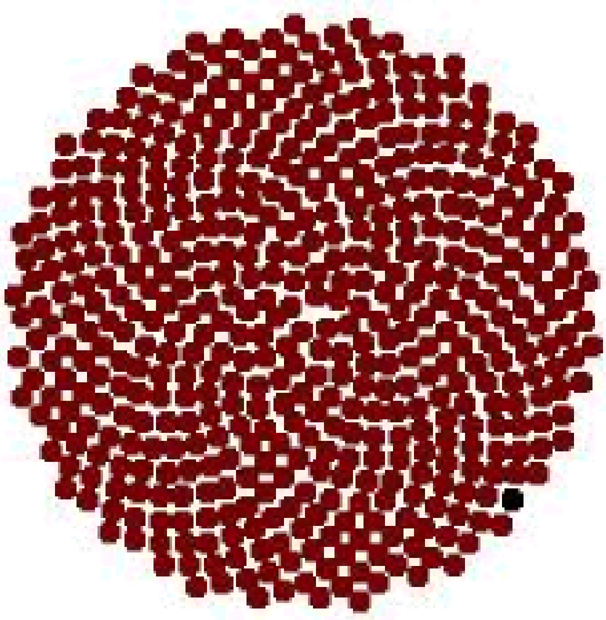
The Fibonacci sequence occurs in many natural phenomena such as the shape of snail shells and the geometry of sunflower heads.

The following image demonstrates the structure of the snail shell using the Fibonacci sequence. It is obtained from arcs connecting the opposite corners of each square starting at the first term of 1 and continuing with an uninterrupted line.



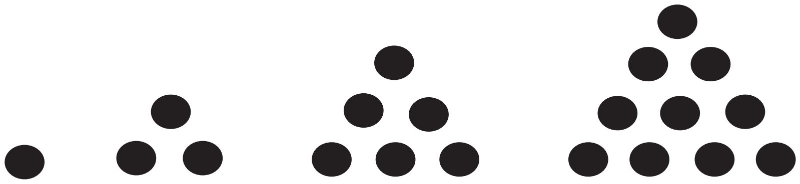
Activity

Take the image of a sunflower which follows and identify the Fibonacci sequence in it using this rectangular construction.



This sequence comprises the natural numbers which can be drawn as dots in a triangular shape.

The first four terms are represented as:



Q21: The triangular number sequence begins 1, 3, 6, 10, 15, ... Find a formula for the nth term.

15 min

Sequence activities

Here are a variety of sequence activities.

Activity

Draw the triangular numbers one by one as right angled triangles with the right angles at a common corner. Deduce, from simple geometry, the answer to the last question.

Activity

Try to find three other sequences with geometrical representations such as the triangular numbers (for example, pentagonal numbers).

Activity

Draw on squared paper, the Fibonacci square representation of the first 12 terms.

10 min

Types of sequence

The graphs of some of the common sequences are shown on the web.

4.8 Convergence and limits

Learning Objective

Apply the tests and limit rules to sequences

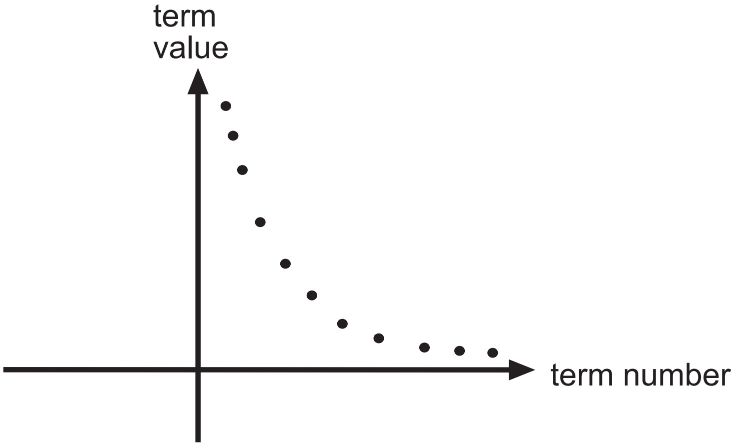
In an earlier section both finite and infinite sequences were defined.

For finite sequences the last term is explicitly stated. For an infinite sequence $\{u_n\}$, there is no last term but it may be possible to say something about the behaviour of the sequence for large values of n.

Consider the sequence $\left\{\frac{1}{n}\right\}$ with terms $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$

The terms in this sequence become smaller and smaller.

This can be clearly seen by plotting some of the terms on a graph.



From this evidence it appears that the sequence tends towards the value 0 for large values of n.

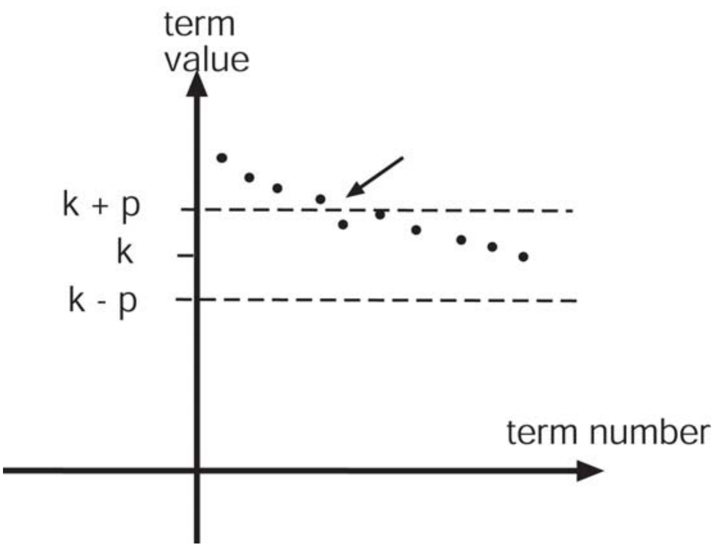
In a similar fashion, it appears that the sequence $\left\{1 + \frac{1}{n}\right\}$ with terms $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$ approaches the value 1 for large n.

This intuitive approach suggests a mathematical rule for determining how sequences behave.

An infinite sequence $\{u_n\}$ tends to the limit k if for all positive numbers p
there is an integer N such that $k - p < u_n < k + p$ for all $n > N$

If the condition is satisfied then $\lim\limits_{n\to\infty} u_n = k$ (The limit of the sequence $\{u_n\}$ as n tends to infinity is k)

Graphically this means that if the terms of the sequence are plotted on a graph then at some value along the x-axis all the subsequent terms of the sequence (to the right) lie between horizontal lines drawn at x = k - p and x = k + p



Here N = 4 since for n = 5 and above $|u_n - k| < p$

By letting p become smaller and smaller it is apparent that the limiting value of the sequence must be k.

A sequence which has such a limit is called convergent.

An infinite sequence $\{u_n\}$ for which $\lim\limits_{n\to\infty} u_n = k$ is called a **convergent sequence** with limit k

Demonstration of convergence

There is a demonstration of convergence on the web which will help to simplify the rather complicated definition of convergence.

5 min

Example Take the sequence $u_n = \left\{\frac{1}{n}\right\}$ and confirm that it is convergent with limit 0

Answer:

First consider some examples for fixed p. Suppose that $p = \frac{2}{11}$ then for each term from u_6, $|u_n| < p$. The integer N is 5

Similarly, if $p = \frac{2}{97}$ then N = 48 as the term u_{49} and all the subsequent ones satisfy $|u_n| < p$

Algebraically $\frac{1}{n} < p \Rightarrow \frac{1}{p} < n$ so for any p if N is chosen as the first integer $\geqslant \frac{1}{p}$ the convergence condition holds and hence the sequence converges to 0

Sequences which converge to the value 0 are given a special name.

A convergent sequence which converges to the limit 0 is called a **null sequence**.

Q22: Are the following null sequences?

a) $\left\{\frac{1}{n^3}\right\}$
b) $\left\{3 + \frac{1}{n^3}\right\}$
c) $\{2n\}$

There are a variety of rules which help to determine limits on more complicated sequences. Here are some of the more common ones.

If $\lim_{n\to\infty} a_n = k$ and $\lim_{n\to\infty} b_n = m$ then the following can be applied.

- **The sum rule for convergent sequences**
  $\lim_{n\to\infty} (a_n + b_n) = k + m$
- **The multiple rule for convergent sequences**
  $\lim_{n\to\infty} (\lambda a_n) = \lambda k$ for $\lambda \in \mathbb{R}$
- **The product rule for convergent sequences**
  $\lim_{n\to\infty} (a_n b_n) = km$
- **The quotient rule for sequences**
  $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{k}{m}$ provided that $m \neq 0$

The second example shown at the start of this section, namely, $\left\{1 + \frac{1}{n}\right\}$ with terms $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots$ can now be thought of as a sum of two sequences.

The first is a constant sequence of 1, 1, 1, 1, 1, ...

The second is the sequence $\left\{\frac{1}{n}\right\}$

This second sequence is a null sequence as it converges to zero. The constant sequence has a limit equal to the constant ... in this case to 1

Using the sum rule, the limit of $\left\{1 + \frac{1}{n}\right\}$ is therefore 1 + 0 = 1

Many sequences do not converge.

Examples

1. The sequence $\{2^n\}$ is not convergent. For any given p and k there is no term in the sequence which can be chosen such that every subsequent term lies between k + p and k - p

However, for any chosen value m there is a point in the sequence where all subsequent terms are larger than m. The sequence is **unbounded**.

The sequence $\{2^n\}$ is an example of a sequence which tends to infinity.

2. The alternating sequence $\{(-1)^n\}$ with terms -1, 1, -1, 1, -1, ... is not convergent.

For $p = \frac{1}{2}$ and any k there is no term in the sequence which can be chosen such that every subsequent term lies between k + p and k - p

However, unlike the previous example $\{2^n\}$ this sequence is not unbounded since all terms lie between -1 and +1

Q23: Using a graphics calculator explore whether the following sequences have a limit.

a) $\left\{\frac{1}{(3n - 1)}\right\}$

b) $\left\{\frac{n^2}{(3n - 1)}\right\}$

c) $\left\{10 - \frac{1}{n}\right\}$

d) $\left\{\frac{(-1)^n}{n}\right\}$

Consider the sequence $\left\{\frac{(2n - 1)(3n + 3)}{4n(n + 2)}\right\}$

This seems a very complicated sequence. The limit, if it exists, is not clear from the expression as it stands. However with some algebraic manipulation the sequence can be transformed into a state in which it is easier to determine if it is convergent.

In this case the following steps make the problem much easier to address.

Multiply out the brackets $\left\{\frac{(2n - 1)(3n + 3)}{4n(n + 2)}\right\} = \left\{\frac{6n^2 + 3n - 3}{4n^2 + 8n}\right\}$

The dominant term is n^2 so divide top and bottom by this.

The new expression is $\left\{\frac{6 + \frac{3}{n} - \frac{3}{n^2}}{4 + \frac{8}{n}}\right\}$

Now the rules can be used. The terms $\frac{3}{n}$, $\frac{8}{n}$, and $\frac{3}{n^2}$ all generate null sequences and the sequence $\left\{\frac{(2n - 1)(3n + 3)}{4n(n + 2)}\right\}$ has a limit of $\frac{6}{4}$

Strategy for finding the limit of a complicated quotient:

- Simplify by removing any brackets.
- Identify the dominant term.
- Divide top and bottom by this dominant term.
- Use the rules to determine the limit, if it exists.

Limits of sequences exercise

10 min

You may wish to try a different exercise on the web which gives randomised questions.

Q24: Determine the limit, if it exists, of the sequence $\left\{\frac{n\,(2n - 2)}{3n^3 - 4}\right\}$

Q25: Determine the limit, if it exists, of the sequence $\left\{\frac{n - 2}{n + 1}\right\}$

Q26: Determine the limit, if it exists, of the sequence $\left\{\sqrt{n + 1}\right\}$

4.9 Definitions of e and π as limits of sequences

Learning Objective

Know the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ and follow the derivations of the number e and of π

4.9.1 The definition of e as a limit of a sequence

Consider the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$

Q27: Using a calculator find and plot the values of the first twenty terms on a graph using the x-axis for the term number and the y-axis for these values. Suggest a limit for this sequence.

Q28: Take the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ and investigate the values of the terms in this sequence with a graphics calculator. Examine a selection of the terms up to number 10000 and hence suggest a limit for this sequence.

Q29: What is the value of e?

Q30: Compare the last three answers and make a statement connecting e and the sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$

The results of the questions are important and worth stating again.

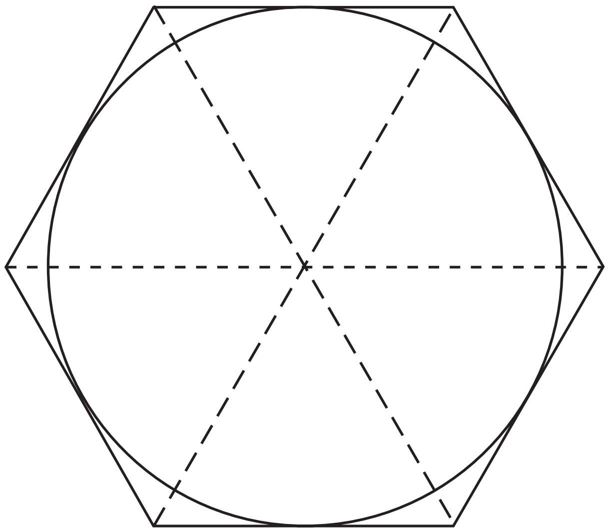
The sequence $\left\{\left(1 + \frac{1}{n}\right)^n\right\}$ converges with a limit equal to e.

That is

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

4.9.2 Definition of π as a limit of a sequence

Take a circle of unit radius and circumscribe a hexagon on it.



By elementary trigonometry in degrees it can be shown that the length of each side of the hexagon is given by the formula $2 \tan\left(\frac{180}{6}\right)$. The perimeter of the hexagon is therefore $6 \times 2 \tan\left(\frac{180}{6}\right)$.

Q31: Suggest a formula for the perimeter of a polygon with twelve sides. From this deduce a formula for the perimeter of a polygon with n sides.

Thus the perimeters of polygons around a unit circle generate a sequence

$\left\{n \times 2 \tan\left(\frac{180}{n}\right)\right\}$ for $n > 6$

Q32: Using a graphics calculator take this function and investigate the sequence of perimeters of polygons around a unit circle for increasing values of n. Explore the values and suggest a limit.

Now consider geometrically the effect of circumscribing a polygon around a unit circle. As the polygon has more and more sides the polygon itself tends closer and closer to the shape of the circle. In other words the limit of the perimeter of polygons circumscribed around a circle is the circumference of the circle.

Q33: Suggest a value for the circumference of the circle from the work done here on polygons and hence give a value for π

Q34: If $\{a_n\}$ is the sequence of perimeter values of polygons around a unit circle, suggest a definition for π based on this sequence.

This is another result worth stating as π can be defined as a limit of a sequence.

$$\pi = \lim_{n \to \infty} \left[n \tan\left(\frac{180}{n}\right)\right]$$

Activity

Consider other ways of defining π using similar techniques to those shown (for example, using areas).

4.10 Series and sums

Learning Objective

Know the term partial sum and use the sigma notation

The link between series and sequences is a simple one.

A series is the sum of the terms in an infinite sequence.

Consider the sequence $u_1, u_2, u_3, u_4, ...$

Let

$u_1 + u_2 = S_2$

$u_1 + u_2 + u_3 = S_3$

and so on.

In this way, $u_1 + u_2 + u_3 + ... + u_n = S_n$

Such a sum is known as a **partial sum** or the **sum to n terms** of a sequence and is denoted by S_n

It is customary to use the sigma sign $\sum$ when describing a series. It gives a useful, accurate shorthand way of writing down a series without having to specify each term. Here are some examples:

- $\sum_{r=1}^{4} (r\ \{r + 2\}) = 3 + 8 + 15 + 24 = 50$ this is a partial sum
- $\sum_{n=2}^{5} \left(\frac{1}{n}\right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{77}{60}$ this is a partial sum
- $\sum_{r=1}^{\infty} \left(2 + \frac{2}{r}\right)$ this is a sum to infinity

The upper and lower limits on the sigma sign indicate the range of values which should be used to construct the series (Recall the use of the sigma sign in the Binomial theorem section of Algebra (unit 1, topic 1).)

The partial sum is the sum of the terms from 1 to n where $n \in \mathbb{N}$ It is denoted by S_n and represented as $S_n = \sum_{r=1}^{n} u_r$

In sigma notation the series $\sum_{r=1}^{\infty} u_r$ is the sum to infinity of the terms of the sequence $\{u_r\}$. It is denoted by S_∞. This concept of sums to infinity for geometric series will be explained later in a separate subsection.

Example Find S_4 for $\sum_{r=1}^{\infty} (2 + r)$

Answer:

S_4 means the partial sum to four terms. So r = 1, 2, 3 and 4

This gives (2 + 1) + (2 + 2) + (2 + 3) + (2 + 4) = 18

Finding partial sums exercise

10 min

There are different questions on finding partial sums on the web if you prefer them.

Q35: Find S_4 for $\sum_{r=1}^{\infty}\left(r + \frac{3}{r}\right)$

Q36: Find the partial sum to five terms for $\sum_{r=1}^{\infty}(r!)$

Q37: Find S_3 for $\sum_{r=1}^{\infty}\left(2 + \frac{3}{r}\right)$

For partial sums, such as those shown in the example and in the exercise, this long-winded method works. It is not so easy however to find the sum to n terms
(the nth partial sum) unless the structure of the sequence is known.

Looking at the last question there is a similarity between $\sum_{r=1}^{\infty}\left(2 + \frac{3}{r}\right)$ and the sequence generated by $\left\{2 + \frac{3}{r}\right\}$

Obviously since sequences are all constructed in different ways, the series relating to them will also differ. Some have special formulae derived from their related sequences which make the calculations of sums to n terms much easier and quicker. Arithmetic and geometric series are two such series of particular interest in this course and these will be dealt with in the next two sections.

There is another aid to finding the sum to n terms. The two combination rules which follow apply to sums to infinity provided that the series a_n and b_n are convergent. The actual sums to infinity will be explored in one of the following subsections but at present the rules can be adapted and used for sums to n terms:

If a_n and b_n form convergent series then:

- **The sum rule for convergent series**

  $$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

- **The multiple rule for convergent series**

  $$\sum_{n=1}^{\infty}(\lambda a_n) = \lambda \sum_{n=1}^{\infty} a_n \text{ for } \lambda \in \mathbb{R}$$

Example Find $\sum_{n=1}^{6}(2n + 3)$

Answer:

$$\sum_{n=1}^{6}(2n+3) = \sum_{n=1}^{6}(2n) + \sum_{n=1}^{6}3$$
$$= 2\sum_{n=1}^{6}n + 3 \times 6$$
$$= 2(1+2+3+4+5+6) + 18$$
$$= 42 + 18 = 60$$

Q38: Find $\sum_{n=1}^{4}\left(\frac{1}{2}n + 3n^2\right)$ using the combination rules

4.11 Arithmetic series

Learning Objective

Find the sum to n terms of an arithmetic series

Recall that an **arithmetic sequence** is one which has the form a, a + d, a + 2d, ... where a is the first term in the sequence and d is the **common difference**.

The nth term of an arithmetic sequence was also defined earlier.

It is $u_n = a + (n - 1)d$

With this information it is possible to find a formula which can be used to provide the sum to n terms of an arithmetic series. An arithmetic series is the sum of the terms of an arithmetic sequence and is also known as an arithmetic progression.

Consider S_n for a general arithmetic series.

$$S_n = \sum_{r=1}^{n}(a + (n-1)d)$$
$$= a + (a+d) + (a+2d) + ... + (a+(n-2)d) + (a+(n-1)d)$$

Also

$$S_n = (a+(n-1)d) + (a+(n-2)d) + ... + (a+2d) + (a+d) + a$$

Add the two expressions for S_n to give

$2S_n = (2a + (n - 1)d) + (2a + (n - 1)d) + ... + (2a + (n - 1)d) + (2a + (n - 1)d)$

That is $2S_n = n(2a + (n - 1)d)$

Hence $S_n = \frac{n}{2}[2a + (n - 1)d]$

The sum to n terms (the nth partial sum) of an arithmetic series is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$ where a is the first term of the sequence, d represents the common difference and $n \in \mathbb{N}$.

Example Find the sum of the arithmetic series 1 + 3 + 5 + 7 + ... + 37

Answer:

Let the nth term be equal to 37 so a + (n - 1)d = 37

but d = 2 and a = 1 so 1 + 2(n - 1) = 37 $\Rightarrow$ n = 19

$S_n = \frac{19}{2}(2 + 18 \times 2) = 361$

Activity

Show that if a = 1 and d = 1, the expression $S_n = \frac{n}{2}\left[2a + (n - 1)d\right]$ reduces to $\frac{1}{2}n\,(n + 1)$

This is a useful formula to use for the sum of n integers.

The sum of the first n integers

There is an on-line animation to show the sum of the first n integers up to and including n = 10.

