Advanced Higher Revision Notes

Unit 1

1.1 Binomial Expansions
1.2 Partial Fractions
2 Differentiation
3 Integration
4 Functions & Curve Sketching
5 Gaussian Elimination

Unit 2

1 Further Differentiation
2 Sequence & Series
3 Further Integration
4 Complex Numbers
5 Proof Theory

Unit 3

1 Vectors, Lines & Planes
2 Matrices & Transformations
3 Further Sequence & Series and MacLaurins
4 1st & 2nd Ordinary Differential Equations
5 Euclidean Algorithm & Further Proof Theory

NB: Order of teaching shall vary, but this is in line with the order of the textbook

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1.1 Binomial Expansions

$$\binom{n}{r} = \left(\frac{n!}{r!(n-r)!}\right) = \binom{n}{n-r}$$

where $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

$n! = n \times (n-1)!$

and $(n-1)! = (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
Etc….

Given $n! = n \times (n-1)!$

Use property with to prove $r(r-1)! = r!$ & $(n-r+1)(n-r) = (n-r+1)!$

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

Binomial Expansion

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + .. + \binom{n}{r} x^{n-r} y^r + .. + \binom{n}{n} y^n$$

Example Find the coefficient Independent of ***x*** in the expansion of $\left(2x + \frac{1}{x}\right)^4$

Independent of x
➔ NO x term
➔ Coeff when x^0

$$\left(2x+\frac{1}{x}\right)^4 = \sum_{r=0}^{4} \binom{4}{r} (2x)^{4-r} \left(\frac{1}{x}\right)^r$$

$$= \sum_{r=0}^{4} \binom{4}{r} (2)^{4-r} (x)^{4-r} \left(x^{-1}\right)^r = \binom{4}{r} (2)^{4-r} (x)^{4-r} \left(x^{-r}\right)$$

$$= \binom{4}{r} (2)^{4-r} (x)^{4-r-r}$$

$$= \binom{4}{r} (2)^{4-r} (x)^{4-2r}$$

Independent when (4 – 2r) = 0
2r = 4
⇨ r = 2

Thus coefficient when r = 2 is, C = $\binom{4}{2} (2)^{4-2} (x)^{4-4}$

$$= \frac{4!}{2!(4-2)!} \times (2)^2$$

$$= \frac{4 \times 3 \times 2!}{2! \times 2!} \times 4 = \frac{12}{2} \times 4 = 6 \times 4 = \mathbf{24}$$

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1.2 Partial Fractions: (7 Types to consider)

1 – Quadratic

$$\frac{4x+1}{(x+1)(x+2)}=\frac{A}{(x+1)}+\frac{B}{(x+2)}$$

2 – Quadratic with repeated factors

$$\frac{2x-1}{(x-3)^2}=\frac{A}{(x-3)}+\frac{B}{(x-3)^2}$$

3 – Cubic

$$\frac{x^2-7}{(x-1)(x+2)(x-4)}=\frac{A}{(x-1)}+\frac{B}{(x+2)}+\frac{C}{(x-4)}$$

4 – Cubic with 2 repeated factors

$$\frac{5x+2}{(x+3)(x-2)^2}=\frac{A}{(x+3)}+\frac{B}{(x-2)}+\frac{C}{(x-2)^2}$$

5 – Cubic with 3 repeated factors

$$\frac{x^2-7x+4}{(x-1)^3}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x-1)^3}$$

6 – Quadratic which can't be factorised

$$\frac{3x^2+2x+1}{(x+1)(x^2+2x+2)}=\frac{A}{(x+1)}+\frac{Bx+C}{(x^2+2x+2)}$$

7 – Higher polynomial on numerator ➔ Need to DIVIDE first

$$\frac{x^3+2}{x(x-3)}=\frac{x^3+2}{x^2-3x}=x+3+\frac{9x+2}{x(x-3)}=x+3+\frac{A}{x}+\frac{B}{(x-3)}$$

Need to use long division before using partial fractions when higher degree of polynomial on numerator. Then solve Partial Fractions as normal.

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2 Differentiation

f(x)	f'(x)
ax^n	nax^{n-1}
sinax	acosax
cosax	-asinax
tanx	sec^2x
$cosecx = \left(\frac{1}{\sin x}\right)$	-cosecxcotx
$secx = \left(\frac{1}{\cos x}\right)$	secxtanx
$cotx = \left(\frac{1}{\tan x}\right)$	$-cosec^2x$
$\ln \lvert x \rvert$	$\frac{1}{x}$
e^{ax}	$a e^{ax}$

Quotient Rule:

$$\frac{f(x)}{g(x)} \text{ or } \frac{u}{v} \qquad \frac{u'v - uv'}{v^2}$$

Product Rule:

f(x)g(x) or uv   u'v + uv'

Chain Rule:

$$(ax+b)^n \qquad n(ax+b)^{n-1}.a = an(ax+b)^{n-1}$$

Other Formulae

$1 + tan^2x = sec^2x$

$sin^2x + cos^2x = 1$

$sin^2x = 1 – cos^2x$

$sinx = \sqrt{(1-\cos^2 x)}$

$cos^2x = 1 sin^2x$

$cosx = \sqrt{(1-\sin^2 x)}$

sin2x = 2sinxcosx

$cos2x = cos^2x – sin^2x$
$= 2cos^2x – 1$
$= 1 – 2 sin^2x$

If $cos2x = 1 – 2sin^2x$

$2sin^2x = 1 – cos2x$

$sin^2x = \frac{1}{2}(1-\cos 2x)$

If $cos2x = 2cos^2x - 1$

$2cos^2x = 1 + cos2x$

$cos^2x = \frac{1}{2}(1+\cos 2x)$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

Parametric: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ & $\frac{d^2y}{dx^2} = \frac{{}^{d}/_{dx}\left({}^{dy}/_{dx}\right)}{{}^{dx}/_{dt}}$

Inverse:

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2} \qquad \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{(a^2-x^2)}} \qquad \frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{(a^2-x^2)}}$$

Differentiating: **Distance s(t) ➔ Velocity, v(t) ➔ Acceleration, a(t)**

s(t) ➔ $\frac{ds}{dt}$ **➔ s''(t) =** $\frac{dv}{dt} = \frac{d^2s}{dt^2}$

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3 Integration Rules

$$\int F'(x)dx = f(x) + c \qquad \int_a^b f(x)dx = F(b) - F(a)$$

$$\int [af(x) + bg(x)]dx = a\int f(x)dx + b\int g(x)dx$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int \cos(ax+b)dx = \frac{1}{a} sin(ax+b) + c \qquad \int \sin(ax+b)dx = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \sec^2(ax+b)dx = \frac{1}{a}\tan\left(ax+b\right) + c \qquad \int \text{-cosec}^2 x dx = \cot x + c$$

$$\int \sec x \tan x dx = \sec x + c \qquad \int -\cos ecx \cot x dx = \cos ecx + c$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c \qquad \int \frac{1}{ax+b}dx = \frac{1}{a}\ln(ax+b) + c$$

$$\int \frac{1}{\sqrt{\left(a^2-x^2\right)}}dx = \sin^{-1}\left(\frac{x}{a}\right) + c \qquad \int e^{ax}dx = \frac{e^{ax}}{a} + c$$

$$\int \frac{-1}{\sqrt{(a^2-x^2)}}dx = \cos^{-1}\left(\frac{x}{a}\right) + c$$

Main Methods of Integration

1. ***Volume of Revolution*** *[if on x-axis use dx, if on y use dy]*
2. ***Higher power on numerator ➔ Division & Partial Fractions***
3. ***Substitution [**Remember to amend integral values and dx]***
4. ***Integration by Parts*** $\int uv' = \{uv - \int u'v\}$ *[diff easier function]*
5. ***Separation of Variables. Often involves ln & remember + C then take exponential of both sides to solve [Let A =*** e^c***]***
6. ***Inverse Trig function [****May appear within partial fractions]*
7. ***Combination of all of the above***

