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2019

AH Mechanics

Worked Solutions

Courtesy of Mr M Pitman

2019

① $m = 4\text{kg}$

$u = 3\underline{i} + 2\underline{j} \text{ ms}^{-1}$

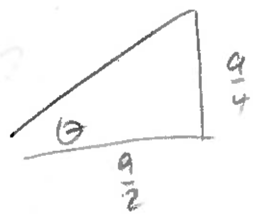
$I = 6\underline{i} + \underline{j} \text{ Ns}$

initial mom $= 12\underline{i} + 8\underline{j} + 6\underline{i} + \underline{j}$

$= 18\underline{i} + 9\underline{j}$

Find $v = \frac{18}{4}\underline{i} + \frac{9}{4}\underline{j} = \frac{9}{2}\underline{i} + \frac{9}{4}\underline{j}$

$|v| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{9}{4}\right)^2} = \frac{9\sqrt{5}}{4} = 5.03 \text{ ms}^{-1}$



$\theta = \tan^{-1}\left(\frac{\frac{9}{4}}{\frac{9}{2}}\right) = \tan^{-1}\frac{1}{2} = 26.565$

$= 26.6°$ above x axis

② a) $f(x) = xe^{-3x}$

$f'(x) = 1e^{-3x} + x \cdot -3e^{-3x}$

$= (1-3x)e^{-3x}$

$f'(-1) = (1-3(-1))e^{-3(-1)}$

$= 4e^3$

b) $g(t) = \frac{3t}{(2t+1)^2}$

$g'(t) = \frac{3(2t+1)^2 - 3t \cdot 2(2t+1) \cdot 2}{(2t+1)^4}$

$= \frac{3(2t+1)^2 - 12t\cancel{(2t+1)}}{(2t+1)^{\cancel{4}\,3}}$

$= \frac{6t+3-12t}{(2t+1)^3}$

$= \frac{3-6t}{(2t+1)^3}$

(3) $v(t) = 4t\underline{i} + (t+1)\underline{j}$ ms^{-1} Range 80m.

$s(t) = \int v(t)dt = 4t\underline{i} + (\frac{1}{2}t^2 + t)\underline{j} + C$

$t=0$ $s(0) = 0$

$s=0$ $0 = 0\underline{i} + 0\underline{j} + C$ $\therefore C = 0$.

$s(t) = 4t\underline{i} + (\frac{1}{2}t^2 + t)\underline{j}$

$s(10) = 40\underline{i} + 60\underline{j}$

$|s(10)| = \sqrt{40^2 + 60^2} = 20\sqrt{13} = \underline{\underline{72.1 \text{ m}}}$

Yes it will still be in range in 10seconds.

(4) $V_{max} = 15 ms^{-1}$ $v(2)$

$a_{max} = 60 ms^{-1}$

$V_{max} = \omega A$ $a_{max} = \omega^2 A$

$15 = \omega A$ $60 = \omega^2 A$

$A = \frac{15}{\omega}$ $60 = \omega^2 \cdot \frac{15}{\omega} = 15\omega \Rightarrow \underline{\underline{\omega = 4}}$

$A = \frac{15}{4}$

$v^2 = \omega^2(A^2 - x^2)$ $V = A\omega \cos \omega t$

$t=2$ $V = \frac{15}{4} \cdot 4 \cdot \cos 4(2)$

$V = 15 \cos 8.$

$V = -2.18 ms^{-1}$

The particle is travelling in the opposite direction to the original movement.