10 min

Arithmetic series partial sums exercise

There is an exercise on the web with randomised questions which may be of help.

10 min

Q39: Find S_8 of the arithmetic series 1 + 0 - 1 - 2 - 3 - ...

Q40: Find S_{13} of the arithmetic series -4 - 2 + 0 + 2 + 4 + ...

Q41: Find S_7 of the arithmetic series -15 + 1 + 17 + 33 + 49 + ...

The techniques of finding a specific term and summing an arithmetic series can be combined to solve more complex problems where only limited information is available.

Example If the third term of an arithmetic series is 11 and S_6 = 78 find S_{22}

Answer:

From the first piece of information u_3 = 11

So using the formula u_n = a + (n - 1)d gives the equation 11 = a + 2d

The second piece of information S_6 = 78 gives another equation in a and d, namely,

78 = 3 (2a + (5)d) = 6a + 15d

These two equations, 11 = a + 2d and 78 = 6a + 15d can be solved simultaneously to give

d = 4 and a = 3

Using the sum formula $S_n = \frac{n}{2}\left[2a + (n - 1)d\right]$ with these values for S_{22} gives

$S_{22} = \frac{22}{2}\left[2 \times 3 + (22 - 1)4\right] = 990$

Extended exercise on arithmetic series

There is a similar exercise on the web with randomised questions if you prefer them.

10 min

Q42: For an arithmetic series, the fifth term is 18 and S_5 is 50. Find S_{30}

Q43: For an arithmetic series S_6 = 78 and S_9 = 171. Find S_{40}

Q44: The third term of an arithmetic series is 21 and S_7 is 161. Find S_{27}

The infinite series 1 + 2 + 3 + 4 + 5 + ... is an arithmetic series of particular interest and use.

This series has a = 1 and d = 1

Thus

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

$$S_n = \frac{n}{2}[2 + n - 1] = \frac{n(n + 1)}{2}$$

This is another important result:

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$$

It can be used in conjunction with the combination rules to simplify the calculations of series sums.

Example Evaluate $\sum_{n=1}^{9} (3n + 2)$

Answer:

By the combination rules

$$\begin{aligned}\sum_{n=1}^{9} (3n+2) &= \sum_{n=1}^{9} (3n) + \sum_{n=1}^{9} (2)\\ &= 3\sum_{n=1}^{9} n + 2\sum_{n=1}^{9} (1)\\ &= \frac{3 \times 9\ (10)}{2} + (2 \times 9)\\ &= \frac{3\ (90)}{2} + 18\\ &= 153\end{aligned}$$

Q45: Calculate $\sum_{n=1}^{6} (2n - 4)$ using the techniques of the example.

Q46: Calculate $\sum_{n=1}^{7} \left(3n - \frac{1}{2}\right)$

4.12 Geometric series

Learning Objective

Find the sum to n terms of a geometric series

You may recall that a **geometric sequence** is one which has the form a, ar, ar^2, ar^3, ar^4, ... where a is the first term in the sequence and r is the **common ratio**.

The nth term of a geometric sequence is $u_n = ar^{n-1}$

Using this information it is possible to find a formula which can be used to provide the sum to n terms of a geometric series. A geometric series is the sum of the terms of a geometric sequence and is also known as a geometric progression.

Consider S_n for a general geometric series.

$$\begin{aligned}S_n &= \sum_{k=1}^{n} \left(ar^{k-1}\right)\\ &= a + ar + ar^2 + ... + ar^{n-2} + ar^{n-1}\\ r\,S_n &= \quad ar + ar^2 + ... + ar^{n-2} + ar^{n-1} + ar^n\end{aligned}$$

thus

$$S_n - rS_n = a - ar^n$$
$$S_n(1 - r) = a(1 - r^n)$$
$$S_n = \frac{a(1 - r^n)}{1 - r}$$

This formula for the sum to n terms of a geometric series only holds for $r \neq 1$

When r = 1, $S_n = a + a + a + ... + a$ for n terms. i.e. $S_n = na$

> The sum to n terms (the nth partial sum) of a geometric series is given by $S_n = \frac{a(1 - r^n)}{1 - r}$ where a is the first term of the sequence, r ($\neq 1$) represents the common ratio and $n \in \mathbb{N}$.

As an aside, at this point it is worth summing up the possibilities which can arise with a general geometric series dependant on the values of r. The range $-1 < r < 1$ will be discussed in the next section.

Behaviour of a general geometric series for values of r, where $r \geq 1$ and $r \leq -1$

$r > 1$	$S_n \to \pm\infty$ as $n \to \infty$
$r < -1$	S_n alternates between +ve and -ve values, $\lvert S_n \rvert \to \infty$
$r = 1$	$S_n = na$
$r = -1$	$S_n = 0$ for even n and $S_n = a$ for odd n

Example Find S_9 of the geometric series 1 + 3 + 9 + 27 + 81 +...

Answer:

Using the formula $S_n = \frac{a(1 - r^n)}{1 - r}$ with a = 1, r = 3 and n = 9 gives

$S_9 = \frac{1(1 - 3^9)}{1 - 3} = 9841$

Geometric series partial sums exercise

10 min

There is a further exercise on the web which you may find useful.

Q47: Find S_8 of the geometric series 2 - 4 + 8 - 16 + 32 + ...

Q48: Find S_{13} of the geometric series -16 - 8 - 4 - 2 - 1 - ...

Q49: Find S_{10} of the geometric series which has a = 2 and r = 5

Q50: Find S_6 for the geometric series with a first term of 36 and a common ratio of $\frac{2}{3}$

The techniques of finding a specific term and summing a geometric series can be combined to solve more complex problems where only limited information is available.

Example Find S_6 for the geometric series which has $u_2 = 3$ and $u_5 = \frac{1}{9}$

Answer:

From the first part of the information $u_2 = 3$

So using the formula $u_n = a\, r^{n-1}$ gives the equation $3 = a\, r$

The second part, $u_5 = \frac{1}{9}$ gives another equation in a and r, namely, $\frac{1}{9} = a\, r^4$

Thus

$\frac{\frac{1}{9}}{3} = \frac{ar^4}{ar}$ so $\frac{1}{27} = r^3 \Rightarrow r = \frac{1}{3}$

Thus a = 9

Using the sum formula $S_n = \frac{a(1 - r^n)}{1 - r}$ with these values and n = 6 gives S_6

$$S_6 = \frac{9\left(1 - \left(\frac{1}{3}\right)^6\right)}{1 - \frac{1}{3}} = \frac{364}{27}$$

Extended exercise on geometric series

There is a further exercise on the web for you to try if you wish.

10 min

Q51: For a geometric series with all terms positive, the fifth term is 27 and the third term is 243. Find S_8

Q52: For a particular geometric series with all terms positive, $S_2 = 3$ and $S_4 = 15$. Find u_9 (Hint: in your working use the formula for the difference of two squares.)

Achilles and the tortoise

Read the following story and then try animation.

5 min

A Greek philosopher called Zeno put forward a now famous paradox concerning Achilles and the Tortoise. He suggested that a tortoise and Achilles were in a race. Achilles could travel at say, 10 times the rate of the tortoise but the tortoise had a head start of say 300 metres. He argued that when Achilles travelled the 300 metres, the tortoise would have moved ahead by 30 metres; when Achilles travelled the next 30 metres, the tortoise would have travelled 3 metres; when Achilles had travelled the next 3 metres the tortoise would have travelled a further 0.3 metres and so on. Thus Achilles could never catch the tortoise. Discuss and find out the flaw in this argument (which looks like a geometric series).

There is a demonstration of this paradox on the web .

4.13 Sums to infinity

Learning Objective

Ascertain the value of a sum to infinity if it exists

It is straightforward to find the limit of a convergent sequence. It is also easy to calculate sums to n terms of an infinite series. In some cases however, it would be useful to know whether an infinite series actually has a sum.

Consider the infinite geometric series 27 + 2.7 + 0.27 + 0.027 + 0.0027 + ...

$S_1 = 27$

$S_2 = 29.7$

$S_3 = 29.97$ and so on.

Activity

Continue to find the sums from S_4 to S_{10} and suggest a value for S_∞

The values of these partial sums form a sequence
27, 29.7, 29.97, 29.997, 29.9997, ... Clearly the sequence approaches 30

That is, the sum to infinity is the limit of the sequence of partial sums.

In this case $S_n \to 30$ as $n \to \infty$ (S_n tends towards a limiting value of 30 as n approaches infinity).

If a limit exists, the series is convergent.

A **convergent series** is one for which the limit of partial sums exists. This limit is called the **sum** and is denoted by S_∞ or $\sum_{n=1}^{\infty} u_n$

A **divergent series** is one which is not convergent. e.g. 1 + 2 + 3 + ...

Consider the general infinite geometric series $a + ar + ar^2 + ar^3 + ar^4 + ...$

The formula for S_n was given in the last section. i.e. $S_n = \frac{a(1 - r^n)}{1 - r}, r \neq 1$

Suppose that $-1 < r < 1$ (i.e.$|r| < 1$)

The terms $(1 - r^n)$ will approach 1 for large n because r^n approaches zero.

So $S_n \to \frac{a}{1 - r}$ as $n \to \infty$

This means that a geometric series is convergent if $|r| < 1$

In other words:

For a convergent geometric series $S_\infty = \frac{a}{1 - r}$ where $|r| < 1$

To conclude, the sum to infinity of a series only exists if the sequence of partial sums is convergent. In the case of geometric series, this only occurs when $|r| < 1$

Example : Finding the sum to infinity

Find S_∞ of the geometric series 625, 125, 25, 5, 1, ...

Answer:

$a = 625$ and $r = {}^1/_5$ so

$S_\infty = \frac{625}{1 - \frac{1}{5}} = \frac{625 \times 5}{4} = 781.25$

Q53: State which of the following geometric series are convergent and for those that are convergent find the sum to infinity.

a) 2 + 4 + 8 + 16 + 32 +...
b) 32 + 16 + 8 + 4 + 2 + ...
c) 32 - 16 + 8 - 4 + 2 - ...
d) $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \frac{1}{48} + ...$
e) $-\frac{1}{3} + 1 - 3 + 9 - 27 + 81 - ...$

Q54: Find the sum to infinity of the geometric series

$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$

Q55: Find the first term of a geometric series which has a sum to infinity of 9 and a common ratio of $\frac{2}{3}$

Q56: Find the sixth term of a geometric series which has a common ratio of $\frac{5}{6}$ and a sum to infinity of 72

Q57: Find the sum to infinity of the geometric series with all terms positive in which $u_2 = 4$ and $u_4 = 1$

Consider the geometric series $\sum_{n=1}^{\infty} ar^n$ where a = 1 and $|r| < 1$

This series takes the form $1 + r + r^2 + r^3 + r^4 + ...$

Using $S_\infty = \frac{a}{1-r}$ gives the sum to infinity for this series as $\frac{1}{1-r}$, which of course can be written as $(1-r)^{-1}$

This alternative expression can be expanded using the Binomial theorem. When the theorem is used with negative powers the expansion is infinite. To ensure that the infinite series converges the condition $|z| < 1$ is needed just as it was in the geometric series formulae. The Binomial expansion gives

$1 + \left(\frac{-1}{1!}\right)(-r) + \left(\frac{-1 \times -2}{2!}\right)(-r)^2 + \left(\frac{-1 \times -2 \times -3}{3!}\right)(-r)^3 + ...$

Clearly this expression reduces to $1 + r + r^2 + r^3 + r^4 + ...$

Example Consider the geometric series $\sum_{n=1}^{\infty} ar^n$ where a = 1 and $r = \frac{1}{2}$

This series takes the form $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$

Using $S_\infty = \frac{a}{1-r}$ gives the sum to infinity for this series as $\frac{1}{1-\frac{1}{2}}$ which of course can be written as $\left(1-\frac{1}{2}\right)^{-1}$

Using the Binomial theorem gives

$1 + \left(\frac{-1}{1!}\right)\left(-\frac{1}{2}\right) + \left(\frac{-1 \times -2}{2!}\right)\left(-\frac{1}{2}\right)^2 + \left(\frac{-1 \times -2 \times -3}{3!}\right)\left(-\frac{1}{2}\right)^3 + ...$

$= 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ...$ as required.

This use of the Binomial theorem only applies to expressions of the form $(1 + x)^n$ where $n \in \mathbb{R}$.

The technique can be adapted however to cope with expressions such as $(a + x)^n$

Note that this next series contains algebraic terms as well as numbers.

The trick is to convert this expression into either $a^n \left(1 + \frac{x}{a}\right)^n$ or into the form $x^n \left(\frac{a}{x} + 1\right)^n$ and use the expansion.

Use $a^n \left(1 + \frac{x}{a}\right)^n$ if $\left|\frac{x}{a}\right| < 1$ and $x^n \left(\frac{a}{x} + 1\right)^n$ if $\left|\frac{a}{x}\right| < 1$

Q58: Use the Binomial theorem to expand $(a + x)^{-1}$ to four terms where $\left|\frac{x}{a}\right| < 1$

Example Show that the expansion of $(3 - x)^{-1}$ is the geometric series with first term $a = \frac{1}{3}$ and common ratio $r = \frac{x}{3}$

Answer:

$(3 - x)^{-1} = 3^{-1}\left(1 - \frac{x}{3}\right)^{-1}$

By the Binomial theorem the expansion is

$\frac{1}{3}\left\{1 + \left(\frac{-1}{1!}\right)\left(-\frac{x}{3}\right) + \left(\frac{-1 \times -2}{2!}\right)\left(-\frac{x}{3}\right)^2 + \left(\frac{-1 \times -2 \times -3}{3!}\right)\left(-\frac{x}{3}\right)^3 + ...\right\}$

This simplifies to $\frac{1}{3}\left\{1 + \frac{x}{3} + \frac{x^2}{9} + \frac{x^3}{27} + ...\right\} = \frac{1}{3} + \frac{x}{9} + \frac{x^2}{27} + \frac{x^3}{81} + ...$

This is a geometric series with first term $a = \frac{1}{3}$ and common ratio $r = \frac{x}{3}$ and converges for $\left| -\frac{x}{3}\right| < 1$ i.e. for $|x| < 3$

Q59: Find the geometric series which relates to the expression $(2 + 3x)^{-1}$ when $|x| > \frac{2}{3}$

The following example shows a numeric expansion using the power of -1. Recall that in the section on the Binomial theorem there were similar examples and a exercise using positive powers.

Example Express $(0.99)^{-1}$ as a geometric series and give an approximate value for it to four decimal places.

Answer:

$(0.99)^{-1} = (1 - 0.01)^{-1}$

By the Binomial theorem the expansion is

$$\begin{aligned}(1 - 0.01)^{-1} &= 1 + \frac{-1}{1!}(-0.01) + \frac{-1 \times -2}{2!}(-0.01)^2 + \frac{-1 \times -2 \times -3}{3!}(-0.01)^3 + ...\\ &= 1 + 0.01 + 0.0001 + 0.000001 + ...\end{aligned}$$

which is a geometric series with the first term a = 1 and the common ration r = 0.01

The series approximates to a value of 1.0101.

(Note: the exact value is given by the repeating decimal 1.010101...)

Q60: Express $(1.05)^{-1}$ as a geometric series and give an approximate value for it to four decimal places.

Partial sums activity on two common series

10 min

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. (This is called the harmonic series.)

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + ... + \frac{1}{16}\right) + ...$$

$$> 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \frac{8}{16} + ... = \infty$$

Calculate the partial sums S_2, S_4, S_8, S_{16}, ... and confirm that $S_{2^n} > 1 + \frac{n}{2}$

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. Explore this fact and suggest a likely limit.

4.14 Summary

At this stage the following topics and techniques should be known:

- Calculation of the nth term of an arithmetic sequence.
- Calculation of a partial sum of an arithmetic series.
- Calculation of the nth term of a geometric sequence.
- Calculation of a partial sum of a geometric series.
- When a sum to infinity exists and how to find it.
- The sequence $\left(1 + \frac{1}{n}\right)^n$ and its limit.
- $\sum_{r=1}^{n} r = \frac{1}{2}n\,(n + 1)$.

4.15 Extended information

Learning Objective

Develop an awareness of the additional material available

There are links on the web which give a variety of web sites on this topic. These sites cover the subject to a higher level and include all the convergence tests which would be useful for those who wish to study this topic further.

ZENO

This Greek philosopher and mathematician lived in the 5th century. His interests lay in paradoxes concerning motion. One of the most famous of these is 'Achilles and the Tortoise'. His arguments and logic are very interesting.

FIBONACCI

This man was probably the greatest mathematician of his time. He was born in Pisa in 1170 and his real name was Leonardo of Pisa. He was an inspired man who produced many mathematical results. The sequence named after him was included in the third section of the book Liber Abbaci published in 1202. This section of the book contained problems on sequences, series and some number theory. Another famous discovery of Fibonacci's was Pythagorean triples.

D'ALEMBERT

D'Alembert was a Frenchman born in 1717. He was a very argumentative man but made significant contributions in Mathematics and Physics. His ideas on limits led to the tests for convergence named after him. These are important results but are beyond the scope of this course. D'Alembert is also credited with being one of the first users of partial differential equations.

4.16 Review exercise

15 min

Review exercise

There is another review exercise with randomised questions on the web.

Q61: Find the 25th term of the arithmetic sequence 17, 26, 35, 44, 53,...

Q62: For the arithmetic sequence 13, 8, 3, -2, -7, ... find S_{28}

Q63: Find the 8th term of the geometric sequence 2, 8, 32, 128, 512, ...

Q64: For the geometric sequence 162, 54,18, 6, 2, ... find an expression for the sum of the first n terms.

4.17 Advanced review exercise

15 min

Advanced review exercise

There is an exercise on the web with randomised parameters.

Q65: Express $(2.95)^{-1}$ as a geometric series showing the first four terms and give an approximate value for it correct to 6 decimal places.

Q66: The compound interest rate is set at 7.5 % per annum. A customer banks $500 at the start of the first year and leaves it for 5 years. At the start of the 6th year the interest rate falls to 5%. He adds $500 at that point and leaves both his deposits and the interest to accumulate for 10 more years.

- Find the two geometric sequences - one for years 1 to 5 and one for years 6 to 16
- State the first term and the common ratio for both.
- Calculate the amount of money in the account at the start of the 17th year (to the nearest cent).

Q67: Find two geometric series for the expression $(3 - 4x)^{-1}$ and state the conditions for which each converges.

Q68: Find the value of $\sum_{n=1}^{\infty}\left(\frac{1}{n^2 + 5n + 6}\right)$

(Hint: use partial fractions.)

4.18 Set review exercise

Set review exercise

15 min

The answers for this exercise are only available on the web by entering the answers obtained in an exercise called 'set review exercise'. The questions may be structured differently but will require the same answers.

Q69: For the arithmetic sequence -11 , -7 , -3 , 1 , ... find :

a) the 14th term

b) the sum of the first 14 terms.

Q70: For the geometic sequence 625 , 125 , 25 , 5 , ... find :

a) the 9th term

b) the sum of the first 6 terms.

Topic 5

Elementary number theory and methods of proof

Contents

Learning Objectives

- *Use standard methods to prove results in elementary number theory.*

Minimum Performance Criteria:

- *Disprove a conjecture by providing a counter-example.*
- *Use proof by contradiction in a simple example.*

5.1 Introduction and definitions

Learning Objective

Explain the need for proofs

Mathematics is justly described as the most secure knowledge assembled by the human race.

This security of mathematical theories is the basis of making predictions and acting upon them in many everyday situations. Examples include the credit-worthiness rating used by banks, the level of payouts on the National Lottery and the maturity value of a pension.

The predictions may not necessarily be reliable but the mathematics behind them is. The extraordinary reliability of mathematical ideas arises from the care that has gone into their construction and development. Mathematics depends upon logical development, precision of expression and taking nothing for granted.

Basic **definitions** appear in the introduction to many subjects. and give a precise meaning to the word, phrase or concept being defined.

Examples

1. The definition of a student is a person who studies.

It does not assume where, when or which subject. It is the precise meaning of the word.

2. The definition of a triangle is a shape formed by three straight lines joined to form a closed path.

Again it is precise.

Three straight lines are required with no mention of size or angles. The lines must form a closed path which ensures that the shape is a triangle.

Activity

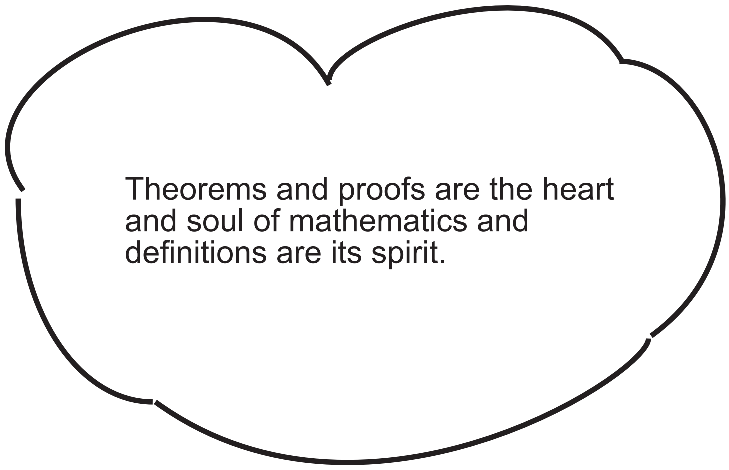
Construct definitions for the words 'mathematics' and 'example'. For example, for the word 'mathematics' will this definition do? 'Mathematics is the study of numbers.' Discuss the precision of these definitions in a group.

