Common Language and Methodology for teaching Algebra
St Ninian's Cluster

Common Methodology - Algebra

Overview

Algebra is a way of thinking, i.e. a method of seeing and expressing relationships, and generalising patterns - it involves active exploration and conjecture. Algebraic thinking is not the formal manipulation of symbols.

Algebra is not simply a topic that pupils cover in Secondary school. From Primary One, staff are involved in helping pupils **lay the foundations for algebra**. This includes:

- Writing equations e.g. 16 add 8 equals?
- Solving equations e.g. $2 + \square = 7$
- Finding equivalent forms
  e.g. $24 = 20 + 4 = 30 - 6$
  $24 = 6 \times 4 = 3 \times 2 \times 2 \times 2$
- Using inverses or reversing e.g. $4 + 7 = 11 \rightarrow 11 - 7 = 4$
- Identifying number patterns
- Expressing relationships
- Drawing graphs
- Factorising numbers and expressions
- Understanding the commutative, associative and distributive laws

CfE Skills: Equations, Patterns and Sequences

Algebra skills:

Pupils should be able to:

Early	First	Second	Third	Fourth
Recognise "+" as the addition sign, "-" as the subtraction sign and "=" as the equals sign.	Identify the missing number in a calculation:	Understand and use function machines and the associated relationship between input and output values. Apply a given rule to an input in terms of +, -, × or ÷ using a single digit. Identify the input given a rule and the output. Describe the rule given the input and output in terms of +, -, × or ÷ Identify and describe the relationship between two sets of numbers Apply an identified rule to calculate the solution for numbers outwith the given sets.	Understand and use the concept of collecting like terms to simplify expressions.	Expand and simplify single brackets Factorise an expression using the highest common factor
Record addition / subtraction equations within ten	Recognise that the equals sign signifies balance in a number sentence.	Understand that letters and symbols can represent numbers. Understand that the equals sign signifies balance in an equation. Understand that the value of a symbol or letter can vary depending on the equation.	Use substitution to evaluate algebraic expressions and formulae.	Construct and solve an extended range of equations involving single brackets, fractions or negative multipliers
Identify missing digits from a sequence	Apply a given rule to an input in terms of +, -, × or ÷ using a single digit.	Simplify algebraic expressions Demonstrate a knowledge of algebraic notation	Demonstrate knowledge of algebraic notation e.g. Collect and simplify like terms Evaluate expressions by substitution, including	Illustrate the solution of an inequality on a number line. Construct and solve inequalities that include a reverse of sign, brackets and fractions.

			integers. Use substitution to evaluate algebraic expressions and formulae across a range of curricular areas	
		Solve simple equations	Understand the concept of an equation as a balance. Solve equations including those with negative and fractional answers Check answer by substituting solution into original equation. Construct algebraic equations from oral, written or graphical information.	

Patterns and sequences skills:

Pupils should be able to:

Early	First	Second	Third	Fourth
Identify and describe patterns in their own and the wider environment	Identify, continue and create 2D shape patterns	Apply an identified rule to calculate the solution for numbers outwith the given sets	Continue a given sequence.	Plot a set (locus) of points, draw the line and determine its equation
Copy and continue repeated patterns using a variety of resources and media	Recognise and continue number sequences within 100.	Explore and continue well-known number sequences e.g. square numbers, triangular numbers, Fibonacci.	Find a rule for a sequence and express in algebraic notation.	Generate points, draw and label a straight line given its equation.
Create more complex repeated patterns using a range of resources and media	Identify and describe the relationship between two sets of numbers e.g. subtract 4, divide by 3	Identify and describe the relationship between two sets of numbers which involves two steps e.g. multiply by 2 and add 3	Generate terms of a sequence using a given formula or rule.	Find an expression for the nth term given a sequence, including sequences in context.
		Continue a given sequence.	Find a specific term in a sequence using the rule e.g. 100th term.	Solve problems by recognising simple relationships and constructing / using simple formulae and equations
		Find a rule for a sequence and express in algebraic notation.		Use a formula to solve problems in context.

Early level

4 + 5 = 9 is the start of thinking about equations, as it is a statement of equality between two expressions.
Move from "makes" towards "equals" when concrete material is no longer necessary.
Pupils should become familiar with the different vocabulary for addition and subtraction as it is encountered. A wall display should be built up.