(5) $$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$$

$$m^2 - 3x + 2 = 0$$

$$(m-2)(m-1) = 0$$

$$m = 2 \quad m = 1$$

$$y = Ae^{2x} + Be^{x} \qquad \frac{dy}{dx} = 2Ae^{2x} + Be^{x}$$

$y = 1$ $\quad 1 = Ae^0 + Be^0 \qquad 3 = 2Ae^0 + Be^0$

$x = 0$

$\frac{dy}{dx} = 3$ $\quad A + B = 1$ ① $\qquad$ ② - ①

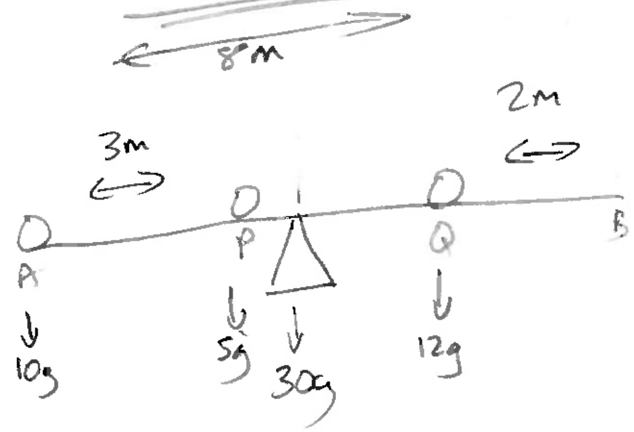
$2A + B = 3$ ②

$A = 2$

$B = -1$

$$\underline{\underline{y = 2e^{2x} - e^{x}}}$$

(6)



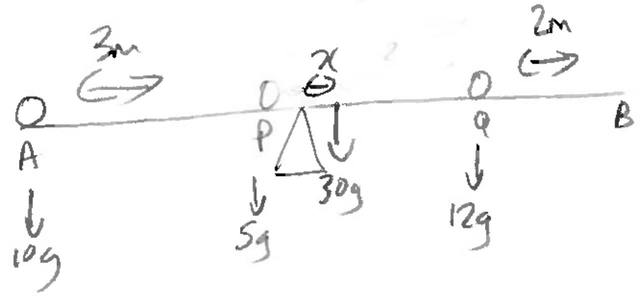
$$\curvearrowright\ (-10g \times 4) + (-5g \times 1) + (12g \times 2)$$

$$= -40g - 5g + 24g$$

$$= -21g$$

$\therefore$ 21g in anticlockwise direction.

b)



$$\circlearrowright = 0 \quad = (-10g \times (4-x)) + (-5g \times (1-x)) + (30g \times x) + (12g \times (2+x))$$

$$= -40g + 10gx - 5g + 5gx + 30gx + 24g + 12gx$$

$$= -21g + 57gx$$

$$57gx = 21g$$

$$x = \frac{21}{57} \text{ m}$$

(7) $f(t) = \ln(\sec 2t + \tan 2t)$

$$f'(t) = \frac{1}{\sec 2t + \tan 2t} \cdot (2\sec 2t \tan 2t + 2\sec^2 2t)$$

$$= \frac{2\left(\frac{1}{\cos 2t} \cdot \frac{\sin 2t}{\cos 2t} + \frac{1}{\cos^2 2t}\right)}{\frac{1}{\cos 2t} + \frac{\sin 2t}{\cos 2t}}$$

$$= \frac{2\left(\frac{\sin 2t}{\cos^2 2t} + \frac{1}{\cos^2 2t}\right)}{\frac{1 + \sin 2t}{\cos 2t}}$$

$$= \frac{2\left(\frac{\sin 2t + 1}{\cos^2 2t}\right)}{\frac{1 + \sin 2t}{\cos 2t}} = 2\left(\frac{1}{\cos 2t}\right)$$

$$= \underline{\underline{2 \sec 2t}}.$$

(8) $a = 2t\sqrt{2t+1} \quad ms^{-2}$

$v \int a\,dt = \int 2t(2t+1)^{\frac{1}{2}}\,dt$

$\int uv' = uv - \int u'v \qquad u = 2t \qquad v' = (2t+1)^{\frac{1}{2}}$

$u' = 2 \qquad v = \dfrac{(2t+1)^{\frac{3}{2}}}{\frac{3}{2} \times 2}$

$= \frac{1}{3}(2t+1)^{\frac{3}{2}}$

$= 2t \cdot \frac{1}{3}(2t+1)^{\frac{3}{2}} - \int 2 \cdot \frac{1}{3}(2t+1)^{\frac{3}{2}}\,dt$