Once their definitions are in place, statements can be made about these objects and notions. Typically a statement will say that an object has a particular property.

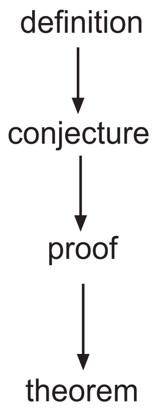
Example In a right angled triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.

This is a statement.

However this statement is called a theorem (Pythagoras' theorem). Why? It is a theorem because the statement was proved many years ago by a mathematician called Pythagoras.



Thus in mathematics the following structure develops:



To be precise:

A conjecture is a precise, unambiguous statement for which a convincing argument is needed.

A **proof** is a logically rigorous and complete argument that a mathematical statement is true.

A **theorem** is a mathematical conjecture which has been proved.

Consider the following statement: ‘Cows are cannibals’. This is a conjecture.

? Proof that cows are cannibals ?

Daisy is a cow

Cows eat daisies

Therefore cows are cannibals

It is plain to see that there is something wrong with this proof. The answer lies in the fact that the proof is not rigorous. It makes assumptions and does not use precise meanings or definitions of words (e.g. daisy).

Mathematicians must be very careful that any proof leaves no room for doubt. The proof of any conjecture must cover every case for which the conjecture is made.

The accepted mathematical **theorems** which exist have to be accurate and precise. The world would be in chaos if they were not.

Example Pythagoras' theorem is not true if the condition 'in a right angled triangle' is omitted.

Activity

Consider the implications of using Pythagoras' theorem in everyday situations without the restriction that it only holds for right angled triangles. (As a starter, think of distances on maps or constructing buildings).

Different types of theorem are occasionally given different names to indicate their logical place in the development of the ideas or their relative importance.

The word **theorem** itself is often reserved for the most important facts and the key parts of the theory while results of lesser importance are called **propositions**.

Occasionally conjectures are proved purely because they are needed in more significant proofs: these subsidiary results are called **lemmas** (subtheorems).

Sometimes a theorem or its proof may help to determine easily that other related statements are true: these statements are called **corollaries** of the theorem.

Q1: Without using addition and by considering the sum of the first 2 integers, then the sum of the first 3 integers, and so on, show that the sum of the first 10 integers is given by the expression $\frac{10\ (11)}{2} = 55$

Hint: try placing dots in a rectangle starting at one corner to find a pattern. Use the geometry of the triangle and rectangle to help.

The sum of the first 10 integers

10 min

Learning Objective

Display structured reasoning to a problem

There is an on-line animation to show the formation of the sum of the first n integers up to and including n = 10

This type of procedure could continue for the sum of 1000000 integers and so on in the hope of finding a general expression for the sum of the first n integers.This procedure in itself however, does not constitute a proof (more of this example later).

Q2: Make a conjecture which gives a formula for the sum of the first n integers.

Activity

The triangle inequality states

$|z + w| \leqslant |z| + |w|$

Consider the statement $|x + y| \leqslant |x| + |y|$ for $x, y \in \mathbb{R}$ and demonstrate that it is true.

5.2 Symbolism and notation

Learning Objective

Use the correct notation and symbols

When writing equations, proofs and mathematics in general, certain symbols are used as a shorthand method for particular phrases.

An example of this is the sigma sign $\sum$ which has been used in previous topics such as Sequences and Series, topic 9 of unit 2, in this Advanced Higher in Mathematics.

There are other symbols which are used in the construction of proofs. Here are three:

The symbol $\Rightarrow$

This symbol means 'implies'.

Implies means that the first statement can be used to logically deduce the next statement. It is denoted by the symbol $\Rightarrow$

Examples

1. The symbol $\Rightarrow$ in everyday context

Today is Sunday 'implies' that tomorrow is a weekday.

This is written as Today is Sunday $\Rightarrow$ tomorrow is a weekday.

2. The symbol $\Rightarrow$ in maths context

The equation $x = 3$ 'implies' that $x^2 = 9$

This is written as $x = 3 \Rightarrow x^2 = 9$

The symbol $\Leftarrow$

This symbol means 'implied by'.

Implied by means that the first statement is a logical consequence of the second statement. It is denoted by the symbol $\Leftarrow$

Examples

1. The symbol $\Leftarrow$ in everyday context

This month begins with the letter J is 'implied by' next month is August.

That is, This month begins with the letter J $\Leftarrow$ next month is August.

2. The symbol $\Leftarrow$ in maths context

The equation $|x + 2| = 3$ is 'implied by' $x = 1$

This is written as $|x + 2| = 3 \Leftarrow x = 1$

(This symbol is not used as frequently as the other two.)

The symbol $\Leftrightarrow$

This symbol means ‘equivalent to’ or ‘if and only if’. It is sometimes shortened to ‘iff’.

> **Is equivalent to** means that the first statement implies and is implied by the second statement. (The first statement is true **if and only if** the second statement is true.) It is denoted by the symbol $\Leftrightarrow$

Examples

1. The symbol $\Leftrightarrow$ in everyday context

Scott and James are two people.

Scott’s father is James ‘if and only if’ Scott is James’s son.

That is, Scott’s father is James $\Leftrightarrow$ Scott is James’s son.

2. The symbol $\Leftrightarrow$ in maths context

The equation $x + 2 = 3$ ‘is equivalent to’ $x = 3 - 2$

That is, $x + 2 = 3 \Leftrightarrow x = 3 - 2$

It is particularly important to note the precise meaning of these symbols and ensure that they are used correctly in any proofs.

It is not possible to interchange $\Rightarrow$ and $\Leftarrow$. If the use of both $\Rightarrow$ and $\Leftarrow$ are valid then the symbol $\Leftrightarrow$ should be used.

‘Today is Sunday’ is not implied by ‘tomorrow is a weekday’.

Similarly ‘this month begins with the letter J’ does not imply that ‘next month is August’.

Although it is possible to use $\Rightarrow$ or $\Leftarrow$ instead of $\Leftrightarrow$ it makes the statements less precise.

Q3: Look at the mathematical examples of $\Rightarrow$ and $\Leftarrow$ and justify why interchanging these symbols is incorrect. An example which fits the equation and contradicts the statements will do.

Which symbol exercise

10 min

There is an exercise and an interactive simulation on the web if you prefer to try it.

Q4: For the following statements insert the correct sign ($\Rightarrow$, $\Leftarrow$ or $\Leftrightarrow$) instead of the questionmarks to make the statements accurate and precise.

a) $\sqrt{4} = 2$?? $2^2 = 4$

b) $a^x = b$?? $\log_a b = x$

c) $a = 2$ and $b = 5$?? ${}^a/_b = {}^2/_5$

d) $n = r$?? $\begin{pmatrix} n \\ r \end{pmatrix} = 1$

e) The inverse function f^{-1} exists ?? The function f is one-to-one and onto.

f) $\{u_n\}$ converges ?? $\sum_{n=1}^{\infty} u_n = 0$

Puzzle on the symbols

15 min

This is a logic type puzzle using the symbols in this section. Try to solve the mystery! If you are unable to do so, discuss it with a friend.

Josie, Dick, Oormi and Glen are in the same class. Each has a unique class identity number of either 1, 2, 3 or 4. Work out their numbers from the following statements but **beware**: one of the statements is untrue.

- Dick has number 1 $\Rightarrow$ Josie has a lower number than Glen
- Josie has a higher number than Oormi $\Leftrightarrow$ Josie has number 3
- Josie has number 2 $\Leftarrow$ Oormi is two numbers lower than Dick

5.3 Definition of a number

Learning Objective

Identify the types of number

Usually in mathematics the first numbers that are encountered are the **natural numbers** ($\mathbb{N}$), i.e. 1, 2, 3, ...

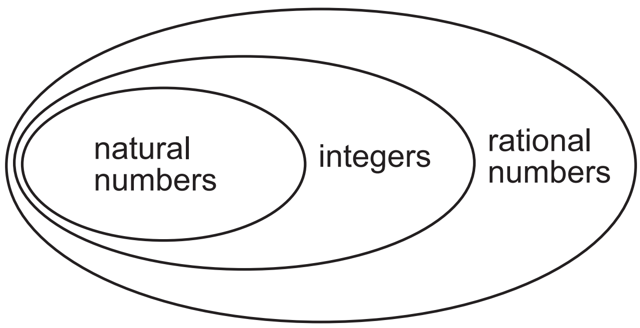
Other numbers are then introduced to cope with various difficulties:

- The **integers** ($\mathbb{Z}$) allow subtractions such as 3 - 5 = -2. These are all the whole numbers both positive and negative and including zero. (Note: the natural numbers are sometimes called the positive integers.)

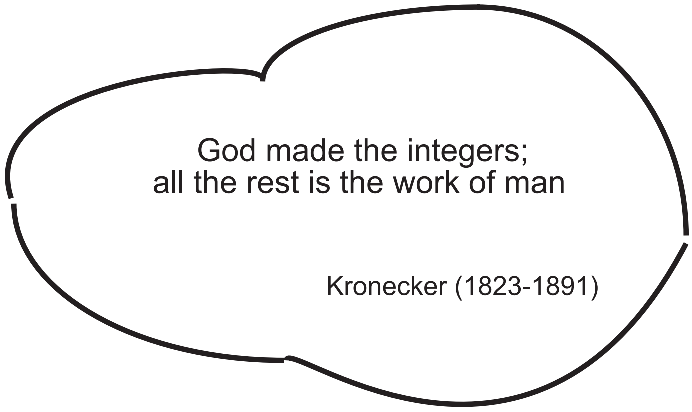
  ($-2 \notin \mathbb{N}$ but $-2 \in \mathbb{Z}$).
- The **rationals** ($\mathbb{Q}$) enable divisions such as $4 \div 6 = {}^2/_3$. The rationals are fractions where the top and bottom are both integers, that is a ratio of two integers. Of course the bottom must be non-zero.

  (${}^2/_3 \notin \mathbb{Z}$ but ${}^2/_3 \in \mathbb{Q}$).

These sets of numbers fit nicely together as shown in the diagram.



Mathematicians over the centuries have argued about the importance or even the existence of such numbers. Here is one view.



In number theory there are two additional types of number which are important. These are **irrational numbers** and **prime numbers**.

An **irrational number** is one which cannot be represented as $^p/_q$ where p and q are integers. It can be expressed as a decimal with an infinite number of decimal places.

Example : Irrational numbers

The following are irrational numbers: $\sqrt{2}$ and e

Q5: Find three more irrational numbers.

Note that it is not always an easy task to judge if a number is irrational.

Prime numbers are natural numbers with a special property.

A **prime number** is a positive integer greater than 1 which has no positive divisors except itself and 1

In other words a prime number has no other factors except itself and 1

Numbers which are not prime are called composite.

The number 1 is neither prime nor composite.

Example : Primes and composites

The number 7 can be divided by 1 and 7 only. The number is prime.

The number 6 can be divided by 1, 2, 3 and 6. It is a composite number.

Imagine trying to list all the prime numbers under 1000. It would be rather time consuming to take each one and attempt to factorise it to find them.

Example 901 factorises to 17×53

These numbers 17 and 53 are both prime but to find this factorisation would take a considerable amount of time even with a calculator.

Sieve of Eratosthenes

There is a shortcut discovered by Eratosthenes, a Greek mathematician, and named after him as 'the sieve of Eratosthenes'. It relies on an interesting property of composite numbers.

'If n is a composite number then n has a factor less than or equal to $\sqrt{n}$'

The proof of this conjecture will follow later.

The sieve works as follows:

- Take the sequence of natural numbers up to n
- Delete all multiples of 2 (but not 2)
- Delete all multiples of 3 (but not 3)
- Delete all multiples of the next available number (5)
- Continue to delete all the multiples of the next available number (but not the number itself)and so on up to the integer after $\sqrt{n}$

At this point all the numbers which remain are primes.

Sieve of Eratosthenes demonstration

There is a web demonstration of the sieve for the first 100 integers.

5 min

Q6: State, with a reason, the number of even primes.

The sieve of Eratosthenes exercise

This activity is also available on the web if you prefer it.

15 min

Q7: Using the grid find all the primes under 100 using the 'sieve of Eratosthenes'.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

What if all the primes up to 10000 were required? Well, this method would work and with not much extra effort. This could be extended to all the primes under 1000000 and so on. There are, in fact, computers set up specifically to find as many primes as possible.

Challenge

It is possible to program a graphics calculator to find the primes under 100 and list them. If time permits, try this using the sieve of Eratosthenes as a starting point for the program.

5.4 Fundamental theorem of arithmetic

Learning Objective

Recall and use the fundamental theorem of arithmetic

Using the sieve of Eratosthenes in the last section should raise some questions about the structure of a composite number. Recall, a composite number is a
positive integer which has factors other than itself and 1

The sieve works by removing each multiple of a prime to leave only the primes. A multiple of a prime is however a composite number.

By looking at the factors of any composite number a very interesting property appears.

Example The composite number 48 has factors of 1, 2, 3, 4, 6, 8, 12, 16, 24 and 48

Taking this a stage further, each of these factors is either a prime or can be factorised.

1 ... leave alone.

2 ... a prime.

3 ... a prime.

4 ... factorises to 1, 2, 2, 4 so 4 = 2×2 ... a product of primes.

6 ... factorises to 1, 2, 3, 6 so 6 = 2×3 ... a product of primes.

8 ... factorises to 1, 2, 4, 8 so 8 = $2 \times 4 = 2 \times 2 \times 2$ ... a product of primes.

Q8: With the remaining factors of 48, namely, 12, 16, 24 and 48, carry out the same process and give each as a product of primes.

It is clear that every factor can be written as a product of primes and this leads to an important theorem.

Every positive integer > 1 is a prime or can be expressed as a unique product of its prime factors.

The proof of this theorem will not be shown in this topic but is available from some of the web links given in the extended information section.

The number 48 expressed as a product of primes is $2 \times 2 \times 2 \times 2 \times 3$ which can be written as $2^4 \times 3$

This is known as the **canonical form** of an integer.

Canonical form example

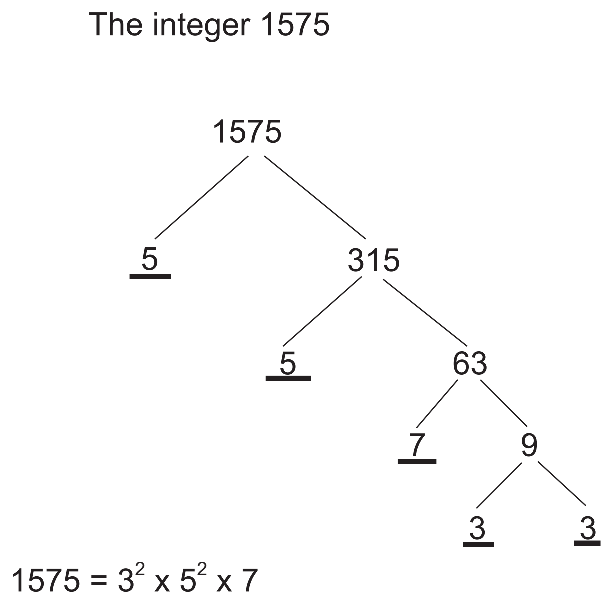
There are examples on the web demonstrating the use of a tree diagram to find the canonical form of an integer.

10 min

Example : Finding the canonical form of an integer

Find the canonical form of the integer 1575

Answer:



There are alternative ways of factorising. The aim is to finish each branch of the tree at a prime number and underline it. It is then simply a case of amalgamating the same prime numbers to give powers.

When an integer is expressed as the product of the powers of primes in ascending order, the integer is in canonical form.

Product of primes and canonical form exercise

Try this exercise now.

There is another of these exercises on the web which you may like to try.

10 min

Q9: Express the following integers as a product of primes:

- 150
- 90
- 140
- 91

Q10: Give the canonical form of the following integers:

- 36
- 105
- 24
- 98

There is no apparent pattern of distribution of primes within the positive integers.

This fact has intrigued many mathematicians and some sub-patterns have emerged.

Twin Primes

Many primes appear in pairs which differ by 2. These are called **twin primes**.

Example (3, 5) (5, 7) (11,13) (17,19) are four pairs of twin primes.

Q11: Find the next five pairs of twin primes.

Goldbach conjecture

This is another famous conjecture (a statement which has still to be proved).

Every even integer greater than 2 is the sum of two primes.

Example The even integer 38 is the sum of 31 and 7, both primes.

Q12: Write the even integers between 11 and 21 as the sum of two primes.

15 min

Fermat and Mersenne primes

If time permits, investigate the **Fermat numbers** and **Mersenne primes**.

Fermat numbers have the form $F_k = 2^{2^k} + 1$

Find the first five Fermat numbers.

Mersenne primes have the form $2^k - 1$ for certain values of $k \in \mathbb{N}$ only.

Find the first three Mersenne primes. (Note that all numbers of the form $2^k - 1$ are known as Mersenne numbers.)

Check the web links in the extended information section for useful sites.

5.5 Methods of proof

Learning Objective

Apply the different methods of proof appropriately

Recall that a **proof** is a logically rigorous and complete argument that a mathematical statement is true.

When confronted with a conjecture and asked to prove it, there are two possible outcomes:

- The conjecture is incorrect; in which case an example to demonstrate that it is incorrect is enough.
- The conjecture is true; in which case a proof is required to show that this is so.

5.5.1 Counter-example

Learning Objective

Use a counter-example when appropriate

If an example can be found to show that a conjecture is untrue, this is called disproving the conjecture by counter-example.

A counter-example is an example that demonstrates that a conjecture is not true.

Example : Counter-example to disprove a conjecture

Disprove the conjecture that if y = 25 - $\frac{1}{x^2}$ then y $>$ 0 for x $\in \mathbb{R}$, 0 $<$ x $\leq$ 4

Answer:

A counter-example is x = $^1/_6$ which gives y = -11 which is not greater than zero. The conjecture is untrue.

Counter-example exercise

10 min

Try this exercise on finding a counter-example

There is a similar exercise on the web if you wish to try it.

Q13: Find a counter-example to disprove the conjecture that x + $x^2 > 0$ for x $\in \mathbb{R}$.

Q14: Disprove the conjecture that all the odd numbers between 2 and 14 are prime by providing a counter-example.

Q15: Disprove the conjecture that if $x^2 \in \mathbb{N}$ then x $\in \mathbb{N}$ by providing a counter-example.

Q16: Disprove the following conjecture: the function f defined by

f (x) = 4x - x^2 is one-to-one where 0 $\leq$ x $\leq$ 4

Extra Help: Disproof - by counterexample

An online exercise is provided to help you if you require additional assistance with this material, or would like to revise this subject.

It is important to realise that finding examples which show the conjecture to be true does **not** constitute a proof except in proof by exhaustion. This is covered in the next section.

5.5.2 Proof by exhaustion

Learning Objective

Apply proof by exhaustion techniques when appropriate.

If it is obvious that a conjecture applies for a limited range of values then it is straightforward to check each possible outcome and determine the truth of the conjecture for each value. This is called proof by exhaustion.

Proof by exhaustion is examining every possible value, case or circumstance to which the conjecture refers and so verifying the conjecture explicity.

Example Prove that $x^2 + 4 \in [4, 20]$ for $x \in \{1, 2, 3, 4\}$

Answer:

Using proof by exhaustion:

- $x = 1$ gives $x^2 + 4 = 5$ and $5 \in [4, 20]$
- $x = 2$ gives $x^2 + 4 = 8$ and $8 \in [4, 20]$
- $x = 3$ gives $x^2 + 4 = 13$ and $13 \in [4, 20]$
- $x = 4$ gives $x^2 + 4 = 20$ and $20 \in [4, 20]$

The conjecture is proved.

Proof by exhaustion exercise

10 min

There is another exercise with different questions on the web if you want to try it.

Q17: Prove the conjecture that there are 2 primes between 20 and 30

Q18: There are two squared integers in the interval [4, 10] Prove this conjecture.

Q19: Prove the conjecture that the sum of any two distinct integers $\in [5, 8]$ lies in the interval [11, 15]

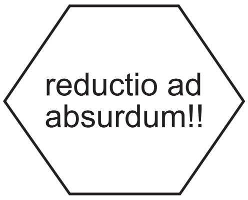
5.5.3 Proof by contradiction

Learning Objective

Use proof by contradiction appropriately

This is a very powerful technique.
Its logical basis is the fact that every mathematical statement is either true or false.

The technique is to assume that the given conjecture is false and, using this, derive a conclusion which **is** known to be false.