Identify missing digits from a sequence.
4,5,6, □,8, 9

First Level

Introduce the term "algebra" when symbols are used for unknown numbers or operators. Identify missing numbers in calculations

2 + □ = 7
2 □ 6 = 8
6 = 3 + □

Use the word "something" or "what" to represent numbers or operators rather than the word "box" or "square" when solving these equations.

Second Level - Function Machines

Use "in" and "out", raising awareness of the terms "input" and "output".
- Apply a given rule to an input in terms of +, -, × or ÷ using a single digit.
- Identify the input given a rule and the output.
- Describe the rule given the input and output in terms of +, -, × or ÷
  e.g. add 5, subtract 3, multiply by 2, divide by 5

E.g. 16---26
18---28
24---34
17---27

What is the rule? Answer: Add 10.

- Understand that letters and symbols can represent numbers.
- Understand that the equals sign signifies balance in an equation.
- Understand that the value of a symbol or letter can vary depending on the equation.
- Simplify algebraic expressions e.g.
  a + a + a + a = 4a

Demonstrate a knowledge of algebraic notation
e.g. $6y = 6 \times y$
Solve simple equations: e.g.

$x + 2 = 6$
$b - 5 = 12$
$2e = 8$
$5t = 30$

<u>Third level – Recognise and explain simple relationships</u>

Using the concept of collecting like terms to simplify expressions.
Use substitution to evaluate algebraic expressions and formulae.

Establish the operation(s) that are an option.

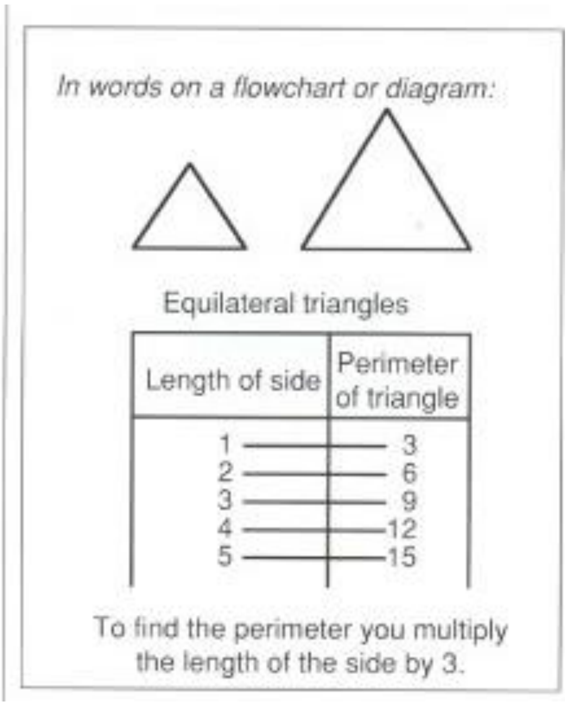
$3 \rightarrow 21$
$8 \rightarrow 56$
$10 \rightarrow 70$

The outputs are larger than the input, so the options are either addition or multiplication.

$18 \rightarrow 9$
$14 \rightarrow 7$
$6 \rightarrow 3$

The outputs are smaller than the input, so the options are either subtraction or division.

In this case, outputs are larger so the options are addition or multiplication. Add 2 works for the first one but
2 add 2 gives 4, we need the answer 6, addition does not work.
For multiplication, look at which table the output values are in.
Ensure you check the answer



• Demonstrate a knowledge of algebraic notation e.g.
$6a^2$
$= 6 \times a \times a$,

$(6a)^2$
$= 6a \times 6a$
$= 36a^2$

• Collect and simplify like terms
e.g. $5a^2 – 3 + 6a + 7 + 2a^2 - 12a$
$= 7a^2 - 6a + 4$

• Evaluate expressions by substitution, including integers.
Use substitution to evaluate algebraic expressions and formulae across a range of curricular areas e.g. V = IR

Solve equations including those with negative and fractional answers e.g.
$1a + 1 = 5$
$2x – 2 = 7$
$7s = 2s + 6$
$6 = 8 – 2c$
$5y – 2 = 3y + 4$
$2d + 5 = 5d + 9$

• Check answer by substituting solution into original equation.
Construct algebraic equations from oral, written or graphical information.

Fourth Level – Collecting like terms (Simplifying Expressions)

The examples below are *expressions* not equations.

Have the pupils rewrite expressions with the like terms gathered together as in the second line of examples 2, 3 & 4 below, before they get to their final answer.
The operator (+, −) and the term ($7x$) stay together at all times. It does not matter where the operator and term ($-7x$) are moved within the expression. (see example 3).