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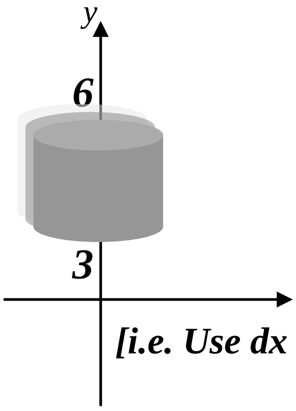
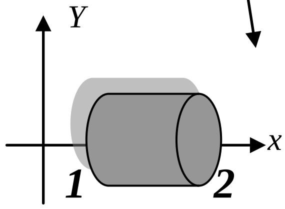
Applied Integration

$V = \int \pi\, y^2\, dx \;\rightarrow$ *if on x-axis*

$V = \int \pi\, x^2\, dy \;\rightarrow$ *if on y-axis*

1. Volumes of Revolution





[i.e. Use dx if rotated on x – axis and dy if rotated on y – axis]

Ex 1: *The container with function y = x² + 2 is rotated around the x-axis between* ***x = 1*** *and* ***x = 2*** *[As around* ***x-axis => use dx*** *and formula with y²]*

$$V = \int \pi y^2 dx$$

$$= \int_1^2 \pi\left(x^2+2\right)^2 dx = \int_1^2 \pi\left(x^4+4x^2+4\right)^2 dx = \pi\left[\frac{x^5}{5}+\frac{4x^3}{3}+4x\right]_1^2$$

$$= \pi\left\{\left[\frac{32}{5}+\frac{32}{3}-8\right]-\left[\frac{1}{5}+\frac{4}{3}-4\right]\right\} = \pi\left[\frac{31}{5}+\frac{28}{3}-4\right] = \frac{293\pi}{15} = 19\frac{8}{15}\pi \quad \text{Units}^3$$

Ex 2: *The container with function y = x² + 2 is rotated around the y-axis between y = 3 and y = 6 (Diagram on LHS). If y = x² + 2 then x² = (y – 2) [As around* ***y-axis => use dy*** *and formula with x²]*

$$V = \int \pi x^2 dy$$

$$= \int_3^6 \pi\left(y-2\right)dy = \pi\left[\frac{y^2}{2}-2y\right]_3^6 = \pi\left\{\left[\frac{36}{2}-12\right]-\left[\frac{9}{2}-6\right]\right\} = \pi\left[\frac{27}{2}-6\right] = \frac{15\pi}{2} \quad \text{Units}^3$$

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Using various methods of Integration:-

2. SAME POWER (OR HIGHER) on NUMERATOR ➔ DIVIDE:

$$\int \frac{x+1}{x+3}dx = \int\left(\frac{(x+1)+2-2}{x+3}\right)dx = \int\left(\frac{x+3}{x+3}-\frac{2}{x+3}\right)dx = \int\left(1-\frac{2}{x+3}\right)dx = x-2\ln|x+3|+c$$

$$\int\left(\frac{x^3+2x^2+x-1}{x+1}\right)dx$$

$$=\int x^2+x+\frac{-1}{(x+1)}dx$$

$$=\frac{x^3}{3}+\frac{x^2}{2}-\ln|x+1|+c$$

Long division:

$$x+1\overline{)x^3+2x^2+x-1}\quad\text{quotient } x^2+x$$

$$\underline{x^3+x^2}$$

$$\underline{x^2+x-1}$$

$$-1$$

*Remember a higher polynomial on the numerator => Long Division*

2B. Partial Fractions:

$$\int\frac{2x-1}{(x+1)(x+4)}dx = \int\left(\frac{A}{(x+1)}+\frac{B}{(x+4)}\right)dx = ...etc..$$

3. Substitution

Given a polynomial ***1 degree higher*** *on numerator,* ***check if denominator can differentiate*** *and cancel out the numerator via* **substitution**

$$\int\frac{2x+2}{x^2+2x}dx = \int \frac{\cancel{2x+2}}{u}\frac{du}{\cancel{2x+2}} = \int\frac{du}{u} = \ln|u|+c = \ln|x^2+2x|+c$$

4. Integration by parts

Use if given ***2 functions combined*** ➔ $\int uv' = \{uv - \int u'v\}$

Functions that repeat or alternate [E.g. e^x, ***cos x, sin x****]*

➔ *set integral to* ***I*** *& use 'loop'/repetition to rearrange & solve integral.*

If integrating complex functions [i.e. ∫tan x dx, ∫ln|x| dx ***etc..***]

➔ *Set up as* ***integration by parts*** *and multiply the function given by* ***1.***

∫ 1. tan x dx & ∫ 1. ln |x| dx

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5. Separation of Variables

$\frac{dy}{dx} = (x+2)^3 y$ — As we have a combination of x and y

$\frac{dy}{y} = (x+2)^3\, dx$ — variables we must separate them.

$\int \frac{dy}{y} = \int (x+2)^3\, dx$ — Then integrate both sides.

$\ln y = \frac{(x+2)^4}{4} + c$ — Only attach constant to RHS.

$y = e^{\left(\frac{(x+2)^4}{4} + c\right)}$ — Then take exponential of each side to obtain y.

$y = e^{\frac{(x+2)^4}{4}} . e^{c}$ — Finally separate using indice rules

$y = Ae^{\frac{(x+2)^4}{4}}$, — Where $A = e^c$

6. Inverse Trigonometric Functions

Remember tan is ONLY inverse function bringing fraction to FRONT
Usually quite obvious when to use, as ***tan, cos and sine inverse*** *functions as they have squares/square roots involved.*

REMEMBER TO CHANGE INTEGRAL VALUES

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c \qquad \int \frac{1}{\sqrt{(a^2-x^2)}}dx = \sin^{-1}\left(\frac{x}{a}\right)+c \qquad \int \frac{-1}{\sqrt{(a^2-x^2)}}dx = \cos^{-1}\left(\frac{x}{a}\right)+c$$

*****In Trig Substitution highly likely that the denominator shall become***

$$\sqrt{(1-\cos^2 x)} = \sin x, \sqrt{(1-\sin^2 x)} = \cos x \text{ or } \sqrt{(1+\tan^2 x)} = \sec x$$

DON'T FORGET TO RE-ARRANGE & REPLACE notation from say, dx, to du or dθ & change DEFINITE INTEGRAL values accordingly so ENTIRE problem is in terms of new variable.

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4 Functions and Curve Sketching

ONE-TO-ONE *If each value in domain (input) images onto only* ***1 value*** *in range*

INVERSE and ONTO *An* ***INVERSE*** *exists if the values in the range fall exactly onto one value in domain, (if co-domain is exactly same as the range* ➔ ***ONTO****)*

If more than one value in the domain gives the same value in range [i.e. $(\pm 2)^2 = 4$]
➔ ***MANY-TO-ONE.*** *{An inverse may exist if the domain is restricted say to $x > 0$}*

Curve Sketching (ASSMTEAS)

Axes ***Find where it cuts the axes*** ***x = 0 (y-axis)***
y = 0 (x-axis)

Symmetry ***Consider Status of Function***

Odd *function is rotated 180° around origin* $f(-x) = -f(x)$
Even *function is reflected on y-axis* $f(-x) = f(x)$
If ***Neither*** *exist cannot assume anything!* $f(-x) \neq f(x)$

Stationary Points or Points of Inflexion, $f'(x)=0$
Max/Min Remember to find y-coordinate from ORIGINAL function.
Turning Pts Use nature table or $f''(x)$ rules to find NATURE of TPts

If $f''(a) > 0$ It is happy ⋃ with a min turning point

If $''(a) < 0$ It is unhappy ⋂ with a max turning point

If $f''(x) = 0$ then need to use the nature table

or

x	$\rightarrow a \rightarrow$
$\frac{dy}{dx}$	
Slope	

Extremes of
Asymptotes *[Think of when undefined and also when $x \rightarrow \pm\infty$]*