This is the Latin name by which such a proof is known. It is thankfully also known as proof by contradiction.

Proof by contradiction assumes the conjecture to be false and shows that this leads to the deduction of a false conclusion.

Examples

1. ‘There is no largest positive integer.’ Prove this conjecture.

Answer:

Assume that the statement is false: that is, assume that there is a largest integer say, P

Since 2 is a positive integer, $\mathbf{P \geq 2}$

But P^2 is a positive integer, thus $P \geq P^2$

Divide by P to give $\mathbf{1 \geq P}$

But $P \geq 2$ and $1 \geq P$ cannot both be true. The statements contradict.

Therefore the assumption that P is the largest integer is false.

The original conjecture that there is no largest positive integer must then be true.

2. ‘$\sqrt{2}$ is not a rational number.’ Prove this conjecture.

Answer:

Assume that $\sqrt{2}$ is rational.

Therefore $\sqrt{2} = {}^{p}/_{q}$ where p and q are positive integers with no common factors. (Otherwise the fraction could be reduced.)

So $2q^2 = p^2 \Rightarrow p^2$ is even.

Now if p is odd, then p^2 is odd.

But here p^2 is even: so **p is even**

and p = 2m for some integer m (this is a proof exercise later).

Thus $2q^2 = 4m^2 \Rightarrow q^2 = 2m^2$

Using the same argument again

$q^2 = 2m^2 \Rightarrow q^2$ is even and so **q is even**.

Thus both p and q are even but both were assumed to have no common factors. This is a contradiction and the assumption that $\sqrt{2}$ is rational is false.

The original conjecture that $\sqrt{2}$ is not a rational number is therefore true.

Note that in the last example, if p and q are two integers with no common factors then p and q are said to be **relatively prime**.

The next example is a very famous theorem which was proved by Euclid around 300BC.

Example 'There are infinitely many primes'. Prove this using proof by contradiction.

Answer:

Suppose that the conjecture is untrue and assume that there are a finite number of primes.

So there is a collection of all primes P_1, P_2, P_3, ..., P_k

Let N = ($P_1 \times P_2 \times ... \times P_k$) + 1

Then N is obviously larger than all P_k

Since P_1, ..., P_k is the collection of all primes then N is composite.

If N is composite then N has a prime divisor Q

But since all the primes P_1, P_2, P_3, ..., P_k leave remainder 1 when dividing N, none of these is equal to Q

This is impossible.

The assumption that there is a finite number of primes is false.

The conjecture that there is an infinite number of primes is true.

Proof by contradiction exercise

20 min

This is a structured exercise in proof by contradiction. Try it now.

There is a more structured exercise on the web if you feel it would help.

Q20: If m^2 is even then m is even. Prove this conjecture using the proof by contradiction method.

Q21: if m^2 = 10 then m is not a rational number. Prove this conjecture using proof by contradiction.

Q22: If n is a composite number then n has a divisor $\leq \sqrt{n}$. Prove this conjecture using proof by contradiction. (This conjecture was stated earlier under the work on the sieve of Eratosthenes.)

5.5.4 Proof by induction

Learning Objective

Apply the techniques of proof by induction when necessary

Proof by induction (or proof by the principle of mathematical induction) is an advanced technique for showing that certain mathematical statements about the positive integers are true.

The proof has two parts: the **basis** and the **inductive step**.

The basis is an explicit check of the result for some positive integer.

The inductive step is as follows:

- Assume the statement to be true for some unspecified value of n, say n = k
- Use this to construct an argument that the statement is true for n = k + 1
- Once this is done the conclusion is that the statement is true for all positive integers $\geq$ the value of the basis step.

It is best described in the following examples.

Examples

1. The summation $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

If n is a positive integer, then $1 + 2 + 3 + ... + n = \frac{1}{2}n(n+1)$

Prove the conjecture by using the principle of mathematical induction.

Answer:

Check an easy value: when n = 1

then $1 = \frac{1}{2} \times 1(1+1) = 1$

Suppose the result is true for n = k then $1 + 2 + 3 + ... + k = \frac{1}{2}k(k+1)$

Consider n = k + 1 then

$$\begin{aligned} 1 + 2 + 3 + ... + k + k + 1 &= \frac{1}{2}k(k+1) + k + 1 \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

The result is true for n = k + 1 if it is true for n = k. But it is true for n = 1 and so by the principle of mathematical induction, the conjecture is true.

2. The Binomial theorem

The theorem states that

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^n$$

Prove the theorem using the principle of mathematical induction.

Answer:

This proof uses two identities which were discussed in topic 1 on Algebra.

These are:

- $\binom{k}{0} = \binom{k+1}{0} = 1$

- $\binom{k}{r-1} + \binom{k}{r} = \binom{k+1}{r}$

Let n =1 then LHS = $(x + y)^n = (x + y)^1 = x + y$

RHS = $\binom{1}{0} x^1y^0 + \binom{1}{1} x^0y^1 = x + y$

So LHS = RHS and the theorem holds for n = 1

Now suppose that the result is true for n = k (where $k \geq 1$)

Then

$$(x + y)^k = \binom{k}{0} x^k + \binom{k}{1} x^{k-1}y + \binom{k}{2} x^{k-2}y^2 + \cdots + \binom{k}{k-1} xy^{k-1} + \binom{k}{k} y^k$$

Consider n = k + 1

$$(x + y)^{k+1} = (x + y)(x + y)^k = x(x + y)^k + y(x + y)^k$$

$$= \binom{k}{0} x^{k+1} + \binom{k}{1} x^ky + \binom{k}{2} x^{k-1}y^2 + \cdots + \binom{k}{k-1} x^2y^{k-1} + \binom{k}{k} xy^k$$
$$+ \binom{k}{0} x^ky + \binom{k}{1} x^{k-1}y^2 + \binom{k}{2} x^{k-2}y^3 + \cdots + \binom{k}{k-1} xy^k + \binom{k}{k} y^{k+1}$$

$$= \binom{k}{0} x^{k+1} + \left[\binom{k}{1} + \binom{k}{0}\right] x^ky + \left[\binom{k}{2} + \binom{k}{1}\right] x^{k-1}y^2 + \cdots$$
$$+ \left[\binom{k}{k-1} + \binom{k}{k-2}\right] x^2y^{k-1} + \left[\binom{k}{k} + \binom{k}{k-1}\right] xy^k + \binom{k}{k} y^{k+1}$$

$$= \binom{k+1}{0} x^{k+1} + \binom{k+1}{1} x^ky + \cdots + \binom{k+1}{k} xy^k + \binom{k+1}{k+1} y^{k+1}$$

Hence if it is true for n = k it is also true for n = k + 1

However it was also true for n = 1

So by the principle of mathematical induction the conjecture is true for all n

3. De Moivre' s theorem

The theorem states that if $z = r(\cos\theta + i\sin\theta)$ then

$z^n = r^n(\cos n\theta + i\sin n\theta)$ for all $n \in \mathbb{N}$.

Answer:

For n = 1

$\{r(\cos\theta + i\sin\theta)\}^1 = r(\cos\theta + i\sin\theta) = r^1(\cos 1\theta + i\sin 1\theta)$.

It is true for n = 1

Now suppose that the result is true for n = k then

$\{r(\cos\theta + i \sin\theta)\}^k = r^k(\cos k\theta + i \sin k\theta)$

Consider n = k + 1 then

$$\begin{aligned}\{r(\cos\theta + i\sin\theta)\}^{k+1} &= \{r(\cos\theta + i\sin\theta)\}\{r(\cos\theta + i\sin\theta)\}^k \\ &= \{r(\cos\theta + i\sin\theta)\}\{r^k(\cos k\theta + i\sin k\theta)\} \\ &= r^{k+1}\{(\cos\theta\cos k\theta - \sin\theta\sin k\theta) + i(\sin\theta\cos k\theta + \cos\theta\sin k\theta)\} \\ &= r^{k+1}\{\cos(\theta + k\theta) + i\sin(\theta + k\theta)\} \\ &= r^{k+1}\{\cos((k+1)\theta) + i\sin((k+1)\theta)\}\end{aligned}$$

So the result is true for n = k + 1 if it is true for n = k. Since it is also true for n = 1, then by the principle of mathematical induction it is true for all $n \in \mathbb{N}$.

Proof by induction exercise

There is a smaller structured exercise on the web if you wish to try it.

20 min

Q23: $n^2 > 2n + 1$ when n is a positive integer greater or equal to 3. Prove this conjecture by the principle of mathematical induction.

Q24: $n < 2^n$ for all $n \in \mathbb{N}$. Prove this conjecture by using induction.

Q25: 8^n is a factor of (4n)! for all $n \in \mathbb{N}$. Prove the result using induction.

Q26: $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} - 1$ for all $n \in \mathbb{N}$.

Q27: By induction, prove that the sum of the squares of the first n positive integers is $\frac{n(n+1)(2n+1)}{6}$

Q28: Prove by induction that $n^2 - 2n - 1 > 0$ for $n \geq 3$

5.6 Summary

At this stage the following ideas and techniques covered in this topic should be known:

- The correct terminology.
- The correct symbolism and notation.
- The different types of numbers and properties of primes and integers.
- Fundamental theorem of arithmetic.
- The methods of proof.

5.7 Extended information

Learning Objective

Display a knowledge of the additional information available on this subject

There are links on the web, which give a selection of interesting web sites to visit. These sites can lead to an advanced study of the topic but there are many areas which will be of passing interest.

Euclid

He was a great mathematician born around 350BC who solved all of his problems with logical reasoning. In his book 'Elements' he devoted chapters 7 to 9 to number theory and some of his results will be explored in the next topic on number theory in unit 3.

Eratosthenes

He lived a century later than Euclid. He worked on prime numbers, giving his name to the method of finding prime numbers, but made major contributions in many fields including music and geography.

Mersenne

He was a major influence in the 17th century with his work on prime numbers. His name is given to a special type of prime (2^p - 1) and to this day no-one is certain how many of these exist.

Fermat

A contemporary of Mersenne, Fermat is probably best known for his 'last theorem' which was proved in 1994. It was also his assertion that a special set of numbers (called Fermat numbers) were all prime, which encouraged Mersenne to consider this further.

Goldbach

He was born in 1690 and worked with Euler on number theory problems. His famous conjecture still remains unproved. He also conjectured that every odd number is the sum of three primes: another result which is still to be proved.

Euler

He lived in the 18th century and made huge contributions in many fields of mathematics. His work on number theory led him to discover that one of the Fermat numbers was not prime.

5.8 Review exercise

Review exercise in elementary number theory

30 min

Try these questions but remember to tackle the print version exercise too.

There are additional questions covering some of the aspects in this topic on the web if you wish to try them.

Q29: Prove by contradiction that if n^3 is odd then n is odd where $n \in \mathbb{Z}$.

Q30: For any integers x, y, w and z, it is conjectured that
$x \geq y$ and $z \geq w \Rightarrow xz \geq yw$. Disprove this conjecture by giving a counter-example.

Q31: Prove by induction that for $x > -1$, $(1 + x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.

Q32: Prove by induction that $n^2 + 3n$ is divisible by 2 for all $n \in \mathbb{N}$.

Q33: Prove that the sum of the first n odd integers = n^2

5.9 Advanced review exercise

Advanced review exercise in elementary number theory

30 min

There are some more questions here to test your skills on the techniques in this topic. You are advised however to try the print exercise as well.

There are additional questions covering some of the aspects in this topic on the web if you wish to try them.

Q34: Prove by induction that $\sum_{r=1}^{n} r = \frac{1}{2}n(n + 1)$

Q35: Prove by induction that $\sum_{r=1}^{n} r^3 = \frac{n^2(n + 1)^2}{4}$ for all $n \in \mathbb{N}$.

5.10 Set review exercise

Set review exercise in elementary number theory

25 min

An on-line assessment is available at this point, which you will need to attempt to have these answer marked. These questions are not randomised on the web. The questions on the web may be posed in a different manner but you should have the required answers in you working.

Q36: Choose a counter-example to disprove the conjecture that all odd integers between 50 and 60 are prime.

- x=53
- x=57
- x=59

Q37: Choose a counter-example to disprove the conjecture that
if $y = 64 - \frac{1}{x^2}$ then $y > 0, x \in \mathbb{R}, 0 < x < 3$

- $\frac{1}{5}$
- $\frac{1}{7}$
- $\frac{1}{9}$

Q38: Prove by contradiction that if n^2 is even then n is even.

Q39: Prove by contradiction that $\sqrt{13} - \sqrt{5} < \sqrt{2}$

Topic 6

End of unit two tests

Contents

Online tests are available on the web. The first two tests are set to provide questions at level ’C’ competency and cover work from the five topics for unit 1.

The third test gives questions at level ’A’ or ’B’. This test does not cover all the topics in unit 1 as the questions for the topic on Algebra include techniques and material from unit 2. These questions will be included in the end of unit 2 tests.

Glossary

adding complex numbers

(a + ib) + (c + id) = (a + c) + i(b + d)

- Add the real parts.
- Add the imaginary parts.

alternating sequence

An alternating sequence is any sequence which has alternate positive and negative terms.

arbitrary constant

An **arbitrary constant** is a constant that occurs in the general solution of a differential equation.

argument of a complex number

The argument θ of a complex number is the angle between the positive x-axis and the line representing the complex number on an Argand diagram. It is denoted arg (z).

arithmetic sequence

An arithmetic sequence is one which takes the form

a, a + d, a + 2d, a + 3d, ... where a is the first term and d is the common difference.

canonical form

When an integer is expressed as the product of the powers of primes in ascending order, the integer is in canonical form.

cartesian equation

The **cartesian equation** of a curve is an equation which links the x-coordinate and the y-coordinate of the general point (x, y) on the curve.

common difference

The common difference in an arithmetic sequence is the difference between any two consecutive terms in the sequence.

common ratio

The common ratio in a geometric sequence is the ratio r = $\frac{u_{n+1}}{u_n}$ of two consecutive terms.

complex number

A **complex number** is a number of the form a + bi where a and b are real numbers and i $= \sqrt{-1}$

The complex number may also be written as a + ib

conjecture

A conjecture is a precise, unambiguous statement for which a convincing argument is needed.

conjugate of a complex number

The **conjugate** of the complex number z = a + ib is denoted by $\bar{z}$

and defined by $\bar{z}$ = a - ib

The conjugate is sometimes denoted by z^*

conjugate roots property

Suppose P (x) is a polynomial with real coefficients. If z = α is a solution of P (x) = 0 then so is z = $\overline{\alpha}$

convergent sequence

An infinite sequence $\{u_n\}$ for which $\lim_{n\to\infty} u_n = k$ is called a **convergent sequence** with limit k

convergent series

A **convergent series** is one for which the limit of partial sums exists. This limit is called the **sum** and is denoted by S_∞ or $\sum_{n=1}^{\infty} u_n$

counter-example

A counter-example is an example that demonstrates that a conjecture is not true.

degree

For a polynomial, the **degree** is the value of the highest power.

de Moivre's theorem

If z = r cos θ + i r sin θ, then $z^n = r^n(\cos n\theta + i \sin n\theta)$ for all $n \in \mathbb{N}$

de Moivre's theorem for fractional powers

$$\{r\,(\cos\theta + i\sin\theta)\}^{p/q} = r^{p/q}\left\{\cos\left(\frac{p}{q}\theta\right) + i\sin\left(\frac{p}{q}\theta\right)\right\}$$

differential equation

A **differential equation** is an equation involving an unknown function and its derivatives.

divergent series

A **divergent series** is one which is not convergent. e.g. 1 + 2 + 3 + ...

equivalent to

Is equivalent to means that the first statement implies and is implied by the second statement. (The first statement is true **if and only if** the second statement is true.) It is denoted by the symbol $\Leftrightarrow$

explicit function

For two variables x and y, y is an **explicit function** of x if it is a clearly defined function of x

exponential decay

Exponential decay occurs when the population decreases at a rate proportional to the size of the population, and we can write

$^{dP}/_{dt} = -kP$

The general solution is $P(t) = Ae^{-kt}$

exponential growth

Exponential growth occurs when the rate of growth of a population is proportional to the size of the population, and we can write

$^{dP}/_{dt} = kP$

The general solution is $P(t) = A\,e^{kt}$

fibonacci sequence

The Fibonacci sequence is formed by taking a first and second term equal to 1. Each subsequent term is formed by adding the two terms immediately before it.

finite sequence

A **finite sequence** is one which has a last term.

fundamental theorem of algebra

Let $P(z) = z^n + a_{n-1}z^{n-1} + ... + a_1z + a_0$ be a polynomial of degree n (with real or complex coefficients).

The **fundamental theorem of algebra** states

that $P(z) = 0$ has n solutions $\alpha_1, ..., \alpha_n$ in the complex numbers

and $P(z) = (z - \alpha_1)(z - \alpha_2)...(z - \alpha_n)$

fundamental theorem of arithmetic

Every positive integer > 1 is a prime or can be expressed as a unique product of its prime factors.

general solution

The **general solution** of a differential equation contains an arbitrary constant and gives infinitely many solutions that all satisfy the differential equation.

geometric sequence

A geometric sequence is one which has the form $a, ar, ar^2, ar^3, ...$ where a is the first term and r is the common ratio.

goldbach conjecture

Every even integer greater than 2 is the sum of two primes.

implicit function

For two variables x and y, y is an **implicit function** of x if it is not an explicit or clearly defined function of x. However, if x is given any value then the corresponding value of y can be found.

implied by

Implied by means that the first statement is a logical consequence of the second statement. It is denoted by the symbol $\Leftarrow$

implies

Implies means that the first statement can be used to logically deduce the next statement. It is denoted by the symbol $\Rightarrow$

improper rational function

Let f (x) be a polynomial of degree n and g (x) be a polynomial of degree m.

When $n \geq m$ then ${}^{f(x)}/_{g(x)}$ is an **improper rational function**.

infinite sequence

An **infinite sequence** is one which continues indefinitely.

initial condition

For a differential equation an **initial condition** is additional information required to determine a particular solution. This could be a coordinate on a curve, a velocity at *t* = 0, the amount of money in a bank account on 1st January 2000, etc.

integrand

For $\int f(x)dx = F(x) + C$, f (x) is the **integrand**

inverse function

Suppose that f is a one-to-one and onto function. For each $y \in B$ (codomain) there is exactly one element $x \in A$ (domain) such that f (x) = y. The **inverse function** is defined as $f^{-1}(y) = x$

irrational number

An **irrational number** is one which cannot be represented as ${}^{p}/_{q}$ where p and q are integers. It can be expressed as a decimal with an infinite number of decimal places.

irreducible quadratic

We say that a quadratic is **irreducible** when it has no real roots. It cannot be factorised.

linear differential equation

A first order differential equation is **linear** if it can be written in the form

${}^{dy}/_{dx} + k(x)\,y = h(x)$

The equation is termed linear as it involves only first order terms in ${}^{dy}/_{dx}$ and y but not terms such as y^2, $y\frac{dy}{dx}$, y^3, cos(y), etc.

modulus r of a complex number

The modulus r of a complex number z = a + ib is written $|z|$ and defined by

$|z| = \sqrt{a^2 + b^2}$

natural logarithm

The **natural logarithm** is the logarithm with base e

The notation we use is $\log_e |x| = \ln |x|$

nth term of a geometric sequence

The nth term of a geometric sequence is given by ar^{n-1} where a is the first term of the sequence, r represents the common ratio and n is the number of the term ($n \in \mathbb{N}$).

nth term of an arithmetic sequence

The nth term of an arithmetic sequence is given by a + (n - 1)d where a is the first term of the sequence, d represents the common difference and n is the number of the term ($n \in \mathbb{N}$).

null sequence

A convergent sequence which converges to the limit 0 is called a **null sequence**.

order

The **order of a differential equation** is that of the highest-order derivative appearing in the equation.

parameter

A **parameter** is a variable that is given a series of arbitrary values in order that a relationship between x and y may be established. The variable may denote time, angle or distance.

parametric equations

Parametric equations are equations that are expressed in terms of an independent variable.
They are of the form

x = f (t)

y = g (t)

where t is the independent variable.

partial sum

The partial sum is the sum of the terms from 1 to n where $n \in \mathbb{N}$ It is denoted by S_n and represented as $S_n = \sum_{r=1}^{n} u_r$

particular solution

The **particular solution** of a differential equation is a solution which is often obtained from the general solution when an initial condition is known.

polar form of a complex number

The polar form of a complex number is $z = r(\cos\theta + i\sin\theta)$ where r is the modulus and θ is the argument.

prime number

A **prime number** is a positive integer greater than 1 which has no positive divisors except itself and 1

principal value of an argument

The principal value of an argument is the value which lies between $-\pi$ and π

proof

A **proof** is a logically rigorous and complete argument that a mathematical statement is true.

proof by contradiction

Proof by contradiction assumes the conjecture to be false and shows that this leads to the deduction of a false conclusion.

proof by exhaustion

Proof by exhaustion is examining every possible value, case or circumstance to which the conjecture refers and so verifying the conjecture explicity.

rational function

When f (x) and g (x) are polynomials then $^{f(x)}/_{g(x)}$ is called a **rational function**.

roots of unity

The nth roots of unity are those numbers that satisfy the equation $z^n = 1$

rule for dividing complex numbers

- Find the conjugate of the denominator.
- Multiply the complex fraction, both top and bottom, by this conjugate to give an integer on the denominator.
- Express the answer in the form a + bi

$$\frac{a + bi}{c + di} = \frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + i(bc - ad)}{(c^2 + d^2)}$$
$$= \frac{(ac + bd)}{(c^2 + d^2)} + \frac{i(bc - ad)}{(c^2 + d^2)}$$

rule for dividing two complex numbers in polar form

- Divide the moduli.
- Subtract the arguments.

$$\frac{r_1\cos\theta_1 + i\, r_1\sin\theta_1}{r_2\cos\theta_2 + i\, r_2\sin\theta_2} = \frac{r_1}{r_2}\{\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\}$$

rule for multiplying complex numbers

- Use the technique for multiplying out two brackets.
  (a + ib)(c + id) = (ac - bd) + i(bc + ad)

rule for multiplying two complex numbers in polar form

- Multiply the moduli.
- Add the arguments.