$= \frac{2}{3}t(2t+1)^{\frac{3}{2}} - \frac{2}{3}\left(\dfrac{(2t+1)^{\frac{5}{2}}}{\frac{5}{2} \times 2}\right) + C$

$v = \frac{2}{3}t(2t+1)^{\frac{3}{2}} - \frac{2}{15}(2t+1)^{\frac{5}{2}} + C$

$t = 0 \quad v = 0$

$0 = 0 - \frac{2}{15} + C \qquad C = \frac{2}{15}$

$v(t) = \frac{2}{3}t\sqrt{(2t+1)}^{3} - \frac{2}{15}\sqrt{(2t+1)}^{5} + \frac{2}{15}$

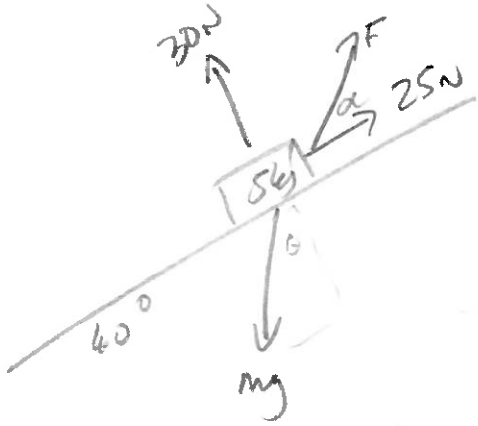
$v(4) = \frac{2}{3} \cdot 4\sqrt{(2(4)+1)}^{3} - \frac{2}{15}\sqrt{(2(4)+1)}^{5} + \frac{2}{15}$

$= \frac{8}{3} \cdot 27 - \frac{2}{15} \cdot 243 + \frac{2}{15}$

$= \dfrac{1080 - 486 + 2}{15}$

$= \dfrac{596}{15} = 39.7 \; ms^{-1}$

(9)



Perp. $5g\cos 40° = 30 + F\sin\alpha$

$F\sin\alpha = 5g\cos 40 - 30$

Parallel $5g\sin 40° = 25 + F\cos\alpha$

$F\cos\alpha = 5g\sin 40° - 25$

$$\frac{F\sin\alpha}{F\cos\alpha} = \tan\alpha = \frac{5g\cos 40 - 30}{5g\sin 40 - 25}$$

$$\underline{\underline{\alpha = 49.2°}}$$

$F\cos 49.2 = 5g\sin 40 - 25$

$$F = \frac{5g\sin 40 - 25}{\cos 49.2}$$

$F = 9.949\ldots$

$\underline{\underline{F = 9.95\,N}}$

(15) $3y + x^2 e^{2y} = 9$

$$3\frac{dy}{dx} + 2x e^{2y} + x^2 \cdot 2e^{2y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}\left(3 + 2x^2 e^{2y}\right) = -2x e^{2y}$$