Example 1

Simplify

$x + 2x + 5x$
$= 8x$

Example 2

Simplify

$4 + 3a + 2 + 5a$
$= 4 + 2 + 3a + 5a$
$= 6 + 8a$

The convention in Maths is to write the answer with the letters first, in alphabetical order, followed by number.

Example 3

Simplify

$3 + 5x + 4 - 7x$
$= 5x - 7x + 3 + 4$
$= -2x + 7$

or

$3 + 5x + 4 - 7x$
$= 3 + 4 + 5x - 7x$
$= 7 - 2x$

This is still correct although the first answer is preferable.

Example 4

Simplify

$5m + 3n - 2m - n$
$= 5m - 2m + 3n - n$
$= 3m + 2n$

or

$5m + 3n - 2m - n$
$= 3n - n + 5m - 3n$
$= 2n + 3m$

Pupils should be able to use these skills to simplify like terms when working with brackets:

- Expand and simplify single brackets

  e.g. $4(a - 2b)$
  $= 4a - 8b$,

  $a(b + 5)$
  $= ab + 5a$

  $4(3a – 5) + 12$
  $= 12a – 20 + 12$
  $= 12a - 8$

- Factorise an expression using the highest common factor

  e.g. $2n + 4$
  $= 2(n + 2)$,

  $9a - a^2$
  $= a(9 - a)$

- Introduce brackets to simplify numerical calculations

  e.g. $4 \times 92 + 96 \times 92 = (4 + 96) \times 92 = 9200$

Use BODMAS to carry out more complex calculations with or without a calculator

Third Level – Evaluating expressions

If $x = 2$, $y = 3$ and $z = -4$

Find the value of:

(*a*) $5x - 2y$
(*b*) $x + y - 2z$
(*c*) $2(x + z) - y$
(*d*) $x^2 + y^2 + z^2$

$$a)\ 5x - 2y$$
$$= 5 \times 2 - 2 \times 3$$
$$= 10 - 6$$
$$= 4$$

This line where the substitution takes place must be shown.
Marks are awarded in examinations for demonstrating this step.

$$b)\ x + y - 2z$$
$$= 2 + 3 - 2 \times (-4)$$
$$= 5 - (-8)$$
$$= 13$$

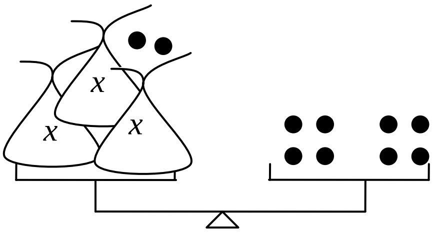
$$c)\ 2(x + z) - y$$
$$= 2(2 + (-4)) - 3$$
$$= 2 \times (-2) - 3$$
$$= -4 - 3$$
$$= -7$$

There is a bracket around the -4 as mathematicians do not write two operators side by side.

$$d)\ x^2 + y^2 + z^2$$
$$= 2^2 + 3^2 + (-4)^2$$
$$= 4 + 9 + 16$$
$$= 29$$

Third/Fourth Level – Solve simple equations

The method used for solving equations is balancing. Each equation should be set out with a line down the right hand side where the method is written, as in the examples below. It is useful to use scales like the ones below to introduce this method as pupils can visibly see how the equation can be solved.



This represents the equation
$3x + 2 = 8$
See example 4 below

Always use a method line

Example 1: Solve $x + 5 = 8$

$$\begin{array}{r|l} x + 5 = 8 & -5 \text{ from both sides} \\ \underline{\underline{x = 3}} & \end{array}$$

In the example shown pupils must state that they will "subtract 5 from both sides." If they only say, "Subtract five," ask them, "Where from?" and encourage them to tell you, "Both sides," on every occasion.

Pupils should be encouraged to check their answer mentally by substituting it back into the original equation.

Example 2: Solve $y - 3 = 6$

$$\begin{array}{r|l} y - 3 = 6 & +\,3 \text{ to both sides} \\ \underline{\underline{y = 9}} & \end{array}$$

Example 3: Solve $4m = 20$

$$\begin{array}{r|l} 4m = 20 & \div \text{ by 4 on both sides} \\ \underline{\underline{m = 5}} & \end{array}$$

Example 4: Solve $3x + 2 = 8$

$$\begin{aligned} 3x + 2 &= 8 && -2 \text{ from both sides} \\ 3x &= 6 && \div \text{ by 3 on both sides} \\ \underline{\underline{x = 2}} \end{aligned}$$

Underline answer

The examples below are more suited to secondary pupils.

