5. DIFFERENTIATION

5.1 BASIC DERIVATIVES

(a) $\frac{dy}{dx} = 8x - 9$

(b) $f(x) = 5 - 4x^3 + x^{\frac{1}{2}}$

$f'(x) = -12x^2 + \frac{1}{2}x^{-\frac{1}{2}}$

$= -12x^2 + \frac{1}{2\sqrt{x}}$

$f'(4) = -12(4)^2 + \frac{1}{2\sqrt{4}}$

$= -191\frac{3}{4} \quad \left(-\frac{767}{4}\right)$

(c) $f(x) = 12x^3 - 5x^{-\frac{1}{3}}$

$f'(x) = 36x^2 + \frac{5}{3}x^{-\frac{4}{3}}$

$= 36x^2 + \frac{5}{3\sqrt[3]{x^4}}$

$f'(-1) = 36(-1)^2 + \frac{5}{3\sqrt[3]{(-1)^4}}$

$= 36 + \frac{5}{3}$

$= 37\frac{2}{3} \quad \left(\frac{113}{3}\right)$

5.2 PRODUCTS AND QUOTIENTS

(a) $y = 10x - 2x^{\frac{3}{2}}$

$\frac{dy}{dx} = 10 - 3x^{\frac{1}{2}}$

$= 10 - 3\sqrt{x}$

(b) $f(x) = 3x^{-1} - x^{-2}$

$f'(x) = -3x^{-2} + 2x^{-3}$

$= -\frac{3}{x^2} + \frac{2}{x^3}$

$f'(4) = -\frac{3}{4^2} + \frac{2}{4^3}$

$= -\frac{5}{32}$

(c) $f(x) = x^{\frac{3}{2}} - 5x^{\frac{1}{2}}$

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - \frac{5}{2}x^{-\frac{1}{2}}$

$= \frac{3}{2}\sqrt{x} - \frac{5}{2\sqrt{x}}$

$f'(4) = \frac{3}{2}\sqrt{4} - \frac{5}{2\sqrt{4}}$

$= 3 - \frac{5}{4}$

$= \underline{\underline{1\frac{3}{4}}} \quad \left(\frac{7}{4}\right)$

5.3 RATES OF CHANGE

(a)(i) $v(t) = h'(t)$

$= 15 - 4t$

$v(2) = 15 - 4(2)$

$= \underline{\underline{7 \text{ ms}^{-1}}}$

(ii) $v(5) = 15 - 4(5)$

$= \underline{\underline{-5}}$

$\because v(5) < 0$, the ball's height is decreasing.