Vertical Asymptote at $x \rightarrow \pm$??, then $y \rightarrow \pm\infty$
(Values which make the denominator undefined)

Non-Horizontal/Non-Vertical/Slant/Oblique at y = ? $x \rightarrow \pm\infty$

- ***Exists when higher power on numerator ➔ find via DIVIDING***
- ***Slant Asymptote is the first section (Quotient) [i.e. line y = ax + b]***

Then use the remainder to determine whether ± for $x \rightarrow \pm\infty$

Sketch Annotate asymptotes, turning points, and where cuts x & y-axes
****** remember by using long division at the start shall make the function far more manageable to differentiate and to determine Non-Vertical Asymptotes***

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5 Gaussian Elimination – 3 Possible Solutions to Consider:

*Work around in an '**L**' shape (from* $a_{21} \rightarrow a_{31}$ *then to* a_{32}*) rearranging the system of equations into **Upper Triangular form** :*

Type 1

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 8 & 9 \end{array}\right)$$

Ox + Oy + kz = N ➔ ***ONLY ONE UNIQUE SOLUTION EXIST***
and can therefore solve for x, y and z.

Type 2

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 9 \end{array}\right)$$

Ox + Oy + Oz = k ➔ ***T****he system of equations does not make sense and is said to be* ***INCONSISTENT ➔ REDUNDANT AS NO SOLUTION EXISTS.***

Type 3

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

0x + 0y + 0z = 0 ➔ *this means parallel planes exist as one has eliminated the other* ➔ ***INFINITE SOLUTIONS EXIST***

ILL-CONDITIONING

A SMALL change in the matrix makes a massive difference in the final solution is called ill-conditioning.

This has rarely came up (2012 only), so potentially a favourite….
Other favourite is using the matrix to find an unknown value?

<u>Advanced Higher Revision Notes</u>

Differentiating Exponential Problems:

<u>Example</u> Differentiate the following $y = 4^{(x^2+2)}$

1. *Take **ln** of each side* $\ln|y| = \ln|4^{(x^2+2)}|$
2. *Rearrange to remove power issue* $\ln|y| = (x^2+2)\ln|4|$
3. *Remember ln |k| is a only a constant* $\ln|y| = \ln|4|x^2 + 2\ln|4|$
4. *Differentiate both sides* $\frac{1}{y}.\frac{dy}{dx} = 2x.\ln|4|$
5. *Need to multiply by y* $\frac{dy}{dx} = (2x.\ln|4|) \times y$
6. *Now express in terms of x* $\frac{dy}{dx} = (2x.\ln|4|) \times 4^{(x^2+2)}$

Inverse Functions:

$$\frac{d}{dx}\left(\tan^{-1}\left(\frac{x}{a}\right)\right) = \frac{a}{a^2+x^2} \qquad \frac{d}{dx}\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = \frac{1}{\sqrt{(a^2-x^2)}} \qquad \frac{d}{dx}\left(\cos^{-1}\left(\frac{x}{a}\right)\right) = \frac{-1}{\sqrt{(a^2-x^2)}}$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

Can invert functions in the following way also, find f'(x) and make $\frac{1}{f'(x)}$

Derivative of an Inverse Function:

1. *DERIVATIVE of inverted function ➔ Change f(x) to f(y) and diff*
2. *Need to find Inverse so switch x ↔ y*
3. *Change back into terms of y*
4. *This is now the inverse $f^{-1}(x)$*
5. *Using rule $\frac{dy}{dx} = \frac{1}{f'(y)}$ write findings of f ' (y) and substitute the inverse of function into **derivative** to represent in terms of x*

<u>Example</u> Find the inverse derivative of $f(x) = x^3$

Inverse ➔ need *f ' (y) rather than f ' (x)* $f(x) = x^3$ ➔ $f(y) = y^3$

$f'(y) = 3y^2$

*Find the inverse function **of $f(x) = x^3$*** ***i.e.***

$y = x^3$

$x = y^3$

$x^{1/3} = y$

➔ $f^{-1}(x) = x^{1/3} = f(y)$

Inverse derivative:

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2} = \frac{1}{3(x^{1/3})^2} = \frac{1}{3x^{2/3}}$$

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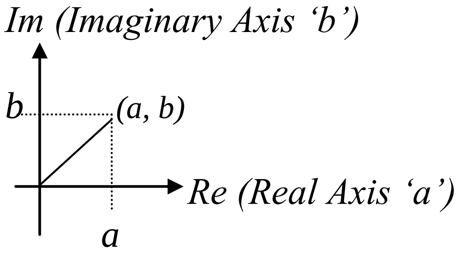
Complex Numbers

$$i = \sqrt{-1} \rightarrow i^2 = -1$$

Cartesian Format, $z = a + ib$

(Use to plot values)

Argand Diagram (a, b)



Use Argand Diagram to find the Modulus |r| & Argument arg(θ)

The distance from the origin to (a, b) is called the modulus of z,

$$|z| = \sqrt{\left(a^2 + b^2\right)}$$

Note that the i is ignored and the modulus is usually referred to as r:

$$r = \sqrt{\left(a^2 + b^2\right)}$$

The argument is represented by:

$$\arg(z) = \theta = \tan^{-1}\left|\frac{b}{a}\right|$$

Together the modulus & argument can then represent the complex number z in Polar form rather than Cartesian ($z = a + ib$)

[Note it is easier to apply calculations with the Polar form.]

Polar form

$$a = r\cos\theta \text{ and } b = r\sin\theta$$
$$z = r(\cos\theta + i\sin\theta)$$

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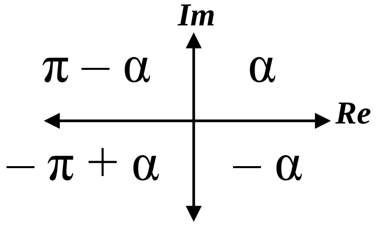
Complex Numbers : To find ARGUMENT

Find 1st Quadrant (Principle argument) then find ***arg(θ)***, using Argand diagram quadrant rules

The ***argument θ*** of the complex number can be found by plotting the Cartesian Coordinate to decide which Quadrant the principle angle θ lies in.

The Principle Argument must lie between $-\pi \le \theta \le \pi$.

- Find the 1st Quadrant angle, say **α**, using $\tan^{-1}\left|\frac{b}{a}\right| = \alpha$
- Plot the Argand diagram to decide which Quadrant ***(a , b)*** is in.
- Use the Quadrant rules to find the value of ***arg(θ)***



Complex Geometric Interpretations (Locus of a Point)

To find the locus of a point is similar to interpreting the centre & radius of a circle. The inequality used with the complex number applies to whether it is on the circumference ($|z| = r$); inside ($|z| < r$) or outside ($|z| > r$) the circle.