$(r_1 \cos \theta_1 + i r_1 \sin \theta_1)(r_2 \cos \theta_2 + i r_2 \sin \theta_2) = r_1 r_2 \{ \cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2) \}$

rule for subtracting complex numbers

(a + ib) - (c + id) = (a - c) + i(b - d)

- Subtract the real parts.
- Subtract the imaginary parts.

second derivative

The **second derivative** of y with respect to x is obtained by differentiating twice.

We write $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

sequence

A **sequence** is an ordered list of terms.

series

A series is the sum of the terms in an infinite sequence.

set of complex numbers

The set of complex numbers is written as $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$

sum to n terms of a geometric series

The sum to n terms (the nth partial sum) of a geometric series is given by $S_n = \frac{a(1 - r^n)}{1 - r}$ where a is the first term of the sequence, r ($\neq$ 1) represents the common ratio and $n \in \mathbb{N}$.

sum to n terms of an arithmetic series

The sum to n terms (the nth partial sum) of an arithmetic series is given by $S_n = \frac{n}{2}[2a + (n - 1)d]$ where a is the first term of the sequence, d represents the common difference and $n \in \mathbb{N}$.

term

Each number in a sequence is called a **term** or an element.

The nth term (or general term) is often denoted by u_n

theorem

A **theorem** is a mathematical conjecture which has been proved.

triangle Inequality

If z and w are complex numbers then $|z + w| \leqslant |z| + |w|$

triangular number sequence

This sequence comprises the natural numbers which can be drawn as dots in a triangular shape.

Hints for activities

Topic 2: Further Integration

Challenge Question 1

Hint 1: Complete the square in the denominator.

Hint 2: The substitution u = x + 1 may help.

Hint 3: Write the integrand as a multiple of $\frac{2u}{u^2 + a^2}$ plus a multiple of $\frac{1}{u^2 + a^2}$

Challenge Question 2

Hint 1:

Rewrite $\frac{1}{x(x^2 + x + 1)}$ in the form $\frac{A}{x} + \frac{Bx + C}{x^2 + x + 1}$

Hint 2: $\int \frac{x + 1}{x^2 + x + 1}dx = \int \frac{x}{x^2 + x + 1}dx + \int \frac{1}{x^2 + x + 1}dx$

Hint 3:

The integral $\int \frac{x}{x^2 + x + 1}dx$ may cause you consternation.

This may help

$$\int \frac{x}{x^2 + x + 1}dx = \frac{1}{2}\int \frac{(2x + 1)}{x^2 + x + 1}dx - \frac{1}{2}\int \frac{1}{x^2 + x + 1}dx$$

Hint 4:

$\int \frac{1}{x^2 + x + 1}dx$ may also cause you to scratch your head. Remember that you can complete the square and then the integral will be a $\tan^{-1}$ function.

Challenge Question 3

Hint 1: Use the same method as for $\int \ln x \, dx$

ie. Let $f = \sin^{-1}x$ and $g' = 1$

Hint 2: You might have difficulty with $\int \frac{x}{\sqrt{1 - x^2}}dx$

Try this, $\int \frac{x}{\sqrt{1 - x^2}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{1 - x^2}}dx$

It is now possible to solve this by substituting u for x^2

Answers to questions and activities

1 Further Differentiation

Revision questions (page 3)

Q1: $3b^2$

Q2: $\frac{1}{2\sqrt{y}}$

Q3: 2x

Q4: There is a connection.

For $x \geq 0$, x and y are inverse functions

$^{dx}/_{dy} = {}^1/_2\ y^{-1/2}$ and $^{dy}/_{dx} = 2x$

$$= \frac{1}{2\sqrt{y}}$$

$$= \frac{1}{2x}$$

$$= \frac{1}{^{dy}/_{dx}}$$

Exercise 1 (page 6)

Q5:

a) $^d/_{dx}(f^{-1}(x)) = {}^1/_{3x}$

b) $^d/_{dx}\ (f^{-1}\ (x)) = {}^1/_2 \exp\ (^x/_2 - 3)$

c) $^d/_{dx}\ (f^{-1}\ (x)) = {}^1/_3 \exp\ (x)$

Q6:

$\frac{dy}{dx} = \frac{1}{2x\sqrt{\ln(x)}}$

Q7:

a) $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{3y^2 + 3}$

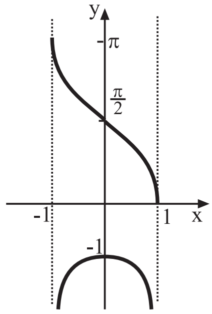
b) x - 15y + 20 = 0

Q8:

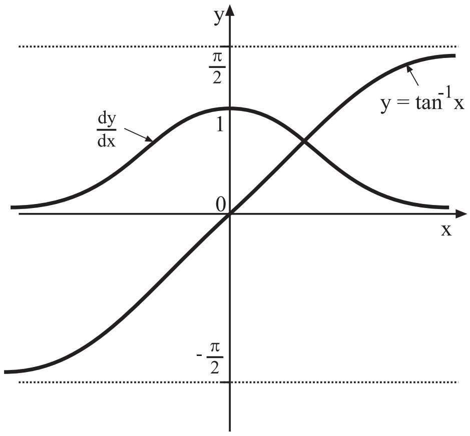
1. $\frac{d}{dx}(f^{-1}(x)) = \frac{1}{5 + 3\cos(3y)}$
2. x - 8y = 0

Answers from page 7.

Q9:



Q10:



Exercise 2 (page 8)

Q11: $\frac{dy}{dx} = \frac{5}{25 + x^2}$

Q12: $\frac{dy}{dx} = \frac{-4}{\sqrt{1 - 16x^2}}$

Q13: $\frac{dy}{dx} = \frac{3}{1 + 9x^2}$

Q14: $\frac{dy}{dx} = \frac{-1}{\sqrt{x(1 - x)}}$

Q15: $\frac{dy}{dx} = \frac{3}{9x^2 + 6x + 2}$

Q16: $\frac{dy}{dx} = \frac{2}{\sqrt{2 - x^2}}$

Q17: $\frac{dy}{dx} = \frac{x}{1 + x^2} + \tan^{-1}x$

Q18: $\frac{dy}{dx} = \frac{\exp(x)}{1 + \exp(2x)}$

Q19: $\frac{dy}{dx} = \frac{-3\sin x}{1 + 9\cos^2 x}$

Q20: $\frac{dy}{dx} = \frac{1}{2x\sqrt{x - 1}}$

Q21: $\frac{dy}{dx} = \frac{1}{x\sqrt{1 - (\ln 3x)^2}}$

Q22: $\frac{dy}{dx} = \frac{1 - 2x\tan^{-1}x}{(1 + x^2)^2}$

Answers from page 9.

Q23: $y = \frac{4x^3 - x^2}{x - 1}, \quad x \neq 1$

Exercise 3 (page 10)

Q24: Implicit

Q25: Explicit, $y = {}^{3x}/_{x+2}$

Q26: Explicit, $y = e^{-x} \ln x$

Q27: Implicit

Q28: Implicit

Q29: Explicit,$y = \frac{\sin 2x}{x^2}$

Exercise 4 (page 12)

Q30: ${}^{dy}/_{dx} = {}^{x}/_{y}$

Q31: ${}^{dy}/_{dx} = {}^{5 - 6x}/_{2y}$

Q32: $\frac{dy}{dx} = \frac{5x - 4}{3y + 1}$

Q33: $\frac{dy}{dx} = \frac{2(4 - 3x)}{5 - 6y}$

Q34: $\frac{dy}{dx} = -\frac{2x + y}{x + 2y}$

Q35: $\frac{dy}{dx} = \frac{4 - 5y}{6 + 5x}$

Q36: $\frac{dy}{dx} = \frac{3(y - x^2)}{(20y - 3x)}$

Q37: $\frac{dy}{dx} = -\frac{2x + 4xy + 5y^2}{2x^2 + 10xy - 6y - 6}$

Q38: $\frac{dy}{dx} = \frac{6x^2}{3 + \sin y}$

Q39: $\frac{dy}{dx} = \frac{2 - \tan y}{3 + x\sec^2 y}$

Q40: $\frac{dy}{dx} = \frac{\sin x\, e^{\cos x}}{\cos y\, e^{\sin y}}$

Q41: $\frac{dy}{dx} = \frac{-3\sin(3x + 2y)}{8y + 2\sin(3x + 2y)}$

Answers from page 14.

Q42: When $y = \cos^{-1}(3x)$

then $\cos y = 3x$

Using implicit differentiation we obtain

$$\frac{d}{dy}(\cos(y))\frac{dy}{dx} = \frac{d}{dx}(3x)$$

$$-\sin(y)\frac{dy}{dx} = 3$$

$$\frac{dy}{dx} = \frac{-3}{\sin(y)}$$

Now we have $\cos(y) = 3x$

Therefore $\cos^2(y) = 9x^2$

and so $\sin^2(y) = 1 - \cos^2(y)$

$= 1 - 9x^2$

$\sin(y) = \sqrt{1 - 9x^2}$

Therefore when $y = \cos^{-1}(3x)$

then $\frac{dy}{dx} = \frac{-3}{\sin(y)}$

$$= \frac{-3}{\sqrt{1 - 9x^2}}$$

Q43:

When $y = \sin^{-1}\left(\frac{3x^2}{2}\right)$

then $\sin(y) = \frac{3x^2}{2}$

Using implicit differentiation we obtain

$$\frac{d}{dy}(\sin(y))\frac{dy}{dx} = \frac{d}{dx}\left(\frac{3x^2}{2}\right)$$

$$\cos(y)\frac{dy}{dx} = 3x$$

$$\frac{dy}{dx} = \frac{3x}{\cos(y)}$$

Now since $\sin(y) = \frac{3x^2}{2}$

then $\sin^2(y) = \frac{9x^4}{4}$

and $\cos^2(y) = 1 - \sin^2(y)$

$= 1 - \frac{9x^4}{4}$

$= \frac{4 - 9x^4}{4}$

$\cos(y) = \sqrt{\frac{4 - 9x^4}{4}}$

Therefore when $y = \sin^{-1}\left(\frac{3x^2}{2}\right)$

then $\frac{dy}{dx} = \frac{3x}{\cos(y)} = \frac{6x}{\sqrt{4 - 9x^4}}$

Exercise 5 (page 15)

Q44:

a) ${}^{dy}/_{dx} = {}^{6}/_{5}$ at (-2, 1)
b) $6x - 5y + 17 = 0$

Q45:

a) ${}^{dy}/_{dx} = -4$ at (1, 4)
b) $4x + y - 8 = 0$

Q46:

a) ${}^{dy}/_{dx} = 1$
b) $x - y + \pi - 1 = 0$

Q47:

a) ${}^{dy}/_{dx} = -3$ at (2, 2)
b) $3x + y - 8 = 0$

Q48:

a) ${}^{dy}/_{dx} = -{}^{11}/_{6}$ at (1, -4)
b) $11x + 6y + 13 = 0$

Q49:

a) ${}^{dy}/_{dx} = 2$

b) $2x - y - 4 = 0$

Q50:

a)

$$\frac{d}{dx}(x^{2/3} + y^{2/3}) = \frac{d}{dx}(2^{2/3})$$

$$\tfrac{2}{3}x^{-1/3} + \tfrac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$x^{-1/3} + y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}}$$

$$= -\left(\frac{y}{x}\right)^{1/3}$$

At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\frac{dy}{dx} = -1^{1/3} = -1$

b) At $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\frac{dy}{dx} = -1^{1/3} = -1$ and the points are (0, 2), (0, -2), (2, 0) and (-2, 0)

Exercise 6 (page 18)

Q51: ${}^{dy}/_{dx} = {}^{3x}/_{4y}$

$$\frac{d^2y}{dx^2} = \frac{3(4y^2 - 3x^2)}{16y^3}$$

Q52:

${}^{dy}/_{dx} = {}^{x+1}/_{y}$

$$\frac{d^2y}{dx^2} = \frac{y^2 - (x+1)^2}{y^3}$$

Q53:

${}^{dy}/_{dx} = -1$

$$\frac{d^2y}{dx^2} = 0$$

Q54:

${}^{dy}/_{dx} = {}^{-y}/_{x + 2y}$

$$\frac{d^2y}{dx^2} = \frac{2y\,(x+y)}{(x+2y)^3}$$

Exercise 7 (page 19)

Q55: ${}^{dy}/_{dx} = 4^x \ln 4$

Q56: $\frac{dy}{dx} = 3^{x^2} x\,(\ln 9)$

Q57: ${}^{dy}/_{dx} = 2x^x (\ln x + 1)$

Q58: $\frac{dy}{dx} = \frac{(6x + 5)(2x + 3)}{2(x + 1)^{3/2}}$

Q59: $\frac{dy}{dx} = \frac{2^x(-2 + \ln 2 + x\ln 4)}{(2x + 1)^2}$

Q60: $\frac{dy}{dx} = \frac{3}{(3 + x)^{1/2}(3 - x)^{3/2}}$

Q61: ${}^{dy}/_{dx} = 9x^2 - 20x - 23$

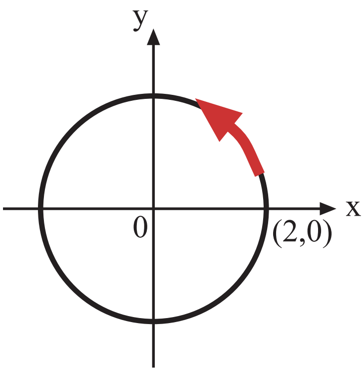
Q62: $\frac{dy}{dx} = \frac{x(21x^2 + 29x - 12)}{2(7x - 3)^{1/2}(1 + x)^2}$

Exercise 8 (page 22)

Q63:

a) $x^2 + y^2 = 4$

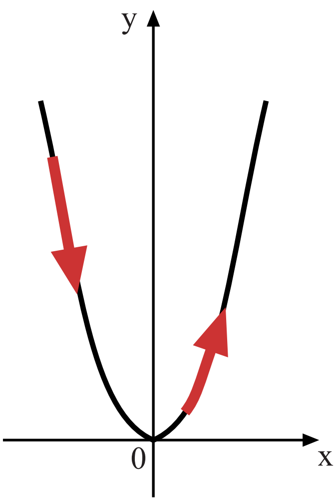
b)



The circle starts at (2,0) and travels round anticlockwise back to the start again.

Q64:

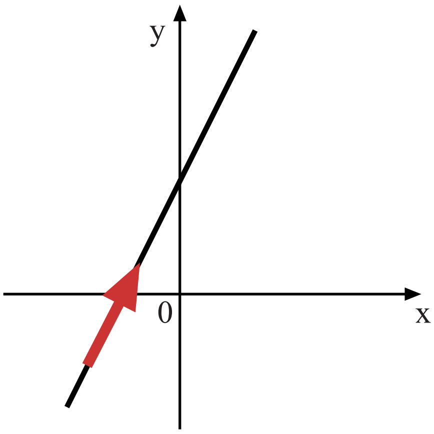
1. $y = x^2$
2.



Q65:

a) $y = 2x + 3$

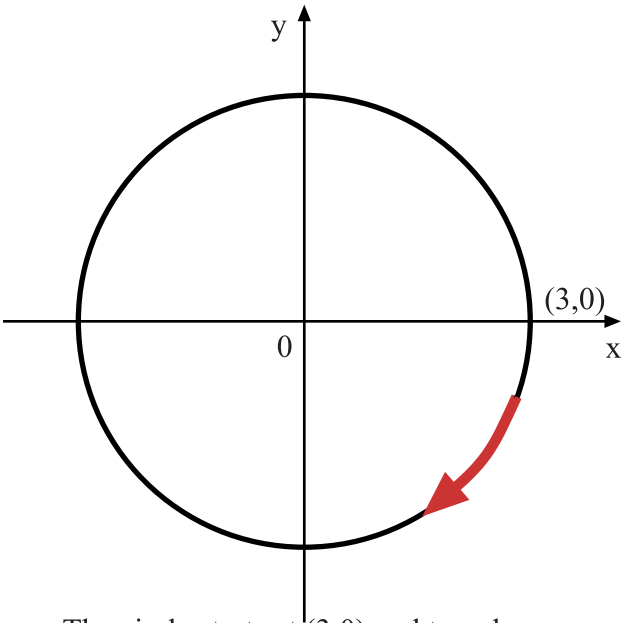
b)



Q66:

a) $x^2 + y^2 = 9$

b)

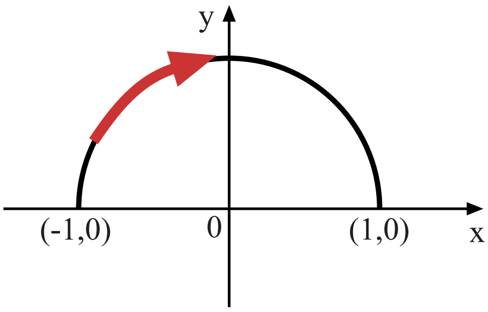


The circle starts at (3,0) and travels clockwise round to the start again.

Q67:

a) $x^2 + y^2 = 1$ and $y \geq 0$

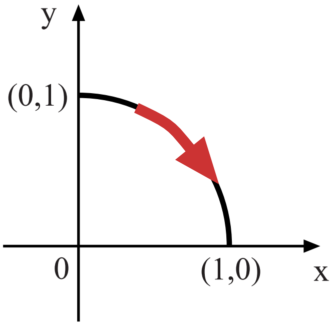
b)



The curve starts at (-1,0) and finishes at (1,0)

Q68:

a) $x^2 + y^2 = 1$, $x \geq 0$ and $y \geq 0$

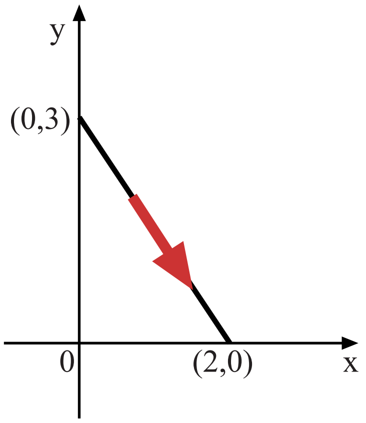
b) 

The curve starts at (0,1)
and ends at (1,0)

Q69:

a) $3x + 2y = 6$, $x \geq 0$ and $y \geq 0$

b)

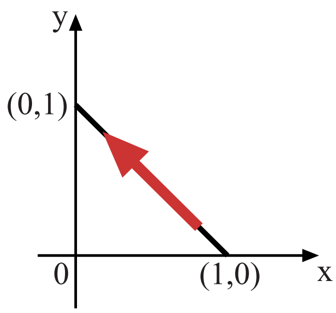


The line starts at (0,3) and ends at (2,0)

Q70:

a) $x = y = 1$, $x \geq 0$ and $y \geq 0$

b)



The line starts at (1,0) and ends at (0,1)

Exercise 9 (page 24)

Q71: $^{dy}/_{dx} = {}^{1}/_{4}t$

Q72: $^{dy}/_{dx} = -\cot t$

Q73: $^{dy}/_{dx} = {}^{1}/_{2}t$

Q74: $\frac{dy}{dx} = 2\left(\frac{t+1}{t-1}\right)^2$

Q75: $\frac{dy}{dx} = \frac{1}{t^2} = \frac{9}{(4-x)^2}$

Q76: $\frac{dy}{dx} = 4t^{3/2} = 4x^3$

Exercise 10 (page 25)

Q77:

a) 1 metre

b)

1. 4 m/s
2. (2t + 3) m/s

c)

i 4 m/s

ii 7 m/s

iii 8.06 m/s

iv

i $^{(2t + 3)}/_{4}$

ii 60.3°

Q78: a) $\sqrt{17}$ **b)** 149°

Exercise 11 (page 27)

Q79: y = -16x + 12

Q80: y = -3x + 10

Q81: $3x + 5y = 15\sqrt{2}$

Q82: y = x - 1

Q83: (16, -4)

Exercise 12 (page 28)

Q84: $\frac{d^2y}{dx^2} = \frac{3}{t^5}$

Q85: $\frac{d^2y}{dx^2} = \frac{3}{4(t+2)}$

Q86: $\frac{d^2y}{dx^2} = -\frac{1}{3\sin^3 t}$

Q87: $\frac{d^2y}{dx^2} = -\frac{3(3t^2 + 2t + 1)}{4(3t + 1)^3}$

Q88: $^{dy}/_{dx} = 3$

Exercise 13 (page 29)

Q89: 12π

Q90: 32π

Q91: t + 6 cos 3t

Q92: 6h

Q93: 3

Q94: $6(3t - 2)^2$

Exercise 14 (page 31)

Q95: -12

Q96: **a)** -6 **b)** 123.7°

Q97: 0

Q98: $^{-3}/_{11}$

Exercise 15 (page 33)

Q99: $^{dV}/_{dt} = 4\pi r^{2} {}^{dr}/_{dt}$

Q100: $^{dS}/_{dt} = 12x {}^{dx}/_{dt}$

Q101: $a = {}^{dv}/_{dt} = (6s^2 + 5)(2s^3 + 5s)$

Q102: **a)** 32π cm^3/s. **b)** 16π cm^2/s.