$$\frac{dy}{dx} = \frac{-2xe^{2y}}{3 + 2x^2 e^{2y}}$$

$y = 0$ $\quad 0 + x^2 e^0 = 9$

$$x^2 = 9$$

$$x = \pm 3 \Rightarrow x = 3.$$

$$\frac{dy}{dx} = \frac{-2 \cdot 3 \cdot e^0}{3 + 2 \cdot 3^2 \cdot e^0}$$

$$= \frac{-6}{3 + 18}$$

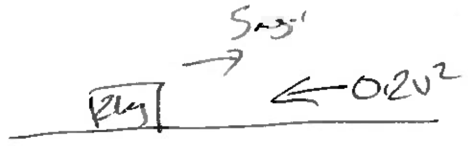
$$= \frac{-6}{21}$$

$$= -\frac{2}{7}$$

(11) $m = 2kg$

$V = 5ms^{-1}$

$F = 0.2v^2$



$$a = v\frac{dv}{dx} = \frac{-0.2v^2}{2} = -0.1v^2$$

$$\frac{dv}{dx} = -0.1v$$

$$\int \frac{1}{v}\,dv = \int -0.1\,dx$$

$$\ln v = -0.1x + C.$$

$t=0 \quad x=0 \quad v=5$

$$\ln 5 = 0 + C$$
$$C = \ln 5.$$

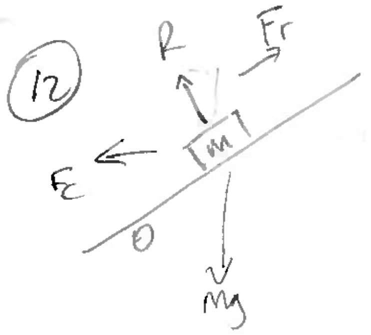
$$\ln v = -0.1x + \ln 5$$

$$\ln\left|\frac{v}{5}\right| = -0.1x$$

$$\frac{v}{5} = e^{-0.1x}$$

$$\underline{v = 5e^{-0.1x}}$$

(12)



$$V = \frac{\sqrt{gr}}{10}\ ms^{-1}$$

horizontally $F_c + R\sin\theta = \mu R\cos\theta$ vertically $mg = R\cos\theta + \mu R\sin\theta$

$$\frac{mv^2}{r} = R(\sin\theta - \mu\cos\theta) \qquad mg = R(\cos\theta + \mu\sin\theta)$$

$$\Rightarrow \frac{R(\sin\theta - \mu\cos\theta)}{R(\cos\theta + \mu\sin\theta)} = \frac{\frac{mv^2}{r}}{mg}$$

$$\frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta} = \frac{v^2}{rg}$$

$$= \frac{\left(\frac{\sqrt{gr}}{10}\right)^2}{rg}$$

$$= \frac{gr}{100rg}$$

$$= \frac{1}{100}$$

$$100(\sin\theta - \mu\cos\theta) = \cos\theta + \mu\sin\theta$$

$$\mu(\sin\theta + 100\cos\theta) = 100\sin\theta - \cos\theta$$

$$\mu = \frac{100\sin\theta - \cos\theta}{\sin\theta + 100\cos\theta}$$

$$\mu = \frac{100\frac{\sin\theta}{\cos\theta} - \frac{\cos\theta}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} + 100\frac{\cos\theta}{\cos\theta}}$$

$$\mu = \frac{100\tan\theta - 1}{\tan\theta + 100}$$ as required

b) $\theta = 25^\circ$ $r = 80km$. $v = 28ms^{-1}$

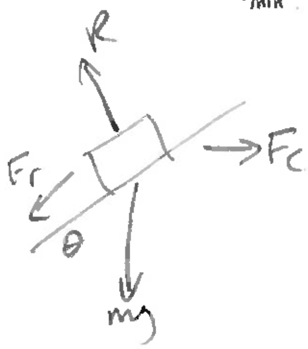
$$\mu = \frac{100\tan 25 - 1}{\tan 25 + 100} = 0.454$$

$$v^2 = rg \cdot \frac{\sin\theta - \mu\cos\theta}{\cos\theta + \mu\sin\theta}$$

$$= 80g \cdot \frac{\sin 25 - 0.454\cos 25}{\cos 25 + 0.454\sin 25}$$

$v^2 = 7.963$

$V_{min} = 2.82 \text{ ms}^{-1}$



For maximum speed

$$\mu R\cos\theta + R\sin\theta = \frac{mv^2}{r}$$

$$R(\mu\cos\theta + \sin\theta) = \frac{mv^2}{r}$$

$$R(\cos\theta - \mu\sin\theta) = mg$$

$$\frac{R(\mu\cos\theta + \sin\theta)}{R(\cos\theta - \mu\sin\theta)} = \frac{\frac{mv^2}{r}}{mg}$$