(b) $v(t) = \frac{1}{4}t^2 + 2t + 1$

$a(t) = v'(t)$

$= \frac{1}{2}t + 2$

$a(5) = \frac{1}{2}(5) + 2$

$= \underline{\underline{4.5 \text{ ms}^{-2}}}$

5.4 EQUATIONS OF TANGENTS

(a) Grad

$\frac{dy}{dx} = 8x - 9$

$m = 8(-2) - 9$

$= -25$

Point

$y = 4(-2)^2 - 9(-2) - 3$

$= 16 + 18 - 3$

$= 31$

$(-2, 31)$

$y - 31 = -25(x + 2)$

$\underline{\underline{y + 25x = -19}}$

(b) Grad

$y = 6 - 5x^{\frac{1}{2}}$

$\frac{dy}{dx} = -\frac{5}{2}x^{-\frac{1}{2}}$

$= -\frac{5}{2\sqrt{x}}$

$m = -\frac{5}{2\sqrt{4}}$

$= -\frac{5}{4}$

Point

$y = 6 - 5\sqrt{4}$

$= -4$

$(4, -4)$

$y + 4 = -\frac{5}{4}(x - 4)$

$\underline{\underline{4y + 5x = 16}}$

(c) Grad

$\frac{dy}{dx} = 12x - 10$

$2 = 12x - 10$

$12 = 12x$

$x = 1$

$m = 2$

Point

$x = 1$

$y = 6(1)^2 - 10(1)$

$= -4$

$(1, -4)$

$y + 4 = 2(x - 1)$

$\underline{\underline{y - 2x = -6}}$

5.5 STATIONARY POINTS

5.5.1 BASIC STATIONARY POINTS

(a) $y = 2x^3 - 6x^2 + 5$

$$\frac{dy}{dx} = 6x^2 - 12x$$

for SPs $\frac{dy}{dx} = 0$

$$0 = 6x^2 - 12x$$

$$0 = 6x(x - 2)$$

$x = 0$

$y = 2(0)^3 - 6(0)^2 + 5$
$= 5$

$\underline{(0, 5)}$

$x = 2$

$y = 2(2)^3 - 6(2)^2 + 5$
$= -3$

$\underline{(2, -3)}$

x	-1	0	1
$\frac{dy}{dx}$	18	0	-6
slope	/	—	\

x	1	2	3
$\frac{dy}{dx}$	-6	0	18
slope	\	—	/

$\therefore$ $(0, 5)$ is a maximum turning point

$(2, -3)$ is a minimum turning point.

(b) $y = 2x^3 - 3x^2 - 12x + 4$

$$\frac{dy}{dx} = 6x^2 - 6x - 12$$

for SPs $\frac{dy}{dx} = 0$

$$0 = 6x^2 - 6x - 12$$

$$0 = 6(x^2 - x - 2)$$

$$0 = 6(x - 2)(x + 1)$$

$x = 2$ $\quad$ $x = -1$

$\underline{(2, -16)}$ $\quad$ $\underline{(-1, 11)}$

x	1	2	3
$\frac{dy}{dx}$	-12	0	24
slope	\	—	/

x	-2	-1	0
$\frac{dy}{dx}$	24	0	-12
slope	/	—	\

$\therefore$ $(2, -16)$ is a minimum turning point

$(-1, 11)$ is a maximum turning point.

5.5.2 INCREASING AND DECREASING

(a) From 5.5.1 (a) . . . $x = 0$, $x = 2$

x	-1	0	1	2	3
$\frac{dy}{dx}$	18	0	-6	0	18
Slope	/	—	\	—	/

increasing where $x < 0$, $x > 2$

(b) From 5.5.1 (b) . . . $x = -1$, $x = 2$

x	-2	-1	0	2	3
$\frac{dy}{dx}$	24	0	-12	0	24
Slope	/	—	\	—	/

increasing where $x < -1$, $x > 3$

decreasing where $-1 < x < 2$

5.5.3 CLOSED INTERVALS

(a) $y = x^3 - 5x^2 + 3x + 1$

$\frac{dy}{dx} = 3x^2 - 10x + 3$

for SPs $\frac{dy}{dx} = 0$

$0 = 3x^2 - 10x + 3$

$0 = (3x - 1)(x - 3)$

$\downarrow \qquad \downarrow$

$x = \frac{1}{3} \qquad \cancel{x = 3}$

outwith interval

End Pt	SP	End Pt.
$x = -1$	$x = \frac{1}{3}$	$x = 2$
$y = -8$	$y = \frac{40}{27}$	$y = -5$
	$(y \approx 1{\cdot}48)$	

max value $= \frac{40}{27}$ (when $x = \frac{1}{3}$)

min value $= -8$ (when $x = -1$)