To obtain a solution in terms of ***x & y*** represent the complex number as $z = x + iy$

We can combine the facts that $|z| = |x + iy|$ and $|z| = |r| = \sqrt{(x^2 + y^2)}$

Thus using these properties: $|z - a - ib| = r$ ➔ *Centre* ***(a, b) &*** *radius*, ***r***

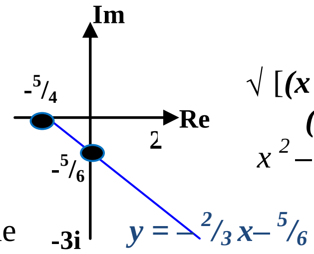
Example 1: Find the equation of the loci for $|z - 2 + 3i| = 7$

	$\lvert z - 2 + 3i\rvert = 7$
Let $z = (x + iy)$	$\lvert (x + iy) - 2 + 3i\rvert = 7$
Collecting real and imaginary	$\lvert (x - 2) + i(y + 3)\rvert = 7$
Remember to drop '*i*' when finding modulus	$\sqrt{[(x-2)^2 + (y+3)^2]} = 7$
Square both sides to find equation of circle	$(x-2)^2 + (y+3)^2 = 49$

⇨ **Circle Centre (2, -3) with radius 7**, & sketch on Argand diagram

Indicate on an Argand diagram the locus which satisfy $|z - 2| = |z + 3i|$

[➔ Let $z = (x + iy)$ as must represent in terms ***of x & y]***



	$\lvert (x + iy) - 2\rvert = \lvert (x + iy) + 3i\rvert$
Collect Real & Imaginary	$\lvert (x - 2) + iy\rvert = \lvert x + i(y + 3)\rvert$
Finding Modulus	$\sqrt{[(x-2)^2 + (y)^2]} = \sqrt{[(x)^2 + (y+3)^2]}$
Square both sides	$(x-2)^2 + y^2 = x^2 + (y+3)^2$
Expand square brackets	$x^2 - 4x + 4 + y^2 = x^2 + y^2 + 6y + 9$
Simplify	$-4x - 5 = 6y$
Express as equation of a line	➔ $y = -{}^2/_3\, x - {}^5/_6$

Advanced Higher Revision Notes

Complex Numbers

Solving a Cubic problem

- Find first solution by Inspection/Synthetic Division i.e. ***(x – a) = 0***
- Take this factor and use long division to obtain a Quadratic remainder
- Factorise or use Quadratic Formula to obtain the final two solutions

Solving a Quartic problem *(below is a prelim solution from 2008/9 to see process)*

- Given the first solution, say ***x = (a + ib)*** By the fundamental theorem of algebra every quartic has 4 solutions and every complex number has a conjugate pair. Thus find the conjugate pair $\bar{x} = a - ib$
- Rearrange both solutions into factors and multiply to obtain a quadratic
  (x – a – ib)(x – a + ib)   [Use a multiplication grid for ease]
- Use this new quadratic expression & divide to obtain a quadratic remainder
- Factorise or use Quadratic Formula to obtain the final two solutions.

Given $z = (2 + i)$ is a root of $z^4 - 4z^3 + 6z^2 - 4z + 5$, find all the roots?

AH 2008/9 Prelim (Unit 1/2) (7)

Q5. $z = (2+i) \Rightarrow \bar{z} = (2-i)$

By the fundamental theorem of algebra a CONJUGATE PAIR exists

* Using 2 solutions $z_1 = (2+i)$ & $z_2 = (2-i)$

* Can then find 2 factors of Quartic: $(z_1 - 2 - i) = 0$ & $(z_2 - 2 + i) = 0$

* Now if we multiply 2 factors ⇒ Obtain a Quadratic
⇒ Use Long division
⇒ Find remaining 2 sols

$(z-2-i)(z-2+i) = z^2 - 4z + 4 - i^2$
$= z^2 - 4z + 4 + 1$
$= z^2 - 4z + 5$

	z	-2	$-i$
z	z^2	$-2z$	$-iz$
-2	$-2z$	$+4$	$+2i$
$+i$	$+iz$	$-2i$	$-i^2$

Now by dividing shall be able to find remainder and solve 4 solutions for $z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$

$$\begin{array}{r|l} & z^2 + 1 \\ z^2 - 4z + 5 & z^4 - 4z^3 + 6z^2 - 4z + 5 \\ & z^4 - 4z^3 + 5z^2 \\ \hline & z^2 - 4z + 5 \\ & z^2 - 4z + 5 \\ \hline & - \end{array}$$

$\therefore z^2 + 1 = 0$
$z^2 = -1$
$z^2 = i^2$
$z = \pm i$ (3)

∴ 4 Solutions are:

$z_1 = 2+i$; $z_2 = 2-i$; $z_3 = i$; $z_4 = -i$

Advanced Higher Revision Notes

De Moivre's Theorem

$$z^n = \left[r\left(\cos\theta + i\sin\theta\right)\right]^n$$
$$z^n = r^n\left(\cos\left(n\theta\right) + i\sin\left(n\theta\right)\right)$$

Given $z = (\cos\theta + i\sin\theta)$ & $w = (\cos\psi + i\sin\psi)$

$zw = (\cos(\theta + \psi) + i\sin(\theta + \psi))$ & $z/w = (\cos(\theta - \psi) + i\sin(\theta - \psi))$

Using Binomial Expansion & De Moivre's *to express in terms* ***of Cos or Sin***

- *Apply from Cartesian to Polar format of Complex number*
- *Rearrange to De Moivre's Theorem so expression is as a 'power'*
- *Expand using Binomial Expansion, take care with **i**.*
- *If asks for Cos → use the real values only*
- *If asks for Sin → use the imaginary values only, **DO NOT include i***
- *Rearrange final expression to required status*

****** VERY POPULAR EXAM STYLE QUESTION***

nth Roots of a Complex Number

- ***Write down the complex number in polar form $z = r(\cos\theta + i\sin\theta)$***
- ***For each other additional root add 2π to the angle each stage***
- *Eg for cube root we would find the polar format for the first root as*

1st Root $z = r[\cos\theta + i\sin\theta]$

2nd Root $z = r[\cos(\theta+2\pi) + i\sin(\theta+2\pi)]$

3rd Root $z = r[\cos(\theta+4\pi) + i\sin(\theta+4\pi)]$

- ***Write down the cube roots of z by taking the cube root of r & dividing each of the arguments by 3***

1st Root $z_1 = r^{1/3}[\cos(\theta/3) + i\sin(\theta/3)]$

2nd Root $z_2 = r^{1/3}[\cos((\theta+2\pi)/3) + i\sin((\theta+2\pi)/3)]$

3rd Root $z_3 = r^{1/3}[\cos((\theta+4\pi)/3) + i\sin((\theta+4\pi)/3))]$

Advanced Higher Revision Notes

n^{th} Roots of Unity : $z^n = 1$

E.g Solve for $z^n = 1$ *OR* $z^n - 1 = 0$

Since NO Imaginary Values & Real = 1 ➔ $\theta = 2\pi$
Thus

$$z^n = [\cos(2\pi) + i\sin(2\pi)]^n = 1$$

It follows that $\cos(^{2\pi}/_n) + i\sin(^{2\pi}/_n)$ *is an* n^{th} *root of unity*

As ***every*** 2π *will therefore result in a root of unity:*

$1 = \cos(^{2\pi}/_n) + i\sin(^{2\pi}/_n)$
$1 = \cos(^{4\pi}/_n) + i\sin(^{4\pi}/_n)$
$1 = \cos(^{6\pi}/_n) + i\sin(^{6\pi}/_n)$
$1 = \cos(^{8\pi}/_n) + i\sin(^{8\pi}/_n) \ldots$ ➔ $\cos(2n\pi) + i\sin(2n\pi) = 1$

Example: *Find the roots when* $z^4 = 1$ ➔ *expect 4 roots*

$$z^4 = (\cos(2\pi) + i\sin(2\pi))^4$$

1^{st} root, $z_1 = [\cos(2\pi) + i\sin(2\pi)]^{1/4}$
2^{nd} root, $z_2 = [\cos(2\pi+2\pi) + i\sin(2\pi+2\pi)]^{1/4}$
3^{rd} root, $z_3 = [\cos(2\pi+4\pi) + i\sin(2\pi+4\pi)]^{1/4}$
4^{th} root, $z_4 = [\cos(2\pi+6\pi) + i\sin(2\pi+6\pi)]^{1/4}$

4 roots are therefore:

$z_1 = \cos(^{2\pi}/_4) + i\sin(^{2\pi}/_4) = \cos(^{\pi}/_2) + i\sin(^{\pi}/_2)) = 0 + i = i$
$z_2 = \cos(^{4\pi}/_4) + i\sin(^{4\pi}/_4) = \cos(\pi) + i\sin(\pi) = -1 + 0i = -1$
$z_3 = \cos(^{6\pi}/_4) + i\sin(^{6\pi}/_4) = \cos(^{3\pi}/_2) + i\sin(^{3\pi}/_2) = 0 - i = -i$
$z_4 = \cos(^{8\pi}/_4) + i\sin(^{8\pi}/_4) = \cos(2\pi) + i\sin(2\pi) = 1 + 0i = 1$