$$\frac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

for max $\quad v^2 = rg \cdot \dfrac{\mu\cos\theta + \sin\theta}{\cos\theta - \mu\sin\theta}$

$$v^2 = 80g \cdot \frac{0.454\cos 25 + \sin 25}{\cos 25 - \mu\sin 25}$$

$$v^2 = 915.29\ldots$$

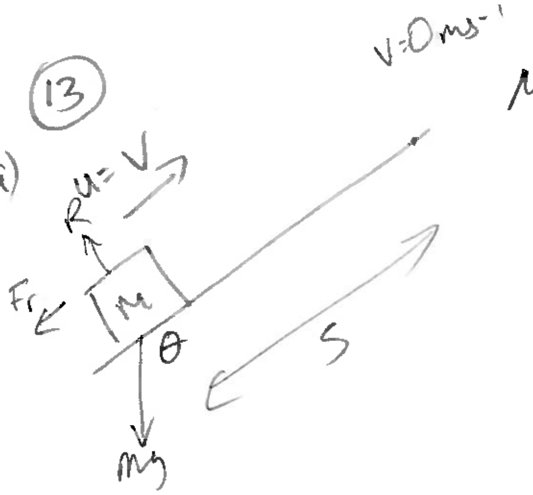
$$V_{max} = 30.25 \text{ ms}^{-1}$$

28 ms⁻¹ is between min + max velocity therefore will not slip.

c) The track is wet therefore reducing the friction.

(13)

a)



$\mu = \mu$

$$ma = -(mg\sin\theta + \mu mg\cos\theta)$$

$$a = -g\sin\theta - \mu g\cos\theta$$

$$v^2 = u^2 + 2as$$

$$0^2 = V^2 + 2\left(-g(\sin\theta + \mu\cos\theta)\right)s$$

$$-V^2 = -2g(\mu\cos\theta + \sin\theta)s$$

$$\underline{\underline{s = \frac{V^2}{2g(\mu\cos\theta + \sin\theta)}}} \quad \text{as required}$$

b) $WD = \frac{1}{8}mV^2$

$$= F_r \cdot d$$

$$= \mu mg\cos\theta \cdot s$$

$$\frac{1}{8}\cancel{m}\cancel{V^2} = \mu \cancel{m} g\cos\theta \cdot \frac{\cancel{V^2}}{2g(\mu\cos\theta + \sin\theta)}$$

$$\frac{1}{8} = \frac{\mu g\cos\theta}{2g(\mu\cos\theta + \sin\theta)}$$

$$2g\mu\cos\theta + 2g\sin\theta = 8\mu g\cos\theta$$

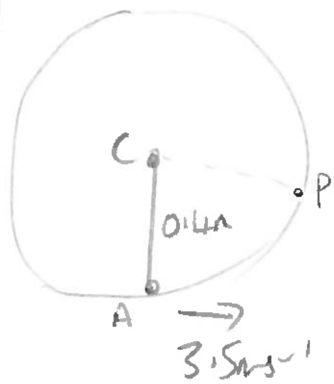
$$6\mu g\cos\theta = 2g\sin\theta$$

$$\mu = \frac{2g\sin\theta}{6g\cos\theta}$$

$$\underline{\underline{\mu = \frac{1}{3}\tan\theta}}$$

14

a)



At A $\quad E_k = \frac{1}{2}mu^2 \quad E_p = 0$.