Q103: $-2\sqrt{3}$ cm/s

Q104: $\frac{7}{4\pi}$ cm/s

Q105: $\frac{dh}{dt} = -\frac{2000}{\pi r^2}$. The water level is dropping at a constant rate of $\frac{2000}{\pi r^2}$ cm/s.

Exercise 16 (page 35)

Q106: $\frac{2250}{\pi}$cm/minute

Q107: 1.5 cm/s

Q108: -1.25ft/second

Q109: 2 m/s

Q110: $18\sqrt{3}$ m/second

Q111: $^{4}/_{3}$ cm^2/ hr

Review exercise in further differentiation (page 39)

Q112: $\frac{-3x^2}{\sqrt{1 - x^6}}$

Q113: $^{dy}/_{dx} = {}^{3x}/_{y}$

Q114: $^{dy}/_{dx} = {}^{3}/_{4}\ t$

Advanced review exercise in further differentiation (page 39)

Q115: $^{1}/_{6}$

Q116: $^{-2}/_{3}$

Q117: $\frac{dy}{dx} = -2\cos 4t, \quad \frac{d^2y}{dx^2} = -2$

Q118: **a)** 6 m/s **b)** $^{3}/_{2}$

Set review exercise in further differentiation (page 39)

Q119: This answer is only available on the web.

Q120: This answer is only available on the web.

Q121: This answer is only available on the web.

2 Further Integration

Revision - Exercise 1 (page 43)

Q1: $(x - 1)(x + 5)(x + 7)$

Q2: $\frac{3}{(x - 3)} - \frac{1}{(x + 5)}$

Q3: $\frac{9}{(x + 2)(x - 1)^2} = \frac{1}{x + 2} - \frac{1}{x - 1} + \frac{3}{(x - 1)^2}$

Q4: $4 \ln (3x + 2) + C$

Q5: $-{}^1/_{x - 1} + C$

Q6: $3 \ln 5$

Exercise 2 (page 45)

Q7:

a) $\sin^{-1}\left(\frac{x}{3}\right) + C$

b) $\frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C$

c) $\frac{1}{2}\tan^{-1}\left(\frac{2x}{5}\right) + C$

d) $\frac{4}{3}\sin^{-1}\left(\frac{3x}{4}\right) + C$

e) $\frac{3\sqrt{5}}{5}\tan^{-1}\left(\frac{\sqrt{5}x}{2}\right) + C$

Q8: **a)** $\pi/2$ **b)** $\pi/3$ **c)** $\pi/3$ **d)** $\pi/16$ **e)** $\pi/12$ **f)** $\sqrt{3}\pi/4$

Exercise 3 (page 47)

Q9:

a) $\ln|x^2 + 9| + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

b) $\frac{3}{2}\ln|x^2 + 16| - \frac{1}{2}\tan^{-1}\left(\frac{x}{4}\right) + C$

c) $\frac{1}{3}\tan^{-1}\left(\frac{x + 1}{3}\right) + C$

d) $\frac{7}{2}\tan^{-1}\left(\frac{x - 3}{2}\right) +$

Q10:

a) $\pi/4$

b) $\frac{1}{2}\ln 2 - \frac{\pi}{4} \approx -0.439$

Challenge Question 1 (page 47)

$\frac{1}{2}\ln(x^2 + 2x + 5) - \frac{3}{2}\tan^{-1}\left(\frac{x + 1}{2}\right) + C$

Exercise 4 (page 49)

Q11:

a) $2 \ln|x - 4| - 2 \ln|x + 1| + C$

b) $2 \ln|x - 2| + \ln|x - 3| + C$

c) $2 \ln|x| - 4 \ln|x + 3| + C$

d) $\ln|x - 1| + \ln|x + 2| - 2 \ln|2x + 1| + C$

Q12: **a)** $(2x - 1)(x + 4)$

b) $\ln|2x - 1| - \ln|x + 4| + C$

Q13: **a)** $(x - 2)(x - 3)(x + 3)$

b) $\ln|x - 2| - 2 \ln|x - 3| + \ln|x + 3| + C$

Exercise 5 (page 51)

Q14:

a) $2 \ln|x| + \ln|x - 1| - \frac{5}{(x - 1)} + C$

b) $\ln|x| - \ln|x - 1| - \frac{2}{x} + C$

c) $3 \ln|x - 1| - 3 \ln|x - 2| - \frac{3}{x - 2} + C$

d) $\frac{3}{2}\ln|2x - 1| + \ln|x - 3| - \frac{12}{x - 3} + C$

Answers from page 52.

Q15:

a) $2 \ln|x + 1| - \ln|x^2 + 4| + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C$

b) $2 \ln|x - 2| - \ln|x^2 + 9| + \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

c) $\ln|x + 1| + \ln|x^2 + 16| - \frac{3}{2}\tan^{-1}\left(\frac{x}{4}\right) + C$

d) $-\ln|x - 3| + \frac{1}{2}\ln|x^2 + 3| + \frac{5}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right) + c$

Challenge Question 2 (page 52)

$\ln|x| - \frac{1}{2}\ln|x^2 + x + 1| - \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x + 1}{\sqrt{3}}\right)$

Exercise 7 (page 54)

Q16:

a) $x + 2 + \frac{3x + 4}{x^2 + x - 6}$

b) $\frac{1}{2}x^2 + 2x + \ln|x + 3| + 2 \ln|x - 2| + C$

Q17:

a) $x - 1 + \frac{9}{x^3 + 3x^2 - 4}$

b) *x - 1)(x + 2)^2

c) $\frac{1}{2}x^2 - x + \frac{3}{x+2} + \ln|x - 1| - \ln|x + 2| + C$

Q18:

a) $3 + \frac{1 - 7x}{x^3 - 2x^2 + 9x - 18}$

b) $(x - 2)(x^2 + 9)$

c) $3x - \ln|x - 2| + \frac{1}{2}\ln(x^2 + 9) - \frac{5}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

Exercise 8 (page 56)

Q19: $-x \cos x + \sin x + C$

Q20: $e^x(x - 1) + C$

Q21: $\frac{1}{80}(2x + 3)^4(8x - 3) + C$

Q22: $\frac{1}{9}\exp(3x)[3x - 1] + C$

Q23: $\frac{x}{2}\sin(2x + 1) + \frac{1}{4}\cos(2x + 1) + C$

Q24: $-\frac{1}{2}(2x - 3)\cos 2x + \frac{1}{2}\sin 2x + C$

Q25: $\frac{1}{4}x^2(2\ln x - 1) + C$

Q26: $2^1/_4$

Q27: $-^9/_{10}$

Q28: 3

Q29: $5\exp(2) - 3$

Q30: $\frac{1}{16}(3 - 2\ln 2)$

Q31: $2^1/_4$

Q32: $-^9/_{10}$

Q33: 3

Q34: $5\exp(2) - 3$

Q35: $\frac{1}{16}(3 - 2\ln 2)$

Exercise 9 (page 58)

Q36: $e^x (x^2 - 2x + 2) + C$

Q37: $\frac{(9x^2 - 2)}{27}(\sin 3x) + \frac{2x}{9}(\cos 3x) + C$

Q38: $\frac{-1 - 4x - 8x^2}{32e^{4x}} + C$

Q39: $e^x (x^3 - 3x^2 + 6x - 6) + C$

Q40: $2 (e^2 - 1) \approx 12.8$

Q41: $\frac{1}{27}(\pi - 2) \approx 0.0423$

Exercise 10 (page 60)

Q42: $\frac{e^{2x}}{5}(\sin x + 2 \cos x) + C$

Q43: $\frac{e^{3x}}{13}(3 \sin 2x - 2 \cos 2x) + C$

Q44: $^1/_2 (\ln x)^2 + C$

Q45: $\frac{e^x}{5}[2\cos(1 - 2x) + \sin(1 - 2x)] + C$

Q46: $x \tan^{-1}x - \frac{1}{2}\ln (x^2 + 1) + C$

Q47: $x \tan^{-1}(3x) - \frac{1}{6}\ln (9x^2 + 1) + C$

Q48: $x \ln (x^2) - 2x + C$

Q49: $x [(\ln x)^2 - 2 \ln x + 2] + C$

Challenge Question 3 (page 60)

$x \sin^{-1} x + \sqrt{(1 - x^2)} + C$

Exercise 11 (page 63)

Q50: $y = 2x^2 + {}^3/_x$

Q51: $18e^6 \approx 7262$ bacteria

Q52: 1 metre.

Q53: 32 m/sec

Exercise 12 (page 65)

Q54:

a) $y = \pm \sqrt{(2e^x + C)}$

b) $y = A\,e^{3x}$

c) $y = \pm\sqrt{(C - \cos x)}$

d) $y = \tan(2x^2 + C)$

e) $y = Ae^{\left(\frac{1}{2}x^2 - x\right)}$

f) $y = \sin^{-1}\left(\frac{1}{3}x^3 + C\right)$

g) $y = A(x + 1) + 1$

h) $y = A\sqrt{(1 + x^2)}$

Q55:

a) $y = e^{x^2}$

b) $y = \ln(2e^{\frac{1}{2}x} - 1)$

c) $y = \frac{2}{\ln|2x - 1| - 1}$

d) $y = \sin\left(\ln|x| + \frac{\pi}{6}\right)$

e) $y = \frac{1}{4}(x + 3)^2 - 1$

Q56:

a) $y = \frac{1 + Ae^{x^2/2}}{1 - Ae^{x^2/2}}$

b) $y = \exp(\sin x - x\cos x - 1)$

c) $y = -{}^{x+2}/_{x}$

Exercise 13 (page 70)

Q57: ${}^{dB}/_{dt}$ - kB, B_0e^{kt}, $B_0 \approx 326$

Q58: $Q = Q_0e^{-kt}$

Q59: ${}^{dP}/_{dH} = -kP$, $P = P_0e^{-kH}$, 49.7 millibars

Q60: 325 days

Exercise 14 (page 75)

Q61: 18.1 minutes longer

Q62: 60°C

Q63: 20 weeks

Q64:

a) $h = \left(\frac{c}{2} - \frac{t}{20}\right)^2$

b) C = 4, and therefore $h = \left(2 - \frac{t}{20}\right)^2$

c) 60 minutes

Q65:

a)

$n = \frac{Ne^{Nkt}}{N - 1 + e^{Nkt}}$

b)

When t = 100 then $n = {}^{1}/_{2}\ N$

Therefore $n = \frac{Ne^{Nkt}}{N - 1 + e^{Nkt}}$

becomes $\frac{1}{2}N = \frac{Ne^{100Nk}}{N - 1 + e^{100Nk}}$

$$2e^{100Nk} = N - 1 + e^{100Nk}$$

$$e^{100Nk} = N - 1$$

$$100Nk = \ln(N - 1)$$

$$k = \{\ln(N - 1)\}/100N$$

c) N

Review exercise in further integration (page 76)

Q66: $4 \ln|x - 4| - \ln|x| + C$

Q67: π

Q68: $y = Ae^{\frac{1}{3}x^3}$

Advanced review exercise in further integration (page 77)

Q69:

a) ${}^{1}/_{x - 2} - {}^{1}/_{x + 2}$

b) $x + \ln|x - 2| - \ln|x + 2| + C$

Q70: $7^{11}/_{15}$

Q71:

a) $c(x) = (x - 2)(x^2 + x + 1)$

b) The discriminant of $x^2 + x + 1$ is -3 which is < 0 so there are no real factors.

c) A = 2, B = -2, C = -1

$$\int \frac{5x + 4}{(x - 2)(x^2 + x + 1)}dx = \ln(x - 2)^2 - \ln(x^2 + x + 1) + C$$

Q72:

1.

$$\frac{d}{dx}\left(\sqrt{1 - x^2}\right) = \frac{-x}{\sqrt{1 - x^2}}$$

2.

$\int \frac{x}{\sqrt{1 - x^2}} dx = -\sqrt{1 - x^2} + C$

$\int 1\, f(x)\, dx = f(x) \int 1 dx - \int f'(x) dx$

$= x\, f(x) - \int x\, f'(x) dx$

3.

$\frac{1}{6}\left(\pi\sqrt{3} - 3\right) \approx 0.407$

Q73:

i

$\frac{-60}{30 - Q} + \frac{60}{15 - Q}$

ii A = 6-, C = 60 ln2 = 41.59 (to 2 decimal places)

- i 13.39 minutes (2 d.p.)
- ii 10.36 grams (2 d.p.)

Set review exercise in further integration (page 79)

Q74: This answer is only available on the web.

Q75: This answer is only available on the web.

Q76: This answer is only available on the web.

3 Complex numbers

Revision exercise (page 83)

Q1: The two roots are $x = {}^1/_3$ and $x = 4$

Q2: $2x^2 - x - 6$

Q3: $16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

Q4: $\cos(a + \pi) = \cos a \cos \pi - \sin a \sin \pi$

But $\sin \pi = 0$ and $\cos \pi = -1$

Hence $\cos(a + \pi) = -\cos a$

Q5: $\frac{(3\sqrt{48} - \sqrt{27})}{(2\sqrt{12} + \sqrt{75})} = \frac{12\sqrt{3} - 3\sqrt{3}}{4\sqrt{3} + 5\sqrt{3}} = 1$

Answers from page 84.

Q6: $\pm 4i$

Q7: $\pm 8i$

Q8: $\pm i2\sqrt{2}$

Q9: There are infinite possibilities here. Check the discriminant ($b^2 - 4ac$) of your equation - if it is < 0 your equation has no real roots.

Answers from page 85.

Q10: $1 \pm 2i$

Q11: $-1 \pm i\sqrt{5}$

Q12: $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

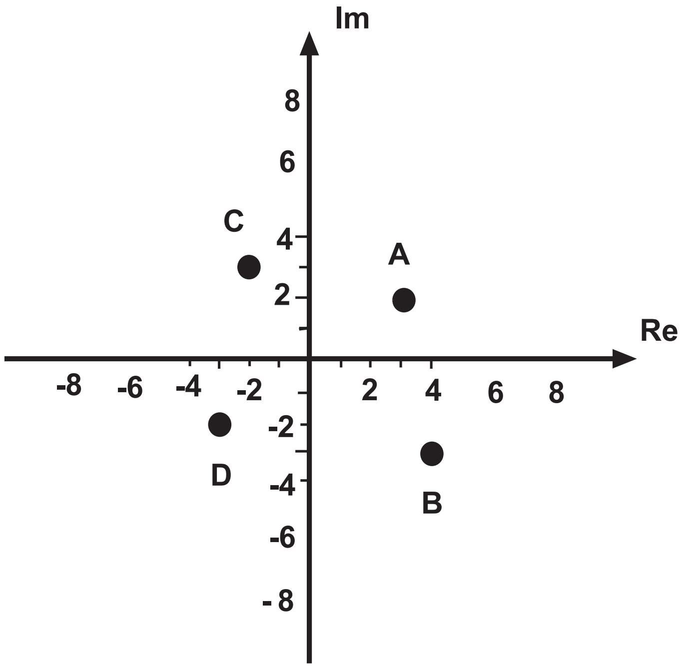
Exercise on finding real and imaginary points (page 85)

Q13: -7

Q14: 0: because the number is 0 + 3i

Plotting complex numbers exercise (page 87)

Q15:



Adding complex numbers exercise (page 88)

Q16: 5 + 4i

Q17: 5

Q18: -i

Q19: 3 + 5i

Subtracting complex numbers exercise (page 90)

Q20: -1 - 4i

Q21: 4 - 4i

Multiplying complex numbers exercise (page 91)

Q22: 3 - i

Q23: -7 - i

Q24: -24

Square root exercise (page 92)

Q25: 1 - 2i and -1 + 2i

Q26: -3 + 2i and 3 - 2i

Q27: 2 - i and -2 + i

Find the conjugate exercise (page 95)

Q28: 3 + 4i

Q29: -2 + i

Q30: -5i

Q31: 3. Think about where the complex number 3 sits in the argand diagram. It lies on the x-axis and so its conjugate is the same.

Q32: The conjugate $\overline{z}$ = 1 - 5i and $z\overline{z}$ = 26

Dividing two complex numbers exercise (page 97)

Q33: This division gives $\frac{17}{25} + \frac{31}{25}i$

Q34: This division gives $\frac{17}{50} - \frac{31}{50}i$

Q35: This division gives $-\frac{17}{26} - \frac{7}{26}i$

The real part is $\frac{17}{26}$ and the imaginary part is $-\frac{7}{26}$

Q36: This division gives $\frac{7}{10} - \frac{1}{10}i$

The real part is $\frac{7}{10}$ and the imaginary part is $-\frac{1}{10}$

Q37: The complex number is -i giving a real part of 0 and an imaginary part of -1

Modulus, argument and polar form exercise (page 104)

Q38: The modulus is $\sqrt{34}$

The principal argument is 1.03 or 59.01°

Q39: $\sqrt{2}\,(\cos(\frac{-\pi}{4}) + i\sin(\frac{-\pi}{4}))$ or, in degrees

$\sqrt{2}\,\{\cos(-45°) + i\sin(-45°)\}$

Q40: The modulus is $\sqrt{20}$ or $2\sqrt{5}$

The principal argument is 2.034 or 116.54°

Q41: The modulus is $\sqrt{17}$

The principal argument is -0.245 or -14.04°

Q42: The modulus is $\sqrt{45}$ or $3\sqrt{5}$

The principal argument is -2.034 or -116.54°

Q43: The modulus is 13

The principal argument is 0.395 or 22.63°

The polar form of this number is

13(cos 0.395 + i sin 0.395) or 13(cos 22.63° + i sin 22.63°)

Q44: $2(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3})$ or 2(cos 120° + i sin 120°)

Q45: The modulus is 2

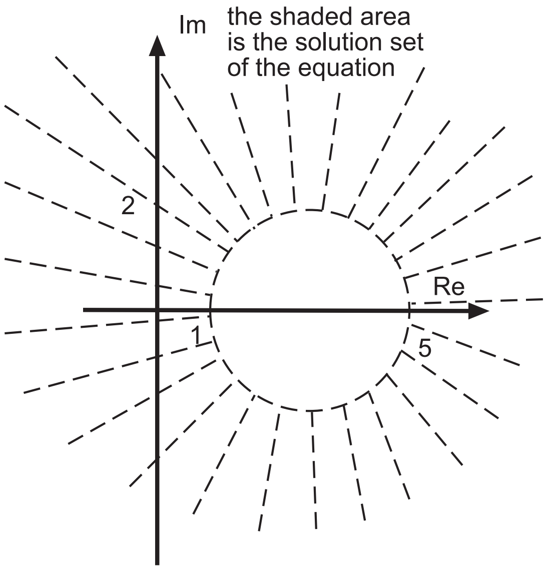
The principal argument is -150° or -2.618

Note that is much better to give the answer here in degrees.

Answers from page 107.

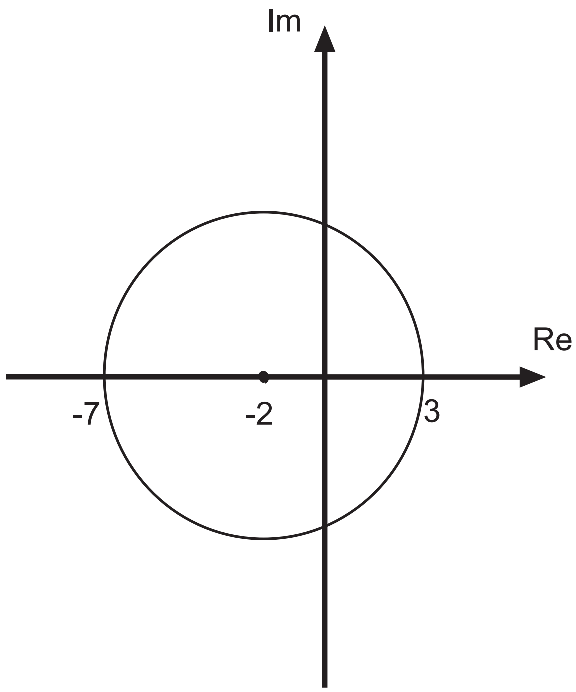
Q46: $|z - 3| > 2$ represents all the points which lie outside the circle with a radius greater than 2 and centre (3, 0).