& can be sketched on an Argand diagram
(4 roots connected would make a square)

Advanced Higher Revision Notes

Further Sequence and Series

Arithmetic Sequences

$u_n = a + (n-1)d$ $\qquad S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

Geometric Sequences

$u_n = ar^{n-1}$ $\qquad S_n = \frac{a(1-r^n)}{1-r}$

$S_\infty = \frac{a}{1-r}$

MacLaurins (Power) Series

$$f(x) = f(0) + f'(0)\frac{x^1}{1!} + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + f^{iv}(0)\frac{x^4}{4!} + .. + f^n(0)\frac{x^n}{n!}$$

Iterative Processes

For iterative processes set $x_{n+1} = g(x_n)$

If asked to show root by sketch split function into 2 easier equations and find intersection, here the value of x will give a good indication of the root you are trying to find.
Eg $x^2 - 2x + 7 = 0$ can be split into $y = x^2$ & $y = 2x - 7$

Algebraically we can determine the function will tend to a given root by finding $g'(x)$ & substitute the root, say x = α into $g'(x)$ i.e. $g'(\alpha)$.

If $|g'(\alpha)| < 1$

Then our value will CONVERGE to the specified root.

If $|g'(\alpha)| \geq 1$

Then our value DIVERGES and we must try a different rearrangement of the function for convergence to occur.

The ORDER of the Sequence/function can be determined by $g'(\alpha)$

$g'(\alpha) \neq 0 \Rightarrow$ The function is FIRST ORDER

$\Rightarrow u_{n+1} = au_n + b$

$g'(\alpha) = 0 \Rightarrow$ The function is SECOND ORDER

$\Rightarrow u_{n+1} = au_{n+1} + bu_n + c$

Advanced Higher Revision Notes

Number Proof Theory

Main types of proof involve:

- Proof by Counter-Example (Substitute values in to prove if true. Especially consider ***negatives and fractions*** to disprove)
- Proof by Exhaustion (Subtitute **EVERY** value in range to solve)
- Proof by Induction ********** The hot favourite!!!
- Proof by Contradiction (Especially square root & Odd/Even questions)

Advanced Higher Revision Notes

Proof by Induction (basic):-

- ***Let n = 1*** and prove true for this case
- ***Assume true for n = k*** & substitute ***k*** in as required
- ***Consider n = k + 1***
- Extend proof for ***n = k*** by adding extra term ***n = k + 1*** to either side & rearrange to obtain original format
- Statement:
  As true for n = 1, assumed true for n = k and by proof by induction also true for n = k + 1, assume true for $\forall\, n \in N$
  {or for whichever set stated, could be $n \in Z^+$, $n \geq 0$ etc..}

Proof by Induction Example:

AH 2008/9 Mini (5)

Q5. $\sum_{r=1}^{n} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3n+2}$

Let n=1: $LHS = \frac{3}{(3-1)(3+2)} = \frac{3}{2\times5} = \frac{3}{10}$

$RHS = \frac{1}{2} - \frac{1}{(3+2)} = \frac{1}{2} - \frac{1}{5} = \frac{5-2}{10} = \frac{3}{10}$

LHS = RHS ✓ true for n=1

Assume true for n=k: $\sum_{r=1}^{n=k} \frac{3}{(3r-1)(3r+2)} = \frac{1}{2} - \frac{1}{3k+2}$

Consider n=k+1

$$\sum_{r=1}^{n=k} \frac{3}{(3r-1)(3r+2)} + \frac{3}{(3(k+1)-1)(3(k+1)+2)} = \left(\frac{1}{2} - \frac{1}{3k+2}\right) + \frac{3}{(3(k+1)-1)(3(k+1)+2)}$$

$$= \left(\frac{1}{2} - \frac{1}{3k+2}\right) + \frac{3}{(3k+3-1)(3k+3+2)}$$

$$= \frac{1}{2} - \frac{1}{3k+2} + \frac{3}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{3}{(3k+2)(3k+5)} - \frac{1\cdot(3k+5)}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{3-3k-5}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{-3k-2}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} + \frac{-(3k+2)}{(3k+2)(3k+5)}$$

$$= \frac{1}{2} - \frac{1}{(3k+5)}$$

$$= \frac{1}{2} - \frac{1}{3(k+1)+2} \quad \text{as required}$$

(5)

As true for n=1, assumed true for n=k and by proof of mathematical Induction also true for n=k+1, conjecture is true $\forall n \in N$.

Advanced Higher Revision Notes

*Harder Proof by Induction (Powers): $8^n - 1$ is divisible by 7

Let $n = 1$: $8^1 - 1 = 7 = 7 \times 1$ ➔ **divisible by 7 so true for n = 1**✓

Assume true for $n = k$: $8^k - 1 = 7m$ (*where m is a positive integer*)

Consider $n = k + 1$: $8^{k+1} - 1 = 8^k . 8^1 - 1$ (*where m is a positive integer*)

(Trick is to $+c - c$) $= 8^k . 8^1 - 1 + 7 - 7$

(as helps find a common factor) $= 8^k . 8^1 - 8 + 7$

(re-create $n = k$ statement) $= 8(8^k - 1) + 7$

(so can replace with $7m$) $= 8(7m) + 7$

(can now show have multiple of 7) $= 7(8m + 1)$ *Let $(8m + 1) = N$ to simplify final expression (not nec, but nice)*

$= 7N$

Statement: *As true for n = 1, assumed true for n = k and by proof by induction also true for n = k + 1, assume true for $\forall\ n \in N$*

*Alternatively if pretty tricky then can rearrange assumption for **n = k** and substitute into problem when considering **n = k + 1***

Let $n = 1$: $8^1 - 1 = 7 = 7 \times 1$ ➔ **divisible by 7 so true for n = 1**✓

Assume true for $n = k$: $8^k - 1 = 7m$ (*where m is a positive integer*)

Consider $n = k + 1$: $8^{k+1} - 1 = 8^k . 8^1 - 1$ (*where m is a positive integer*)

(if $8^k - 1 = 7m$) $= (7m + 1) . 8^1 - 1$

(Then $8^k = (7m + 1)$) $= 8(7m + 1) - 1$

(use this to replace 8^k) $= 56m + 8 - 1$

(Simplify expression) $= 56m + 7$

(can now show have multiple of 7) $= 7(8m + 1)$ *Let $(8m + 1) = N$ to simplify final expression (not nec, but nice)*

$= 7N$

Statement: *As true for n = 1, assumed true for n = k and by proof by induction also true for n = k + 1, assume true for $\forall\ n \in N$*

Induction & Inequalities

- Use similar steps
- Will find it tricky to rearrange to final answer as inequality
- So leave some space
- As you know what the final answer with ***n = k+1*** looks like
- Write this expression below your given space
- Then consider what has happened in space between this
- As you must justify the inequality.

Advanced Higher Revision Notes

Proof by Contradiction:

- Assume their conjecture is **false**
- Assume you are true with a contradicting statement
- Try to prove, but you will have an error ➔ **CONTRADICTION!!**
- Hence initial conjecture is true!!

'√2 is not a rational number.' Prove this conjecture.

Assume statement is false:

Assume that √2 is rational.

Try to prove

Therefore let $\sqrt{2} = {}^{p}/_{q}$ — where p and q are no common factors.

Squaring both sides gives $2 = {}^{p^2}/_{q^2}$

$2q^2 = p^2 \Rightarrow p^2$ is even.

Now if p is odd, then p^2 is odd.