$$= \frac{49}{8}m$$

At B $\quad E_k = 0 \quad E_p = mgh$

$$= mg(r - r\cos\theta)$$

$$= mgr(1-\cos\theta)$$

$$\cancel{m}gr(1-\cos\theta) = \frac{49}{8}\cancel{m}$$

$$g \cdot 0.4(1-\cos\theta) = \frac{49}{8}$$

$$1-\cos\theta = \frac{25}{16}$$

$$\cos\theta = 1 - \frac{25}{16}$$

$$\underline{\underline{\theta = 124.23^\circ}}$$

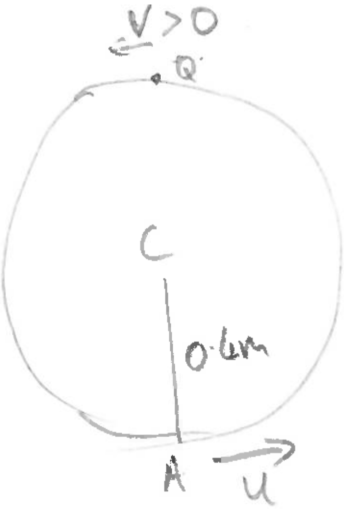
$$\cos\theta = \frac{x}{r}$$

$$x = r\cos\theta$$

$$h = r - x$$

$$h = r - r\cos\theta$$

b)



At Q $V > 0\ ms^{-1}$ to exit pipe

$\therefore$ E_k at A > E_p at Q

$$\frac{1}{2}mu^2 > mgh$$

$$\frac{1}{2}u^2 > 0.8g$$

$$u^2 > 1.6g$$

$$u > \sqrt{1.6g}$$

c) The ball and pipe are smooth so thee is no friction.

15

$S_u < 3m$.

At x $V_H = 0\ ms^{-1}$

$u_H = u\sin\theta$

$$v^2 = u^2 + 2as$$

$$0 = (u\sin\theta)^2 + 2(-g)s$$

$$S_V = \frac{u^2\sin^2\theta}{2g} < 3$$

$$\sin^2\theta < \frac{6g}{u^2}$$

$$\sin\theta < \sqrt{\frac{6g}{u^2}}$$

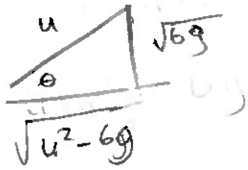
$$\sin\theta < \frac{\sqrt{6g}}{u}$$ as required

b) at x $v=0$

$$v = u + at$$
$$0 = u\sin\theta - gt$$
$$t_x = \frac{u\sin\theta}{g}$$

at x $s_v = 3$

$$\sin\theta = \frac{\sqrt{6g}}{u}$$



$$\cos\theta = \frac{\sqrt{u^2-6g}}{u}$$

$\therefore$ total time $t = \dfrac{2u\sin\theta}{g}$

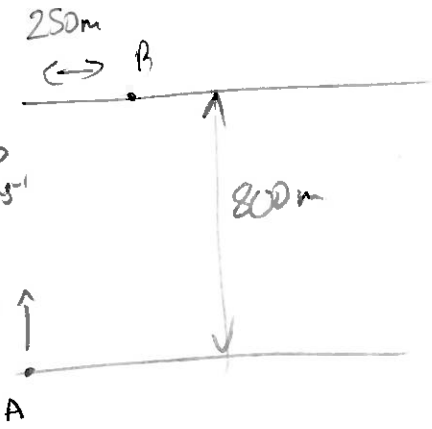
$$s_h = u_h t$$
$$s_x = u\cos\theta \cdot \frac{2u\sin\theta}{g}$$
$$s_x = \frac{2u^2\sin\theta\cos\theta}{g}$$
$$s_h = \frac{2\cancel{u^2} \cdot \frac{\sqrt{6g}}{\cancel{u}} \cdot \frac{\sqrt{u^2-6g}}{\cancel{u}}}{g}$$
$$= \frac{2}{g} \cdot \sqrt{6g(u^2-6g)}$$
$$= \frac{2}{g}\sqrt{6gu^2 - 36g^2}$$
$$= 2\sqrt{\frac{6gu^2-36g^2}{g^2}}$$
$$= 2\sqrt{\frac{6(u^2-6g)}{g}}$$
$$= 2\sqrt{\frac{36(u^2-6g)}{6g}}$$
$$= 12\sqrt{\frac{u^2-6g}{6g}}$$ as required