Example 5: Solve $10 - 2x = 4$

$$\begin{aligned} 10 - 2x &= 4 && + 2x \text{ to both sides} \\ 10 &= 4 + 2x && -4 \text{ from both sides} \\ 6 &= 2x && \div \text{ by 2 on both sides} \\ 3 &= x \\ \underline{\underline{x = 3}} \end{aligned}$$

Write answer in the form $x =$

Example 6: Solve $3x + 2 = x + 14$

$$\begin{aligned} 3x + 2 &= x + 14 && - x \text{ from both sides} \\ 2x + 2 &= 14 && -2 \text{ from both sides} \\ 2x &= 12 && \div \text{ by 2 on both sides} \\ \underline{\underline{x = 6}} \end{aligned}$$

NB Secondary:
Always deal with the variable before the constants, ensuring that the variable is written with a positive coefficient. This avoids errors when dividing by negatives and also avoids learning rules for dealing with inequations.

Equations with fractions:

Example 7:

$$\frac{x}{3} = 7$$

$$3 \times \frac{x}{3} = 3 \times 7$$

$$x = 21$$

Method

- Multiply each side by 3
- Check 21÷ 7 = 3

Language

- Always get rid of fractions first
- What type of fractions do we have in this equation?
- What do we multiply thirds by to get rid of them?
- Multiply each side of the equation by3

Example 8

$$\frac{x}{2} + \frac{1}{3} = 2$$

$$3x + 2 = 12$$

$$3x = 10$$

$$x = \frac{10}{3}$$

Method

- Multiply each side by 6
- Subtract 2 from each side
- Divide each side by 3

Language

- What kind of fractions do we have here?
- How can we get rid of $\frac{1}{2}$ and $\frac{1}{3}$?

Other equations at this stage should include ones where *x* is a negative number or fraction. Pupils should be encouraged to write their answers as a fraction and not as a decimal. Use the language add, subtract, multiply (not times) and divide.
Also when referring to the number ‘−5’ we say ‘negative 5’ **NOT** ‘minus 5’ as minus should be treated as an operation (verb).

Third/Fourth Level – Solve inequations

Example 1:

Solve the inequation $x + 3 > 6$ choosing solutions from {0, 1, 2, 3, 4, 5, 6}

$x + 3 > 6$ | -3 from both sides
$x > 3$

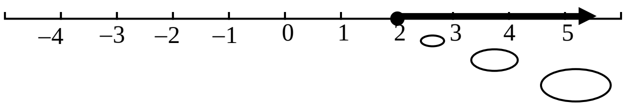
$x = \{4, 5, 6\}$

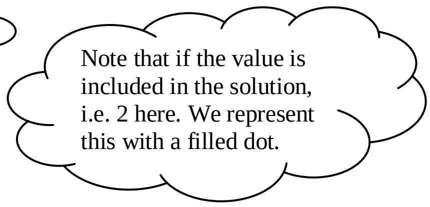
In the following examples a range of answers is not given therefore the answer should always be shown on a number line in preparation for more complex inequations at National Qualification level.

Example 2

Solve $x + 5 \geq 7$

$x + 5 \geq 7$ | -5 from both sides
$x \geq 2$

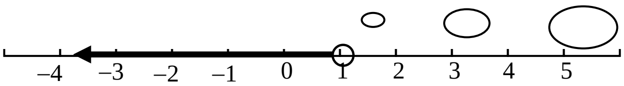


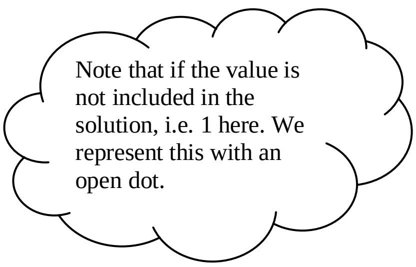


Example 3

Solve $x + 3 < 4$

$x + 3 < 4$ | -3 from both sides
$x < 1$





Third / Fourth Level – Using formulae

Using Formulae

Most students should be able to cope with using simple formulae. Those who find it difficult might find it helpful if the formula is written with boxes to insert the relevant numbers.

e.g. $D = ST$. If $S = 40$ and $T = 2$ find D. $D = \overset{S}{\boxed{40}} \times \overset{t}{\boxed{2}} = 80$