But p^2 is even

As p is even, let p = 2m for some integer m

$p^2 = (2m)^2$

$p^2 = 4m^2 = 2q^2$

So if $2q^2 = 4m^2$

then $q^2 = 2m^2 \Rightarrow q^2$ is even

$\Rightarrow$ q is even

Find flaw

Statement

Initially we said p & q had no common factors & were irreducible. However here p & q are both even and have a common factor of 2. **CONTRADICTION!**

Thus by proof of contradiction √2 must be irrational.

Advanced Higher Revision Notes

Euclidean Algorithm

The Greatest Common Divisor (gcd) [or highest common factor]

$78 = 1.42 + 36$

$42 = 1.36 + \mathbf{6}$

$36 = 6.\underline{\mathbf{6}} + 0$ Therefore the **gcd(42, 78) = 6**

Obtain values of x & y using Euclidean Algorithm:

E.g. Find the values of x & y which satisfy the following Euclidean Algorithm

$$\gcd(2695, 1260) = 2695x + 1260y$$

- **First find the gcd**

$2695 = 2.1260 + 175$
$1260 = 7.175 + 35$
$175 = 5.35 + 0$ Therefore the **gcd(2695, 1260) = 35**

- **Now starting at the second last line work backwards:-**

Then from line 2 if: $1260 = 7.175 + 35$
Then

$$35 = 1260 - 7.175$$
$$= 1260 - 7.(2695 - 2.1260)$$
$$= 1260 - 7.2695 + 14.1260$$
$$\Rightarrow 35 = \underline{-7.2695 + 15.1260}$$

Thus $\gcd(2695, 1260) = 2695x + 1260y$ **&** $\gcd(2695, 1260) = 35$

$$\rightarrow 2695x + 1260y = 35$$

and $-7 . 2695 + 15 . 1260 = 35 \Rightarrow \underline{x = -7 \;\&\; y = 15}$

Diophantine *(if equation looks similar but RHS has changed)*

From above $2695x + 1260y = 35$ **has solutions** $x = -7$ & $y = 15$
If $2695x + 1260y = 105$ → **Original solutions x 3** → $x = -21$ & $y = 45$

M Newlands, 2014

Advanced Higher Revision Notes

11 Vectors

Length of a vector

$$\underline{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \qquad |\underline{p}| = \sqrt{\left(a^2 + b^2 + c^2\right)}$$

Component form of a vector

$$a\underline{i} + b\underline{j} + c\underline{k}$$

Direction Ratios & Cosine Ratios

Let $\underline{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\underline{i} + b\underline{j} + c\underline{k}$

The vector $\underline{p}$ makes an angle of α with the x-axis,

an angle of β with the y-axis and an angle of γ with the z-axis.

Thus,

$$\cos\alpha = \frac{a}{|\underline{p}|} \quad ; \cos\beta = \frac{b}{|\underline{p}|} \quad ; \cos\gamma = \frac{c}{|\underline{p}|}$$

THE DIRECTION COSINES are the values:

$$\frac{a}{|\underline{p}|} \;;\; \frac{b}{|\underline{p}|} \text{ and } \frac{c}{|\underline{p}|}$$

The DIRECTION RATIOS are the ratios of a : b : c

Advanced Higher Revision Notes

The Scalar Product

Let $\underline{p} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$; $\underline{q} = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$ Then $\underline{p} \bullet \underline{q}$ = ad + be + cf

Scalar Product Properties

Property 1 → ***a . (b + c)= a . b + a . c***

Property 2 → ***a . b = b . a***

Property 3 → $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 \geq 0$

Property 4 → ***a . a = 0 if and only if (iff) a = 0***

Property 5 → *For **non-zero** vectors**, a** and **b** are **perpendicular iff a . b = 0***

*In geometric form, **the scalar product of two vectors a** and **b** is defined as*

$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ *where* θ *is the angle between **a** and **b**,* $0 \leq \theta \leq 180^o$

Vector Product Properties

Property 1 → ***a* x *(b + c)= a* x *b + a* x *c***

Property 2 → ***a* x *b = -b* x *a (i.e. AB = -BA)***

Property 3 → ***a* x *a*** $= |\mathbf{a}|^2 \geq 0$

Property 4 → ***a* x *(b* x *c)*** $\neq$ ***(a* x *b)* x *c***

Property 5 → ***If a . (a* x *b) = 0** and **b . (a* x *b) = 0***

The vector a* x *b is perpendicular to both a & b

The Vector product *in geometric form of **a** and **b** is defined with*

Magnitude of $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ *where* θ *is the angle between **a** and **b**,* $0 \leq \theta \leq 180^o$

***Direction perpendicular to both a** and **b** as determined by the Right Hand Rule.*

*[NB **a* x *b = 0 iff a** and **b** are parallel]*

Advanced Higher Revision Notes

Vector form of the equation of a line

$$\underline{r} = \underline{a} + \lambda \underline{d}$$

where $\underline{a} = \overrightarrow{OP}$, $\underline{d}$ *is a vector parallel to the required line and* λ *is a real number.*

i.e. the vector equation is $\underline{r} = (\underline{a}_i + \underline{a}_j + \underline{a}_k) + \lambda(\underline{d}_i + \underline{d}_j + \underline{d}_k)$

Parametric form of the equation of a line

The parametric equations of a line through the point $P = (a_1, \quad a_2, \quad a_3)$ *with direction* $\underline{d} = d_1 i + d_2 j + d_3 k$ *are*

$x = a_1 + \lambda d_1$, $\quad y = a_2 + \lambda d_2$, $\quad z = a_3 + \lambda d_3$

where λ *is a real number.*

Symmetric form of the equation of a line

If $\quad x = a_1 + \lambda d_1$, $\quad y = a_2 + \lambda d_2$, $\quad z = a_3 + \lambda d_3$

are parametric equations of a line,

the symmetric equation of the line is:

$$\frac{x - a_1}{d_1} = \lambda \ , \qquad \frac{y - a_2}{d_2} = \lambda \ , \qquad \frac{z - a_3}{d_3} = \lambda$$

These symmetric equations are also known as the CARTESIAN EQUATIONS OF A LINE.

- *Note that if any of the* denominators are zero *then the corresponding numerator is also zero.*

This means the vector is PARALLEL to an axis.

- *Note that if a denominator is* 1, *the form of equations requires that it should be left there.*
- *If* 2 *lines are parallel their direction ratios are proportional*

Advanced Higher Revision Notes

Vector form of the equation of a line

If you are given the position vector, ***a*** & the direction***, d*** ➔ ***r = a + λd***

If you are **NOT** told the direction, but are given 2 points e.g. A(1, 2, 3) & B(4, 5, 6) , find the direction by finding the directed line segment $\overrightarrow{\mathbf{AB}}$:

A = (1, -2, 3) & B(4, 5,-7)

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 4 \\ 5 \\ -7 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ -10 \end{pmatrix} = \underline{d}$$

Vector Equation of line

$$r = \underline{a} + \lambda \underline{d}$$

$$\underline{r} = (i - 2j + 3k) + \lambda(3i + 7j - 10k)$$

Parametric form of the equation of a line

$$x = 1 + 3\lambda \ , \qquad y = -2 + 7\lambda \ , \qquad z = 3 - 10\lambda$$

Symmetric form of the equation of a line

$$If \quad \lambda = \frac{x-1}{3} \quad = \frac{y+2}{7} \quad = \frac{z-3}{-10}$$

Advanced Higher Revision Notes

Point of Intersection between 2 lines

- *Lines must be in parametric form first ($x = a + \lambda d$, etc)*
- *Use λ for L_1 & μ for L_2*
- *Set your x, y and z parametrics equal to each other*
- *Use simplest equation & rearrange so $\lambda =$ to form of μ*
- *Then use with other equations to solve for λ & μ*
- *If when replacing these into initial parametrics both lines have exactly the same values for x, y & z then the lines intersect at this point, otherwise they don't intersect (see example)*