This can be shown as:

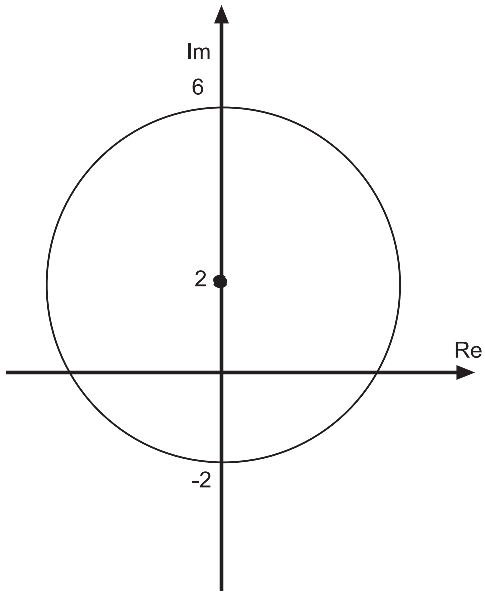


Geometric representation exercise (page 109)

Q47: This is a circle with radius of 5 units and centre (-2, 0)



Q48: This is a circle with radius of 4 units and centre (0, 2)



Q49: This is a circle with radius of 4 units and centre (1, 0)

Q50: The equation is $|z + 3i| = 4$

Q51: This is a line with equation 4y = 6x + 5

$|z + 2i| = |z + 3|$

$\Rightarrow x^2 + (y + 2)^2 = (x + 3)^2 + y^2$

$\Rightarrow 4y = 6x + 5$

Q52: The equation is $|z - 2i| < 2$

Solving complex equations exercise (page 113)

Q53: There are three roots. One is real: x = 1, and two are complex conjugates: x = -1 + 2i and x = -1 - 2i

Q54: There are three roots. One is real: x = -1, and two are complex conjugates: $x = \frac{-1}{2} + i\frac{\sqrt{11}}{2}$ and $x = \frac{-1}{2} - i\frac{\sqrt{11}}{2}$

Q55: b = -3 + i and c = 8 + i

The trick is to multiply the two factors (x - 2 + 3i) and (x - 1 - 2i) together and equate coefficients.

Q56: The four roots are $x = 1 + i\sqrt{6}$, $1 - i\sqrt{6}$, i and -i

Q57: The four roots are x = -1 + i, -1 - i, 2 + i and 2 - i

Answers from page 114.

Q58: The answer is

$12\,(\cos\,(30 + 60)^\circ - i\sin\,(30 + 60)^\circ) = 12\,(\cos 90^\circ + i \sin 90^\circ) = 12i$

De Moivre's expansion exercise (page 116)

Q59: De Moivre's theorem gives $27\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -27i$

Q60: De Moivre's theorem gives $16(\cos 3\pi + i \sin 3\pi) = -16$

Further De Moivre expansion exercise (page 116)

Q61:

The polar form of z = (-1 - i) is $\sqrt{2}\ \left\{\cos\ \left(\frac{-3\pi}{4}\right) + i\sin\ \left(\frac{-3\pi}{4}\right)\right\}$

so $z^4 = 4\{\cos(-3\pi) + i\sin(-3\pi)\} = -4$

The real part is -4 and the imaginary part is zero.

Q62:

The polar form of $z = (-\sqrt{3} + i)$ is $2\ \left\{\cos\ \left(\frac{5\pi}{6}\right) + i\sin\ \left(\frac{5\pi}{6}\right)\right\}$

so $z^3 = 2^3\ \left\{\cos\ \left(\frac{15\pi}{6}\right) + i\sin\ \left(\frac{15\pi}{6}\right)\right\} = 8i$

Answers from page 117.

Q63: $\frac{81i}{2}$

Q64: $4\sqrt{2}\left\{\cos\left(\frac{-7\pi}{12}\right) + i\sin\left(\frac{-7\pi}{12}\right)\right\}$

Answers from page 118.

Q65: $16\sin^4 x - 20\sin^2 x + 5$

Q66: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Hence

$\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$

Q67: $16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

Answers from page 119.

Q68:

By De Moivre's theorem

$\left\{27\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\right\}^{1/3} = 3\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} + i\,\frac{3}{2}$

Q69: De Moivre's theorem gives $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = 2\sqrt{3} + 2i$

Answers from page 121.

Q70: $z = 81\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z = 81\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$

$z = 81\left(\cos\frac{9\pi}{2} + i\sin\frac{9\pi}{2}\right)$

$z = 81\left(\cos\frac{13\pi}{2} + i\sin\frac{13\pi}{2}\right)$

$r^{1/4} = 3$

The fourth roots of $81\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$ are

$3\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right), 3\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right),$

$3\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$ and $3\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right)$

Q71: The modulus of $\sqrt{3} + i$ is 2 and the argument is $\frac{\pi}{6}$

The sixth roots are $\sqrt[6]{2}\left(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36}\right)$, $\sqrt[6]{2}\left\{\cos\left(\frac{13\pi}{36}\right) + i\sin\left(\frac{13\pi}{36}\right)\right\}$,

$\sqrt[6]{2}\left\{\cos\left(\frac{25\pi}{36}\right) + i\sin\left(\frac{25\pi}{36}\right)\right\}$, $\sqrt[6]{2}\left\{\cos\left(\frac{37\pi}{36}\right) + i\sin\left(\frac{37\pi}{36}\right)\right\}$,

$\sqrt[6]{2}\left\{\cos\left(\frac{49\pi}{36}\right) + i\sin\left(\frac{49\pi}{36}\right)\right\}$ and $\sqrt[6]{2}\left\{\cos\left(\frac{61\pi}{36}\right) + i\sin\left(\frac{61\pi}{36}\right)\right\}$

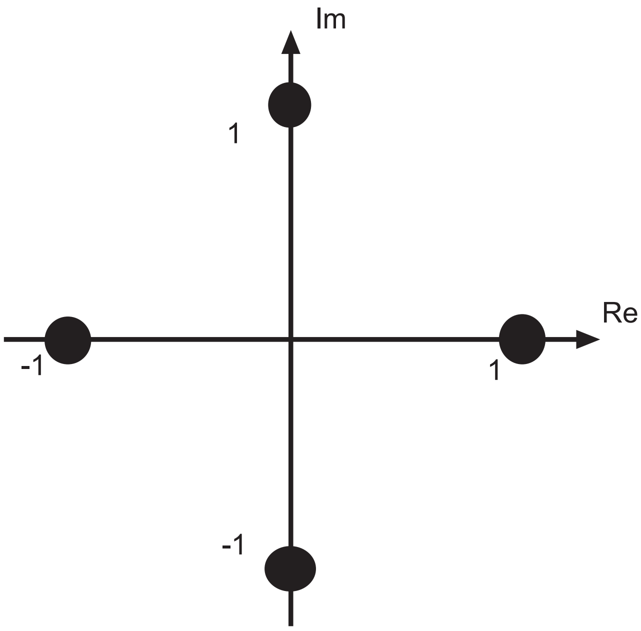
Answers from page 122.

Q72: The solutions are

z = cos 0 + i sin 0 , cos $\frac{\pi}{2}$ + i sin $\frac{\pi}{2}$, cos π + i sin π and cos $\frac{3\pi}{2}$ + i sin $\frac{3\pi}{2}$

i.e. z = 1, -1, i and -i

On an Argand diagram this gives

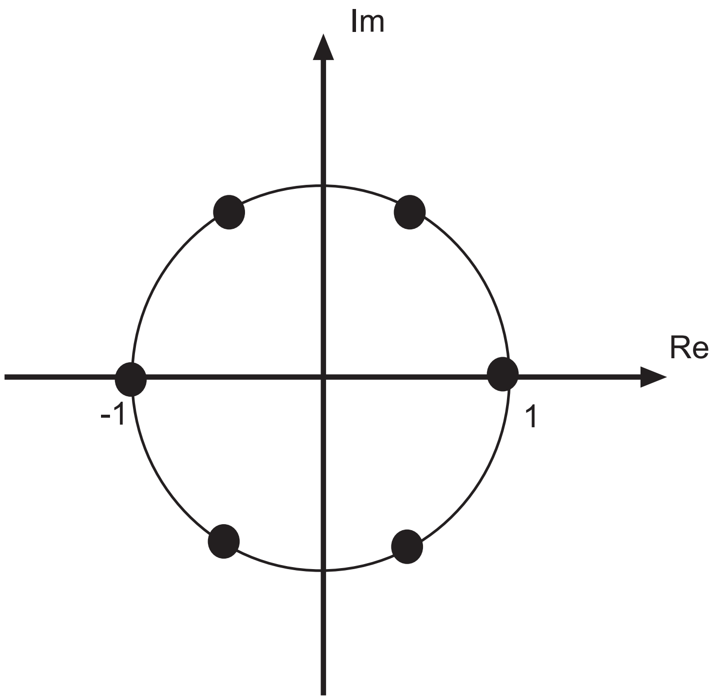


Q73: The solutions are

cos 0 + i sin 0, cos $\frac{\pi}{3}$ + i sin $\frac{\pi}{3}$, cos $\frac{2\pi}{3}$ + i sin $\frac{2\pi}{3}$, cos π + i sin π,

cos $\frac{4\pi}{3}$ + i sin $\frac{4\pi}{3}$ and cos $\frac{5\pi}{3}$ + i sin $\frac{5\pi}{3}$

The Argand diagram gives



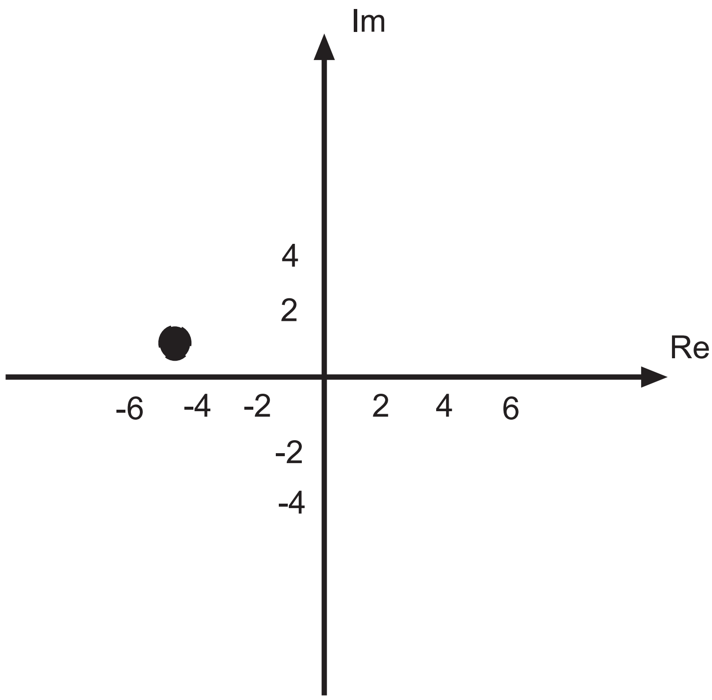
Modulus properties exercise (page 124)

Q74: The answers are:

1. $zw = \frac{-3-4\sqrt{3}}{2} - i\left(\frac{3\sqrt{3}-4}{2}\right)$
2. $\overline{z}\,\overline{w} = \overline{zw} = \frac{-3-4\sqrt{3}}{2} + i\left(\frac{3\sqrt{3}-4}{2}\right)$
3. $|z| = 5$
4. $z\overline{z} = |z|^2 = 25$

Review exercise (page 128)

Q75: zw = -5 + i



Q76: The modulus is 2

The principal argument is $-\frac{\pi}{3}$

Q77: $x = -1, 1 \pm i\sqrt{6}$

Q78: -32

Q79: $|z + 2i| = 4$

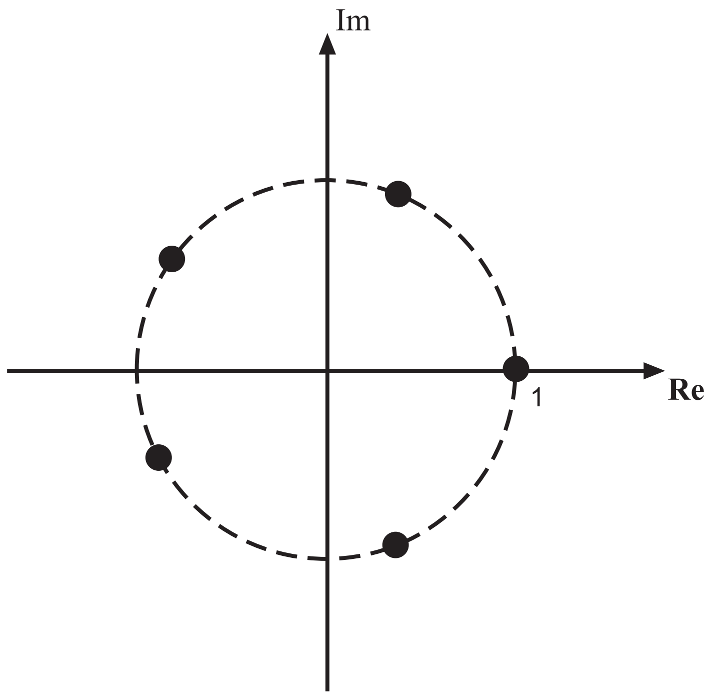
Advanced review exercise (page 129)

Q80:

$\cos 0 + i \sin 0$, $\cos \frac{2\pi}{5} + i \sin\frac{2\pi}{5}$,

$\cos \frac{4\pi}{5} + i \sin\frac{4\pi}{5}$, $\cos \frac{6\pi}{5} + i \sin\frac{6\pi}{5}$

and $\cos \frac{8\pi}{5} + i \sin\frac{8\pi}{5}$



Q81: Using the Binomial Theorem

$z^3 = (\cos\theta + i\sin\theta)^3$

$= \cos^3\theta - 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

The real part is $\cos^3\theta - 3\cos\theta\sin^2\theta$

Using De Moivre's Theorem

$z^3 = \cos 3\theta + i\sin 3\theta$

The real part of z^3 is $\cos 3\theta$

Equating the real parts gives

$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Q82: The modulus is $\frac{1}{\sqrt{2}}$, the argument is $\frac{-\pi}{12}$ and n = 6

Q83: The roots are $1 - i\sqrt{2}$, $1 + i\sqrt{2}$, $-i$ and i

Set review exercise (page 129)

Q84: This answer is only available on the web.

Q85: This answer is only available on the web.

Q86: This answer is only available on the web.

4 Sequences and Series

Revision exercise (page 132)

Q1: The first four terms are $64x^6 - 576x^5 + 2160x^4 - 4320x^3$

Check the section on the Binomial theorem and finding coefficients in topic 1 of this course.

Q2: The value after n years is u_n

where $u_0 = 1000$ and $u_{n+1} = 1.07 \times u_n$

Hence $u_1 = 1070$,

$u_2 = 1144.90$,

$u_3 = 1225.043$,

$u_4 = 1310.79601$,

$u_5 = 1402.551731$

So the final value is 1402 pounds and 55 pence (to the nearest penny).

Q3: 5.695

$u_2 = 0.3 \times 5 + 4 = 5.5$,

$u_3 = 0.3 \times 5.5 + 4 = 5.65$,

$u_4 = 0.3 \times 5.65 + 4 = 5.695$

Q4: $\frac{1}{(n^3 - n)}$

Find a common denominator $(n^3 - n)$, calculate new numerators and simplify.

Answers from page 133.

Q5: 10

Answers from page 134.

Q6: 1, 2, 3, 10, 29

Finding the nth term of a given sequence (page 135)

Q7: 4

Q8: $u_2 = 3$, $u_3 = {}^{7}/_{3}$, $u_4 = {}^{17}/_{7}$,

and $u_5 = {}^{41}/_{17} = 2{}^{7}/_{17}$ or 2.412

Q9: 3, 12, 27, 48

Answers from page 136.

Q10: a = 3 and d = -2. Using the definition for an arithmetic sequence the first four terms are 3, 1, -1, -3

Answers from page 137.

Q11:

a) -12, -9, -6, -3 This is an arithmetic sequence with a = -12 and d = 3
b) 5, 15, 45 ,55, 85, 95, 125, ... This is not an arithmetic sequence. The difference between terms 1 and 2 is 10 but between terms 2 and 3 the difference is 30. There is no common difference.
c) 9, 14, 20, 27, ..., 64, 76, ... This is not an arithmetic sequence. There is no common difference.
For example, the first difference is 5 and the second difference is 6
d) 0, 1, 2, 3, 4, 5, ..., 45, ..., 90, 91 This has a = 0 and d = 1 and is an arithmetic sequence.
e) 0.01, 0.001, 0.0001, 0.00001, 0.000001, ... This has no common difference and is not an arithmetic sequence.
f) 1, 8, 27, 64, 125, ... This is not an arithmetic sequence. It has no common difference.

Arithmetic sequence exercise (page 137)

Q12: The first term is a = 1 and d = $\frac{-1}{2}$

Using the formula gives $u_{16} = 1 + 15 \times \frac{-1}{2} = \frac{-13}{2}$

Q13: d = 3 and a = 3 so the sequence is defined by {3 + 3(n - 1)} = {3n}

Q14: d = 4 and a = -21 which gives the sequence {-21 + 4(n - 1)} = {4n - 25}. The tenth term is 15

Q15: The sequence is {4 - 2(n - 1)} = {6 - 2n} and $u_{40} = -74$

Answers from page 138.

Q16: a = -1 and r = -3. Using the definition for a geometric sequence the first four terms are -1, 3, -9, 27

Answers from page 139.

Q17:

a) -1, 1, -1, 1, -1, 1, -1, 1, -1, ... This has a = -1 and r = -1 and is a geometric sequence.
b) 5, -10, 20, -40, 80, ... This has a = 5 and r = -2 and is a geometric sequence.
c) 1, 4, 9, 16, 25, 36, 49, ... This is not a geometric sequence since the ratio of two consecutive terms is not the same throughout the sequence.
d) 2.2, 3.3, 4.4, 5.5, ..., 9.9 This is not a geometric sequence since there is no common ratio.

e) 0.01, 0.001, ..., 0.0000001, 0.00000001, ... This has a = 0.01 and r = 0.1 and is a geometric sequence.

f) 64, 32, 16, 8, 4 This has a = 64 and r = 0.5 and is a geometric sequence.

Geometric sequence exercise (page 139)

Q18: The first term is a = 0.1 and r = 4

Using the formula gives $u_{16} = 0.1 \times 4^{15} = 107374182.4$

Q19: r = 2 and a = 1 so the sequence is defined by

$\{1 \times 2^{(n-1)}\} = \{2^{n-1}\}$

Q20: a = 500 and r = 1.055

The sequence is $\{500 \times 1.055^{(n-1)}\}$ and the term $u_7 = 500 \times 1.055^6 = \$689.42$

Answers from page 141.

Q21: $\frac{n(n+1)}{2}$

Answers from page 144.

Q22:

a) This is a null sequence.

b) This is not a null sequence because $u_n > 3$ for all n

c) This is not a null sequence because $2n > 1$ for all n

Answers from page 145.

Q23: The answers are:

a) This sequence is a null sequence and so has a limit of zero.

b) This sequence tends to infinity.

c) This sequence has a limit of 10.

d) This sequence is a null sequence. It is also an alternating sequence.

Limits of sequences exercise (page 146)

Q24: The limit exists and is zero (it reduces to $^0/_3 = 0$).

Q25: The limit exists and is 1

Q26: This sequence tends to infinity.

Answers from page 146.

Q27: The calculator gives the value of the twentieth term as 2.6533 (to 4 decimal places). The graph should indicate that the points are levelling out and a limit of under 3 is reasonable to suggest from the graph.

Q28: The term numbers 1000, 2000, 3000 can be examined. At term number 6000 and thereafter the values of the terms of the sequence are equal to 2.7181 (to 4 decimal places). It is reasonable at this stage to suggest that the sequence has a limit close to 2.7181

Q29: 2.718281828...

Q30: In symbols this is $\lim_{n \to \infty} (1 + \frac{1}{n})^n = e$.

This is read as 'the limit of the sequence one plus one over n all to the power n as n tends to infinity is e.'

Answers from page 147.

Q31: A reasonable formula for the perimeter of a polygon with twelve sides is $12 \times 2 \tan(\frac{180}{12})$ and for a polygon with n sides is $n \times 2 \tan(\frac{180}{n})$

Answers from page 147.

Q32: A sensible limit is 6.2832. Each term after number 566 gives this value (to four decimal places).

Answers from page 147.

Q33: The circumference of the circle is $6.2832 = 2\pi \times 1$. Thus π has a value of 3.1416 using this method.