Transforming Formulae

Students are taught the balance method of transforming formulae which involves carrying out the same operation on both sides of the equation. The operation being completed is written to the right of the line of working. Only more able students (S4 Intermediate 2) are likely to be able to do this.

e.g. (*i*) $V = IR$. Change the subject of the formula to R.

$$\begin{array}{rl|l} V &= IR & \\ \dfrac{V}{I} &= R & \div I \end{array}$$

(*ii*) $v^2 = u^2 + 2as$. Make a the subject of the formula.

$$\begin{array}{rl|l} v^2 &= u^2 + 2as & \\ v^2 - u^2 &= 2as & -u^2 \\ \dfrac{v^2 - u^2}{2s} &= a & \div 2s \end{array}$$

Rules such as the triangle rule, work for specific types of question and may be useful in some subjects but students should be encouraged to use the above method if possible. If the triangle method is used as a starting point the pupil will develop little understanding of the process involved and may attempt to apply such techniques inappropriately.

When using a formula, students may find it easier to substitute known values before carrying out the transformation.

e.g. The volume of a sphere is given by the formula $V = \frac{4}{3}\pi r^3$.

Find the radius of the sphere when the volume is 75 cm^3.

$$
\begin{aligned}
V &= \tfrac{4}{3}\pi r^3 \\
75 &= \tfrac{4}{3}\pi r^3 && \times 3 \\
225 &= 4\pi r^3 && \div 4\pi \\
\frac{225}{4\pi} &= r^3 && \sqrt[3]{} \\
\sqrt[3]{\frac{225}{4\pi}} &= r \\
r &\approx 2\cdot 62 \text{ cm}
\end{aligned}
$$

<u>Second/Third Level – Use and devise simple rules</u>

Pupils need to be able to use notation to describe general relationships between 2 sets of numbers, and then use and devise simple rules.

Pupils need to be able to deal with numbers set out in a table horizontally, set out in a table vertically or given as a sequence.
A method should be followed, rather than using "trial and error" to establish the rule.

Pupils have already been asked to find the rule by establishing the single operation used. (+5, ×3, ÷2)

<u>Example 1:</u> Complete the following table, finding the n^{th} term.

> In this example, the output values are still increasing, however, addition or multiplication on their own do not work, so this must be a two-step operation.

Input	1	2	3	4	5		n
Output	5	7	9	11	13		?

Look at the outputs. These are going up by 2 each time. This tells us that we are multiplying by 2. (This means × 2.)
Now ask:
1 multiplied by 2 is 2, how do we get to 5? Add 3.
2 multiplied by 2 is 4, how do we get to 7? Add 3.
This works, so the rule is:

Multiply by 2 then add 3.

Check using the input 5: $5 \times 2 + 3 = 13$

We use n to stand for any number
So the n^{th} term would be $\boldsymbol{n \times 2 + 3}$ which is rewritten as

$$\boldsymbol{2n + 3}$$

Example 2: Find the 20th term.

Input	Output
1	7
2	10
3	13
4	16
5	19
6	22
n	$3n + 4$
20	

+3
+3

To find the 20th term it is best to find the n^{th} first.

Look at the output values. These are going up by 3 each time. This tells us that we are multiplying by 3. (This means × 3.)
Now ask:
1 multiplied by 3 is 3, how do we get to 7? Add 4.
2 multiplied by 3 is 6, how do we get to 10? Add 4.
This works so the rule is

Multiply by 3 then add 4.

Check using 6: $6 \times 3 + 4 = 22$

We use n to stand for any number
So the n^{th} term would be $\boldsymbol{n \times 3 + 4}$ which is rewritten as

$\boldsymbol{3n + 4}$

To get the 20th term we substitute $n = 20$ into our formula.

$3n + 4$
$=3\times20 + 4$
$=60 + 4$
$=64$

<u>Example 3</u>:

For the following sequence find the term that produces an output of 90.

Input	Output	
1	2	
2	10	+8
3	18	+8
4	26	
5	34	
6	42	
N	$8n - 6$	
	90	

We go through the same process as before to find the n^{th} term, which is $\mathbf{8n - 6}$.

Now we set up an equation.

$$\begin{aligned} 8n - 6 &= 90 \quad &| + 6 \\ 8n &= 96 \quad &| \div \text{ by } 8 \\ n &= 12 \quad &| \end{aligned}$$

Therefore the 12^{th} term produces an output of 90.