Ex: *Find the point of intersection between the two lines.*

$$L_1:\ \frac{x-2}{1} = \frac{y+2}{3} = \frac{z+1}{5}$$
$$L_1: x = 2+\lambda;\ y = -2+3\lambda;\ z = -1+5\lambda$$
$$L_2: x = 1+\mu;\ y = -1+\mu;\ z = 2+\mu$$
$$x \Rightarrow 2+\lambda = 1+\mu$$
$$y \Rightarrow -2+3\lambda = -1+\mu$$
$$z \Rightarrow -1+5\lambda = 2+\mu$$

Taking the parametrics for x we have

$$2+\lambda = 1+\mu$$
$$\lambda = -1+\mu$$

Taking the parametrics for y & substituting for λ:

$$-2+3\lambda = -1+\mu$$
$$-2+3(-1+\mu) = -1+\mu$$
$$-2-3+3\mu = -1+\mu$$
$$2\mu = 4$$
$$\underline{\underline{\mu = 2}}$$

We can now equate these values by substitution

$$\mu = 2 \quad \& \quad \lambda = -1+\mu \quad \Rightarrow \lambda = -1+2$$
$$\underline{\underline{\lambda = 1}}$$

$\underline{\mu = 2 \quad \& \quad \lambda = 1:}$

$$x \Rightarrow 2+\lambda = 2+1 = 3 \qquad x \Rightarrow 1+\mu = 1+2 = 3$$
$$y \Rightarrow -2+3\lambda = -2+3 = 1 \qquad y \Rightarrow -1+\mu = -1+2 = 1$$
$$z \Rightarrow -1+5\lambda = -1+5 = 4 \qquad z \Rightarrow 2+\mu = 2+2 = 4$$

As both lines result in same values the point of intersection is therefore (3,1,4)

Advanced Higher Revision Notes

Angle between 2 lines

If we have the symmetric form we may obtain both directions and find the angle between the 2 lines using:

$$\cos\theta = \frac{\underline{d_1}.\underline{d_2}}{|d_1||d_2|}$$

Equation of a plane

$$\underline{r}.\underline{n} = \underline{a}.\underline{n}$$

n is the normal (perpendicular) to plane, ***a*** is position vector

Ex: Find the **Vector Equation** of a Plane throug **(-1, 2, 1)** with normal $\underline{n} = \underline{i} - \underline{3j} + \underline{2k}$

$\underline{r}.\underline{n} = \underline{a}.\underline{n}$ ➔

$$Let \quad \underline{a} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \; ; \; \underline{n} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$Then \; \underline{r} \bullet \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = -1 - 6 + 2 = -5$$

$$\Rightarrow \quad \underline{r} \bullet (i - 3j + 2k) = -5$$

The Cartesian Equation *is similar except x, y & z are used and the normal is represented by the values in front of x, y & z*

$$\Rightarrow \quad x - 3y + 2z = -5$$

Advanced Higher Revision Notes

Using 3 points to find the Equation of a Plane

If the normal is NOT given we must find it by using the vector product.

E.g. Find Cartesian equation of a plane given the points
A(1, 2, 1); B(-1, 0, 3) & C(0, 5, -1)

$$Let \quad \underline{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \;;\; \underline{b} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \;;\; \underline{c} = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix}$$

$$\overrightarrow{AB} = b - a = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AC} = c - a = \begin{pmatrix} 0 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

$$Then \;\; \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} i & j & k \\ -2 & -2 & 2 \\ -1 & 3 & -2 \end{pmatrix} = (4-6)i - (4-(-2))j + (-6-2)k$$

$$\Rightarrow \overrightarrow{AB} \times \overrightarrow{AC} = -2\underline{i} - 6\underline{j} - 8\underline{k}$$

$$\Rightarrow \underline{n} = \underline{i} + 3\underline{j} + 4\underline{k} \;\; \textit{as normal is represented as a multiple of this}$$

$$Then \quad r.n = a.n \Rightarrow r \bullet \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} = 1 + 6 + 4 = 11$$

$$\Rightarrow \quad \underline{r} \bullet (i + 3j + 4k) = 11$$

<u>The Cartesian Equation</u> $\Rightarrow \quad x + 3y + 4z = 11$

<u>*Advanced Higher Revision Notes*</u>

<u>Vectors: Plane & Planes</u>

The angle between 2 planes can be found by first finding both normals:

Eg P_1: $2x + 3y - 5z = 1$ ➔ $\underline{n} = (2, 3, -5)$
P_2: $3x - 4y + 7z = 2$ ➔ $\underline{m} = (3, -4, 7)$

<u>Angle between 2 planes:</u> $$\cos\theta = \frac{\underline{n}\bullet\underline{m}}{|\underline{n}|\,|\underline{m}|}$$

<u>Intersection of 3 planes: Gaussian Elimination/Algebraic Manipulation</u>

Easiest method is to use algebraic manipulation

E.g. *Given 2 planes* $\mathbf{4x + y - 2z = 3}$ *and* $\mathbf{x + y - z = 1}$ *, find the line of intersection if it exists.*

P_1: $4x + y - 2z = 3$
P_2: $x + y - z = 1$
$P_1 - P_2$: $3x - z = 2$

P_1: $4x + y - 2z = 3$
$2P_2$: $2x + 2y - 2z = 2$
$P_1 - 2P_2$: $2x - y = 1$

Represent x in terms of y, and separately in terms of x to obtain the symmetric equation of a line

$$3x = z + 2 \qquad x = \frac{z+2}{3}$$

$$2x = y + 1 \qquad x = \frac{y+1}{2}$$

$$\text{Let } x = \lambda \;\Rightarrow\; \lambda = \frac{x-0}{1} = \frac{y+1}{2} = \frac{z+2}{3}$$

<u>3 possible Solutions when investigating 3 planes intersecting:</u>

- ***Intersect at a point ➔ No lines parallel so obtain an exact solution***
- ***Infinite solutions on a line 0 0 0 | 0 ➔ equations cancel as parallel planes and equal (coincident) remaining 2 lines can be rearranged to find the line of intersection, with infinite solutions existing***
- ***2 parallel and unequal, therefore not intersection 0 0 0 | k***

Advanced Higher Revision Notes

Vectors: Lines & Planes

The angle between a line and a plane can be found by first finding the direction of the line and the normal to the plane:

Eg P_1: $2x + 3y - 5z = 1$ ➔ $\underline{n} = (2, 3, -5)$

L_1: $\frac{x-4}{1} = \frac{y+7}{-2} = \frac{z+2}{3}$ ➔ $\underline{d} = (1, -2, 3)$

Angle between the line and plane:	$\cos\theta = \frac{\underline{n}.\underline{d}}{\lvert\underline{n}\rvert\,\lvert\underline{d}\rvert}$

Intersection of a Line and a Plane

4 Main Steps

- ***Change line into parametric form if not given***
- ***Substitute these for x, y & z into the plane***
- ***Solve for λ***
- ***Substitute this value for λ back into parametric to find point***

L_1: $\frac{x-1}{1} = \frac{y-2}{1} = \frac{z-3}{2}$ ➔	$x = 1 + \lambda;\quad y = 2 + \lambda;\quad z = 3 + 2\lambda$
P_1: $x - y - 2z = -15$	

$$x - y - 2z = -15$$
$$(1 + \lambda) - (2 + \lambda) - 2(3 + 2\lambda) = -15$$
$$1 + \lambda - 2 - \lambda - 6 - 4\lambda = -15$$
$$-7 - 4\lambda = -15$$
$$-4\lambda = -8$$
$$\Rightarrow \lambda = 2$$

$x = 1 + \lambda = 1 + 2 = 3$
$y = 2 + \lambda = 2 + 2 = 4$
$z = 3 + 2\lambda = 3 + 4 = 7$ ➔ ***Line & Plane intersect at (3, 4, 7)***