(b) $y = x^4 + 8x^3 - 32x^2 + 1$

$\frac{dy}{dx} = 4x^3 + 24x^2 - 64x$

for SPs $\frac{dy}{dx} = 0$

$0 = 4x(x^2 + 6x - 16)$

$= 4x(x + 8)(x - 2)$

$\downarrow \qquad \downarrow \qquad \downarrow$

$x = 0 \qquad \cancel{x = -8} \qquad x = 2$

outwith interval

End Pt	SP	SP	End Pt
$x = -5$	$x = 0$	$x = 2$	$x = 5$
$y = -1174$	$y = 1$	$y = -47$	$y = 826$

max value $= 826$ $(x = 5)$

min value $= -1174$ $(x = -5)$

5.5.4 OPTIMISATION PART (B)

(a) $V(x) = 2x + 20000x^{-1}$

$V'(x) = 2 - 20000x^{-2}$

$= 2 - \dfrac{20000}{x^2}$

for SPs $V'(x) = 0$

$0 = 2 - \dfrac{20000}{x^2}$

$\dfrac{20000}{x^2} = 2$

$20000 = 2x^2$

$10000 = x^2$

$\pm 100 = x$

$\because x > 0, \; x = 100$

x	99	100	101
$V'(x)$	-0.04	0	0.04
slope	\	—	/

$\therefore x = 100$ minimises the volume.

(b) $P = -x^3 + 6x^2 + 1440x - 800$

$P' = -3x^2 + 12x + 1440$

for SPs $P' = 0$

$0 = -3(x^2 - 4x - 480)$

$0 = -3(x - 24)(x + 20)$

$x = 24$ $\quad$ $\cancel{x = -20}$

$\because x > 0, \; x = 24$

x	23	24	25
P'	129	0	-135
slope	/	—	\

$x = 24$ maximises profit.

The company should produce 24 units.

5.6 ALWAYS/NEVER INCREASING/DECREASING

(a) $y = x^3 + x$

$\frac{dy}{dx} = 3x^2 + 1$

$\because x^2 \geq 0$

$3x^2 + 1 > 0$

$\therefore \frac{dy}{dx} > 0$

$\therefore$ Curve is always increasing □

(b) $y = 8 - 5x - 2x^3$

$\frac{dy}{dx} = -5 - 6x^2$

$\because x^2 \geq 0$

$-6x^2 \leq 0$

$-6x^2 - 5 < 0$

$\therefore \frac{dy}{dx} < 0$

$\therefore$ curve is always decreasing □

(c) $y = 3x^3 + 3x^2 + x + 5$

$\frac{dy}{dx} = 9x^2 + 6x + 1$

$= 9\left[x^2 + \frac{6}{9}x\right] + 1$

$= 9\left[x^2 + \frac{2}{3}x\right] + 1$

$= 9\left[\left(x + \frac{1}{3}\right)^2 - \frac{1}{9}\right] + 1$

$= 9\left(x + \frac{1}{3}\right)^2 - 1 + 1$

$= 9\left(x + \frac{1}{3}\right)^2$

$\because \left(x + \frac{1}{3}\right)^2 \geq 0$

$9\left(x + \frac{1}{3}\right)^2 \geq 0$

$\therefore \frac{dy}{dx} \geq 0$

So curve is never decreasing. □

(d) $y = \frac{1}{3}x^3 - 2x^2 + 6x - 4$

$$\frac{dy}{dx} = x^2 - 4x + 6$$

$$= (x-2)^2 - 4 + 6$$

$$= (x-2)^2 + 2$$

$$\because (x-2)^2 \geq 0$$

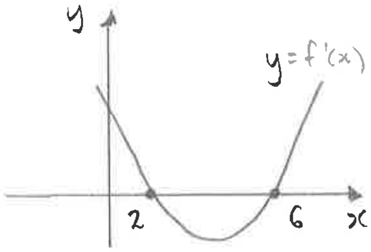
$$(x-2)^2 + 2 > 0$$

$$\therefore \frac{dy}{dx} > 0$$

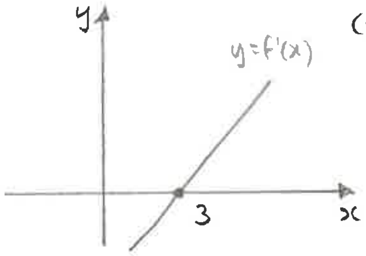
$\therefore$ curve is always increasing. ◻

5.7 DERIVATIVE GRAPHS

(a)



(b)



(c)