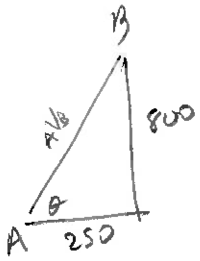
ii) $u^2 > 6g$

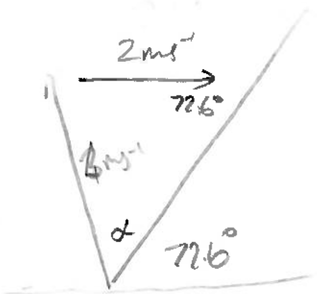
(16)

a) $V_W = 2\,ms^{-1}$

$V_B = 4\,ms^{-1}$







$\theta = \tan^{-1}\left(\frac{800}{250}\right) = 72.64°$.

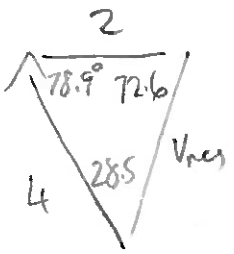
$$\frac{\sin\alpha}{2} = \frac{\sin 72.6}{4}$$

$$\alpha = \sin^{-1}\left(\frac{\sin 72.6}{2}\right)$$

$$\alpha = 28.5°$$

Angle from bank 101.1° anticlockwise

b) $t = 60$ secs $V = 3\,ms^{-1}$

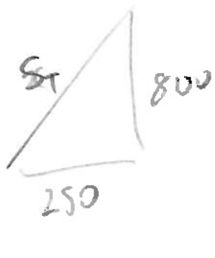


$V_{res}^2 = 2^2 + 4^2 - 2 \times 2 \times 4 \times \cos 78.9$

$V_{res}^2 = 16.91$.

$V_{res} = 4.11\,ms^{-1}$

in 60secs $S = 4.11 \times 60 = 246.6\,m$.

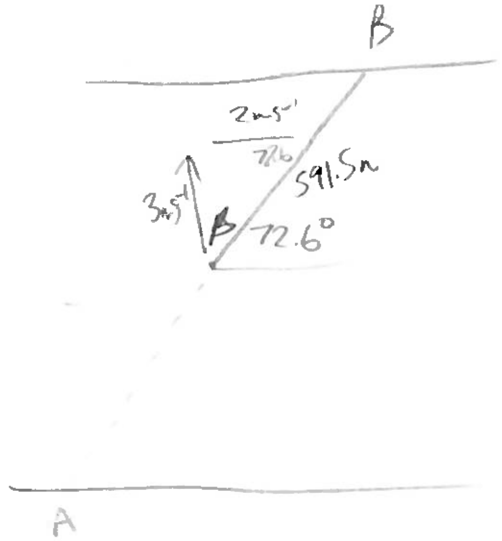


$S_T = \sqrt{800^2 + 250^2}$

$= 838.1\,m$

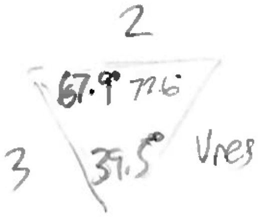
$S_{rem} = 591.5\,m$

(ii)



$$\frac{\sin \beta}{2} = \frac{\sin 72.6}{3}$$

$$\beta = 39.5^\circ$$



$$V_{res}^2 = 2^2 + 3^2 - 2 \times 2 \times 3 \times \cos 67.9$$

$$V_{res}^2 = 8.484$$

$$V_{res} = 2.91 \text{ ms}^{-1}$$

$$t = \frac{591.5}{2.91} = 203 \text{ seconds.}$$

Total time = 263 seconds

17 a) $\int e^t \sec^2(e^t)\, dt = e^t \cdot \frac{1}{e^t} \tan(e^t) + C$

$$= \tan e^t + C$$

b) $s = \tan e^t + C$ $\quad v = e^t \sec^2 e^t$

as $e^t \neq 0$ and $\sec e^t \neq 0$

then $v \neq 0$.