12 Matrix Algebra

Matrix Laws/Properties

1. **Addition Law** *(**Need same order for this to work**)*

Property 1 → A + B = B + A

2. **Commutative Law**

Given $A_{(r_1 \times c_1)}$ & $B_{(r_2 \times c_2)}$

Can only multiply if $c_1 = r_2$

i.e. $(3\times 2)\times(2\times 4) = (3\times 4)$ matrix solution.

but $(2\times 4)\times(3\times 2) \neq$ possible solution as $c_1 \neq r_2$

Property 2 → AB ≠ BA

3. **Assosciative Law** ***(**Need to comply with multiplication rules**)***

Property 3 → ABC = A(BC) = (AB)C

4. **Distributive Law**

Property 4 → A(B + C) = AB + AC

5. Transpose Law

A^{T} *or* A' Transpose

This is when we interchange rows and columns

i.e. $\begin{pmatrix} 1 & 4 & 6 & 3 \\ 2 & 5 & -1 & -5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 \\ 4 & 5 \\ 6 & -1 \\ 3 & -5 \end{pmatrix}$

$\left(A^{T}\right)^{T} = A \quad or \quad \left(A'\right)' = A$

$As, \quad r \rightarrow c \rightarrow r \quad \& \quad c \rightarrow r \rightarrow c$

Property 7 $\quad AB^{T} = B^{T}A^{T}$

Property 8 $\quad \left(A+B\right)^{T} = A^{T} + B^{T}$

Property 9 $\quad \left(AB\right)^{-1} = B^{-1}A^{-1}$

Using this we can then show,

$$\begin{aligned} (AB)(AB)^{-1} &= AB\left(B^{-1}A^{-1}\right) \\ &= A\left(BB^{-1}\right)A^{-1} \\ &= A(I)A^{-1} \\ &= AA^{-1} \\ &= I \end{aligned}$$

As $AA^{-1} = I$, similarly $(AB)(AB)^{-1}$ *should* $= I$

Transformations Matrices - 4 TO KNOW!!!

Advanced Higher Revision Notes

Rotation (Anti-Clockwise)	$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Reflection (-90 $\leq \theta \leq$ 90)	$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$

Dilation (Scaling) for x & y values only	$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

General Transformation ($\times 2$ Sim Eqns)	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Advanced Higher Revision Notes

Determinant and Inverse Matrices

If we wish to find the inverse matrix we must first obtain the determinant of the function often called ***det(A)***

$$If \quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ then the determinant of A is called

$$\det(A)=\frac{1}{ad-bc}$$ in any 2×2 matrix

It is a little more complicated for a 3×3 matrix

$$A=\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}=\begin{pmatrix} \boxed{a} & \cdots & \cdots \\ \cdots & e & f \\ \cdots & h & i \end{pmatrix}-\begin{pmatrix} \cdots & \boxed{b} & \cdots \\ d & \cdots & f \\ g & \cdots & i \end{pmatrix}+\begin{pmatrix} \cdots & \cdots & \boxed{c} \\ d & e & \cdots \\ g & h & \cdots \end{pmatrix}$$

$$\det(A)=\frac{1}{a(ei-fh)-b(di-fg)+c(dh-eg)}$$

Once we know how to find the determinant we can easily find the inverse of the 2×2 matrix as follows

$$If \quad A=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \; ; \quad \det(A)=\frac{1}{ad-bc}$$ then inverse is

$$A^{-1}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}=\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}=\frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Again it is a little more complicated for a 3×3 matrix

Better to set up a EROs problem with $\left[\mathrm{A}\middle|\mathrm{I}\right]$ and use Gaussian Elimination to change the LHS into I, $\left[\mathrm{I}\middle|\mathrm{A}^{-1}\right]$

The RHS will then be the Inverse of the matrix.

$$\left[\mathrm{A}\middle|\mathrm{I}\right]=\left(\begin{array}{ccc|c} a & b & c & 1\,0\,0 \\ d & e & f & 0\,1\,0 \\ g & h & i & 0\,0\,1 \end{array}\right)\rightarrow\left[\mathrm{I}\middle|\mathrm{A}^{-1}\right]$$

Lastly

When the transpose equals the inverse the matrix is Orthogonal.

$A^{\mathrm{T}} = A^{-1} \Rightarrow \det A=\pm 1$

Advanced Higher Revision Notes

Further Ordinary Differential Equations

First Order Differential Equations ($^{dy}/_{dx}$ only)

$\frac{dy}{dx}+P(x)y=f(x)$	Rearrange into this format to determine P(x)
$I(x)=e^{\int P(x)dx}$	Use P(x) to find the Integrating Factor,
Then,	(**no constant here, leave 'c' until end**)
$I(x)y=\int I(x)f(x)dx$	Then rearrange to find y, and ensure when integrating at this stage to include constant, c.

Second Order Differential Equations

Homogenous 2^{nd} Order Diff Eqns => RHS = 0

By using this property we can find the **Auxiliary Equation** and solve it to find what is known as the **Complimentary Function**

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ Is homogeneous format used to find Auxiliary Equation.

$am^2+bm+c=0$ Substituting m^2; m & c in place of $\frac{d^2y}{dx^2};\frac{dy}{dx}$ & y

Then factorise (may require the quadratic formula)

$$m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

There are 3 Complementary Solutions, (y_c) possible

- $b^2-4ac>0 \Rightarrow$ 2 real distinct roots say m_1 & m_2

Then solution is of the format:

$$y=Ae^{m_1}+Be^{m_2}$$

- $b^2-4ac=0 \Rightarrow$ A repeated root, say m

Then solution is of the format:

$$y=Ae^{m}+Bxe^{m}$$

- $b^2-4ac<0 \Rightarrow$ 2 complex roots, say $(p\pm iq)$

Then solution is of the format:

$$y=e^{px}\left(A\cos(qx)+BSin(qx)\right)$$

Note here that the value of q is taken only, the sign and *i* are ignored.

Advanced Higher Revision Notes

Non-Homogenous 2nd Order Differential Equations → Find a Particular Integral (Need the Complimentary Function & a Particular Integral)

$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = f(x)$ Is Non-homogeneous $(\neq 0)$.

Depending on the RHS value for $f(x)$ will determine what we shall set y=?

- If is a numerical value i.e. $f(x) = 3$, then we shall set $y = a$
- If it is a linear function i.e. $f(x) = 2x + 5$, then we shall set $y = ax+b$
- If it is a quadratic function i.e. $f(x) = 5x^2 + 3x - 4$, then we shall set $y = ax^2+bx+c$
- If it is a cubic function i.e. $f(x) = 2x^3 + 7x - 8$, then we shall set $y = ax^3+bx^2+cx+d$
- If it is an exponential function i.e. $f(x) = e^{2x}$, then we shall set $y = ke^{2x}$,
  In general for any exponential value, say r then $y = ke^{rx}$
- If is a trig function i.e. $f(x) = 2\cos(3x)$, then we shall set $y = p$
- If it is a combination of any two or more we treat each separately i.e. $f(x)+g(x)$

When the required value of y has been chosen we then carry out <u>*Second Order Differentiation*</u>:

$y = k[\,f(x)\,]$ By differentiating twice we can then substitute

$\frac{dy}{dx} = ?$ into the LHS for values of

$\frac{d^2y}{dx^2} = ?$ $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy = k[\,f(x)]$

After substituting, it is then possible to rearrange and solve the unknowns on the RHS

This process finds the Particular Integral, (y_p)

The final solution to the 2^{nd} *O.D.E. problem consists of combining the homogeneous solution with the non-homogeneous*

i.e $y = y_c + y_p$

<u>CARE with the Exponential functions</u>

If the Auxiliary Equation has similar roots to that of the RHS f(x) value we must make additional steps:

- If there is a single root of the auxiliary equation (m_1 or m_2) which resembles $f(x) = ke^{rx} \Rightarrow y_p = kxe^{rx}$
- If there is a repeated root of the auxiliary equation (m) which resembles $f(x) = ke^{rx} \Rightarrow y_p = kx^2e^{rx}$