Q34: The suggested formula is $\pi = \frac{1}{2} \lim_{n \to \infty} \{a_n\}$ where $a_n = 2n \tan\left(\frac{180}{n}\right)$ (a_n is the perimeter of a circumscribed polygon with n sides).

Finding partial sums exercise (page 149)

Q35: 16.25

This is the answer to (1 + 3) + (2 + 1.5) + (3 + 1) + (4 + 0.75)

Q36: 153

This is the answer to

$(1) + (1 \times 2) + (1 \times 2 \times 3) + (1 \times 2 \times 3 \times 4) + (1 \times 2 \times 3 \times 4 \times 5)$

Q37: 11.5

This is the answer to (2 + 3) + (2 + 1.5) + (2 + 1)

Answers from page 150.

Q38: 95

$$\sum_{n=1}^{4}\left(\frac{1}{2}n + 3n^2\right) = \frac{1}{2}\sum_{n=1}^{4} n + 3\sum_{n=1}^{4} n^2$$
$$= \frac{1}{2}(1 + 2 + 3 + 4) + 3(1 + 4 + 9 + 16)$$
$$= 5 + 90 = 95$$

Arithmetic series partial sums exercise (page 151)

Q39: a = 1 and d = -1

With n = 8 this gives $S_8 = \frac{8}{2}(2 + 7 \times -1) = -20$

Q40: a = -4 and d = 2

With n = 13 this gives $S_{13} = \frac{13}{2}(-8 + 12 \times 2) = 104$

Q41: a = -15 and d = 16

With n = 7 this gives $S_7 = \frac{7}{2}(-30 + 6 \times 16) = 231$

Extended exercise on arithmetic series (page 151)

Q42: 1800

Q43: 3240

Q44: 1161

Answers from page 153.

Q45: 18

$$\sum_{n=1}^{6}(2n - 4) = 2\sum_{n=1}^{6} n - 4\sum_{n=1}^{6}(1)$$
$$= \frac{2\times(6\times 7)}{2} - (4\times 6)$$
$$= 42 - 24$$
$$= 18$$

Q46: 80.5

Geometric series partial sums exercise (page 154)

Q47: a = 2, r = -2 and n = 8

$S_8 = \frac{2(1 - (-2)^8)}{1 - (-2)} = -170$

Q48: a = -16, r = $^1/_2$ and n = 13

This gives $S_{13} = \frac{-16(1 - (^1/_2)^{13})}{1 - ^1/_2} = -\frac{8191}{256}$

Note that in the cases where r is given as a fraction it is best to give the answer in the same form.

Q49: 4882812

Q50: $\frac{2660}{27}$

Extended exercise on geometric series (page 155)

Q51: $r^2 = \frac{27}{243} = \frac{1}{9}$, so $r^2 = \pm\frac{1}{3}$

Since all the terms are positive r = $\frac{1}{3}$ and a = 3^7

So $S_8 = 1 + 3 + 3^2 + ... + 3^7 = \frac{1 - 3^8}{1 - 3} = 3280$

Q52: 256

Answers from page 156.

Q53:

a) This is not convergent since r = 2 i.e. r is outwith the range $-1 < r < 1$
b) This is convergent with a = 32 and r = $\frac{1}{2}$ giving $S_\infty = 64$
c) This is convergent with a = 32 and r = $-\frac{1}{2}$ giving $S_\infty = \frac{64}{3}$
d) This is convergent with a = $\frac{1}{3}$ and r = $\frac{1}{2}$ giving S_∞ as $\frac{2}{3}$
e) This is not convergent since r = -3 i.e. r is outwith the range $-1 < r < 1$

Q54: a = 1 and d = $\frac{1}{2}$ which gives $S_\infty = 2$

Q55:

$9 = \frac{a}{1 - r} = \frac{a}{1 - {}^2/_3} = 3a$ So a = 3

Q56: $\frac{3125}{648}$

Q57: $4 = ar$, $1 = ar^3$ so $r^2 = \frac{1}{4}$. Since $r > 0$ then $r = \frac{1}{2}$, a = 8 and sum to infinity = 16

Answers from page 158.

Q58:

$$\begin{aligned}(a + x)^{-1} &= a^{-1}\left(1 + \frac{x}{a}\right)^{-1} \\ &= a^{-1}\left[1 + \left(\frac{-1}{1!}\right)\left(\frac{x}{a}\right) + \left(\frac{-1 \times -2}{2!}\right)\left(\frac{x}{a}\right)^2 + \left(\frac{-1 \times -2 \times -3}{3!}\right)\left(\frac{x}{a}\right)^3 + ...\right] \\ &= \frac{1}{a} - \frac{x}{a^2} + \frac{x^2}{a^3} - \frac{x^3}{a^4}\end{aligned}$$

Answers from page 158.

Q59: The series is $\frac{1}{3x} - \frac{2}{9x^2} + \frac{4}{27x^3} - \frac{8}{81x^4} + ...$

Answers from page 158.

Q60:

$$\begin{aligned}(1 + 0.05)^{-1} &= 1 - (0.05) + (0.05)^2 - (0.05)^3 + ... \\ &= 1 - 0.05 + 0.0025 - 0.000125 + ... \\ &= 0.9524\end{aligned}$$

Review exercise (page 160)

Q61: 233

Q62: -1526

Q63: 32768

Q64: $S_n = \frac{162(1 - \left(\frac{1}{3}\right)^n)}{1 - \frac{1}{3}}$

Advanced review exercise (page 160)

Q65: The geometric series is

$$(3 - 0.05)^{-1} = 3^{-1}\left(1 + \frac{0.05}{3} + \frac{0.0025}{9} + \frac{0.000125}{27} + \ldots\right)$$
$$= \frac{1}{3} + \frac{0.05}{9} + \frac{0.0025}{27} + \frac{0.000125}{81} + \ldots$$

and this approximates to 0.338983

Q66:

- For years 1 to 5 inclusive the geometric sequence is $\{500 \times 1.075^n\}$ and for years 6 to 16 the geometric sequence is $\{1217.81 \times 1.05^n\}$
- The first sequence has a = 500 and r = 1.075
  The second sequence has a = 1217.81 and r = 1.05
- The amount at the end of the 16th year (the start of the 17th year) is $1983.68

Q67: For the condition $\left|\frac{-4x}{3}\right| < 1$ the series is

$\frac{1}{3} + \frac{4x}{9} + \frac{16x^2}{27} + \frac{64x^3}{81} + \ldots$

For the condition $\left|\frac{-3}{4x}\right| < 1$ the series is $-\frac{1}{4x} - \frac{3}{16x^2} - \frac{9}{64x^3} - \frac{27}{256x^4} - \ldots$

Q68: $^1/_3$

The trick is to find the partial fractions, use the combination rules, write out the first few terms of each series and subtract.

$$\sum_{n=1}^{\infty}\left(\frac{1}{n^2 + 5n + 6}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)$$
$$= \sum_{n=1}^{\infty}\left(\frac{1}{n+2}\right) - \sum_{n=1}^{\infty}\left(\frac{1}{n+3}\right)$$
$$= \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots\right) - \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \ldots\right)$$
$$= \frac{1}{3}$$

Set review exercise (page 161)

Q69: This answer is only available on the web.

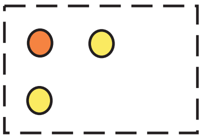
Q70: This answer is only available on the web.

5 Elementary number theory and methods of proof

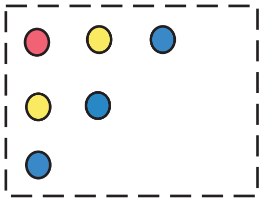
Answers from page 166.

Q1: If the integers are placed in a rectangle starting at one corner the pattern becomes obvious.

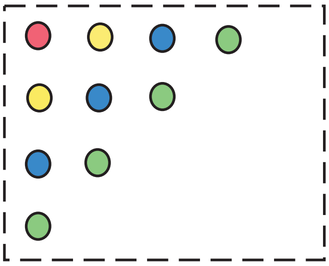
1 + 2



1 + 2 + 3



1 + 2 + 3 + 4



and so on

The rectangles are 1 unit wider than deep.

The third rectangle shows the sum of four integers in a rectangle of 5×4

It follows that 10 integers can be placed in a rectangle of size 11×10

This is the area of the rectangle and the diagrams show that it is possible to mirror the dots into the second half of the rectangle and fill it.

Thus the sum of 10 integers is $\frac{10\,(10 + 1)}{2} = 55$

Answers from page 166.

Q2: The sum of the first n integers is given by the formula $\frac{n\,(n + 1)}{2}$

Answers from page 168.

Q3: The statement $x = 3$ 'implies' that $x^2 = 9$ changes to the statement

$x = 3$ is 'implied by' $x^2 = 9$

This is not correct as $x^2 = 9$ can also give $x = -3$ as a solution.

The statement $|x + 2| = 3$ is 'implied by' $x = 1$ changes to the statement

$|x + 2| = 3$ 'implies' that $x = 1$

This is not correct as $|x + 2| = 3$ can also give $x = -5$

Which symbol exercise (page 168)

Q4:

a) $4\sqrt{} = 2 \Rightarrow 2^2 = 4$

b) $a^x = b \Leftarrow \log_a b = x$

c) a = 2 and b = 5 $\Rightarrow {}^a/_b = {}^2/_5$

d) $n = r \Rightarrow \begin{pmatrix} n \\ r \end{pmatrix} = 1$

e) The inverse function f^{-1} exists $\Leftrightarrow$ The function f is one-to-one and onto.

f) $\{u_n\}$ converges $\Leftarrow \sum_{n=1}^{\infty} u_n = 0$

Answers from page 170.

Q5: There are many. Here are three: π, $\sqrt{3}$, $\sqrt[3]{2}$

Answers from page 171.

Q6: There is only one even prime (the number 2) because even numbers are all divisible by 1, 2 and themselves, contradicting the definition of a prime.

The sieve of Eratosthenes exercise (page 171)

Q7: The prime numbers which are those not scored out should be as shown:

	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			

Answers from page 172.

Q8: $12 = 2 \times 2 \times 3$

$16 = 2 \times 2 \times 2 \times 2$

$24 = 2 \times 2 \times 2 \times 3$

$48 = 2 \times 2 \times 2 \times 2 \times 3$

Product of primes and canonical form exercise (page 173)

Q9: The answers are:

- $2 \times 3 \times 5 \times 5$
- $2 \times 3 \times 3 \times 5$
- $2 \times 2 \times 5 \times 7$
- 7×13

Q10: The answers are:

- $36 = 2^2 \times 3^2$
- $105 = 3 \times 5 \times 7$
- $24 = 2^3 \times 3$
- $98 = 2 \times 7^2$

Answers from page 174.

Q11: The next five are: (29, 31) (41, 43) (59, 61) (71, 73) (101,103)

Answers from page 174.

Q12: 12 = 7 + 5, 14 = 3 + 11, 16 = 13 + 3, 18 = 11 + 7, 20 = 13 + 7

Counter-example exercise (page 175)

Q13: If $x = 0$ then $x + x^2 = 0$ and the conjecture is false.

Q14: The number 9 is odd but 9 is composite.

Q15: There are many possible answers. Here is one.

If $x^2 = 2$, which is a natural number, then $x = \sqrt{2}$, which is an irrational number (as stated earlier) and $\notin \mathbb{N}$.

Q16: A counter-example which disproves the conjecture is $x = 0$ and $x = 4$. They both give the same image of 0.

Proof by exhaustion exercise (page 176)

Q17: Check each number in turn.

$20 = 4 \times 5$, not prime

$21 = 3 \times 7$, not prime

$22 = 2 \times 11$, not prime

23 = prime, number 1

$24 = 3 \times 8$, not prime

$25 = 5 \times 5$, not prime

$26 = 2 \times 13$, not prime

$27 = 3 \times 9$, not prime

$28 = 4 \times 7$, not prime

29 = prime, number 2

$30 = 3 \times 10$, not prime

The numbers 23 and 29 are the only primes in that range. The conjecture is proved.

Q18: Each integer 4, 5, 6, 7, 8, 9, 10 has to be checked. By doing so it reveals that 4 and 9 are the only squares and the conjecture is true.

Q19: Here it is necessary to take every pairing for a proof by exhaustion. This means 5 + 6 = 11, 5 + 7 = 12, 5 + 8 = 13, 6 + 7 = 13, 6 + 8 = 14, 7 + 8 = 15 and check that each answer $\in [11, 15]$ which they do. The conjecture is proved.

Proof by contradiction exercise (page 178)

Q20: Assume that the conjecture is false. So assume that m^2 is even but that m is odd.

Let m = 2t + 1 where t is an integer

then $m^2 = (2t + 1)^2 = 4t^2 + 4t + 1 = 2(2t^2 + 2t) + 1 = 2s + 1$ where $s = 2t^2 + 2t$

This is in the form of an odd integer and contradicts the assumption that m^2 is even.

Thus the conjecture is true.

Q21: Assume the conjecture is false and make a new assumption that m is rational.

Therefore $m = {}^p/_q$ where p and q are relatively prime.

So $10q^2 = p^2 \Rightarrow p^2$ is even since $10q^2 = 2 \times (5q^2)$

By the recently proved conjecture that if m^2 is even then m is even.

Then **p is even** and p = 2k for some integer k

Thus $10q^2 = 4k^2 \Rightarrow 5q^2 = 2k^2$

Using the same argument again

$5q^2 = 2m^2 \Rightarrow q^2$ is even and so **q is even**.

Thus both p and q are even but both were assumed to be relatively prime. This is a contradiction and the assumption that m is rational is false.

The original conjecture that m is not a rational number is therefore true.

Q22: Suppose that the conjecture is false. Make the assumption that if n is composite then all divisors $> \sqrt{n}$

Let **n = ab** where a and b are positive integers both greater than $\sqrt{n}$

So $n = ab > \sqrt{n} \times \sqrt{n} = n$ That is, **ab > n**

This gives a contradiction that n = ab and ab > n. The assumption is false and therefore the conjecture is true.

Proof by induction exercise (page 181)

Q23: Check the first easy value of n = 3

then $9 > 7$ and the result is true.

Assume that the result is true for n = k

then $k^2 > 2k + 1$

Consider n = k + 1

then $(k + 1)^2 = k^2 + 2k + 1 > 2k + 1 + 2k + 1 > 2k + 1 + 1 + 1 = 2(k + 1) + 1$

So $(k + 1)^2 > 2(k + 1) + 1$

The statement is true for n = k + 1 if it is true for n = k but it is true for n = 3.

So by the principle of mathematical induction the conjecture is true for all n greater than or equal to 3.

Q24: Check the easy value of n = 1

then $1 < 2^1 = 2$

Assume that the conjecture is true for n = k

then $k < 2^k$

Consider n = k + 1

so $k + 1 < 2^k + 1 < 2^k + 2 < 2^{k+1}$

The result holds for n = k + 1 when it is true for n = k. But it is also true for n = 1 so by the principle of mathematical induction the conjecture is true.

Q25: Check n = 1

then 4! = 24 and 8 is a factor of 4!

Assume the result is true for n = k then (4k)! has a factor 8^k

that is $(4k)! = 8^k \times m$

Consider n = k + 1

then $(4(k + 1))! = (4k + 4)(4k + 3)(4k + 2)(4k + 1)(4k)!$

But this equals $8(k + 1)(4k + 3)(2k + 1)(4k + 1) \times 8^k \times m$

thus $(4(k + 1))! = 8^{k+1}(k + 1)(4k + 3)(2k + 1)(4k + 1) \times m$

that is $(4(k + 1))!$ has a factor 8^{k+1}

Thus the result holds for n = k + 1 if it is true for n = k

But it is true for n = 1. So by the principle of mathematical induction the conjecture is true for all $n \in \mathbb{N}$.

Q26: Check for n = 1

then $1 + 2^1 = 3 = 2^2 - 1$

The result is true for n = 1

Assume that the result is true for n = k

then $1 + 2 + 2^2 + ... + 2^k = 2^{k+1} - 1$

Consider n = k + 1

then $1 + 2 + 2^2 + ... + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$

So the result is true for n = k + 1 if it is true for n = k

But it is true for n = 1 and so by the principle of mathematical induction the conjecture is true for all $n \in \mathbb{N}$.

Q27: Check when n = 1

Then $1 = \frac{1(1+1)(2+1)}{6} = 1$. The result is true for n = 1

Assume true for n = k

Then $1 + 4 + 9 + ... + (k-1)^2 + k^2 = \frac{k(k+1)(2k+1)}{6}$

Consider n = k + 1

Then $1 + 4 + 9 + ... + (k-1)^2 + k^2 + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+2)(2(k+1)+1)}{6}$$

So the result is true for n = k + 1 if it is true for n = k. But it is true for n = 1 and so by the principal of mathematical induction the conjecture is true for all $n \in \mathbb{N}$.

Q28: Check for n = 3

then $3^2 - 2 \times 3 - 1 = 2$ and $2 > 0$

the result holds for n = 3

Assume that the conjecture is true for n = k

then $k^2 - 2k - 1 > 0$

Consider n = k + 1

then $(k+1)^2 - 2(k+1) - 1 = (k^2 - 2k - 1) + 2k - 1$

but $(k^2 - 2k - 1) + 2k - 1 < k^2 - 2k - 1 > 0$

Thus the result is true for n = k + 1 if it is true for n = k

But it is also true for n = 3 and so by the principle of mathematical induction the conjecture is true for all $n \geq 3$

Review exercise in elementary number theory (page 183)

Q29: Assume that the conjecture is false then n^3 is odd but n is even

Let n = 2m then

$n^3 = 8m^3$ which is even.

This contradicts the new statement and so the original conjecture is true.

Q30: There are many answers.

If x = 2, y = -3, z = 2 and w = -4 then x z = 4 and y w = 12. This gives an example to disprove the conjecture.

Q31: Check for n = 1

$(1 + x)^1 = 1 + x = 1 + 1 \times x$ The result holds.

Assume true for n = k

then $(1 + x)^k \geq 1 + kx$

Consider n = k + 1

then $(1 + x)^{k+1} \geq (1 + kx)(1 + x) = 1 + (k + 1)x + kx^2 \geq 1 + (k + 1)x$

Thus it is true for n = k + 1 if it is true for n = k

But it is true for n = 1 and so by the principle of mathematical induction the conjecture is true for all $n \in \mathbb{N}$ when $x > -1$

Q32: Check n = 1

1 + 3 = 4 and $2|4$ (note this symbol means 'divides' and this reads '2 divides 4').

Thus 1 + 3 = 2m for some integer m

Assume true for n = k

so $k^2 + 3k = 2 \times m$

Consider n = k + 1

then $(k + 1)^2 + 3(k + 1) = 2 \times m + 2k + 4 = 2(m + k + 2)$

Thus the conjecture is true for n = k + 1 if it is true for n = k

But it is true for n = 1 and so by the principle of mathematical induction the result is true for all $n \in \mathbb{N}$.

Q33: Check for n = 1

The sum of the first odd integer is 1 which is 1^2

Assume the conjecture is true for n = k

then $1 + 3 + 5 + ... + (2k - 1) = k^2$

Consider n = k + 1

then $1 + 3 + 5 + ... + (2k - 1) + (2k + 1) = k^2 + 2k + 2 = (k + 1)^2$

The result is true for n = k + 1 if it is true for n = k

But it is true for n = 1 and so by the principle of mathematical induction the conjecture is true for all $n \in \mathbb{N}$.

Advanced review exercise in elementary number theory (page 183)

Q34: Check an easy value: when n = 1

then $1 = \frac{1}{2} \times 1(1 + 1) = 1$

Suppose the result is true for n = k then $1 + 2 + 3 + ... + k = \frac{1}{2}k(k + 1)$

Consider n = k + 1 then

$$1 + 2 + 3 + ... + k + (k + 1) = \frac{1}{2}k(k + 1) + k + 1$$
$$= \frac{(k + 1)(k + 2)}{2}$$

The result is true for n = k + 1 if it is true for n = k. But it is true for n = 1 and so by the principle of mathematical induction, the conjecture is true.

Q35: Check the conjecture when n = 1

$1 = \frac{1^2(1 + 1)^2}{4} = 1$ The result holds.

Suppose that the result is true for n = k then

$$\sum_{r=1}^{k} r^3 = \frac{k^2(k + 1)^2}{4}$$

Consider n = k + 1 then

$$\sum_{r=1}^{k+1} r^3 = \frac{k^2(k + 1)^2}{4} + (k + 1)^3$$
$$= \frac{(k + 1)^2[k^2 + 4k + 4]}{4}$$
$$= \frac{(k + 1)^2(k + 2)^2}{4}$$

The result is true for n = k + 1 if it is true for n = k. But it is true for n = 1 and so by the principle of mathematical induction, the conjecture is true for all $n \in \mathbb{N}$.

Set review exercise in elementary number theory (page 183)

Q36: This answer is only available on the web.

Q37: This answer is only available on the web.

Q38: This answer is only available on the web.

Q39: This answer is only available on the web.