Higher

Mathematics

UNIT 2

Mathematics 2

HSN22000

Contents

OUTCOME 1

Polynomials and Quadratics

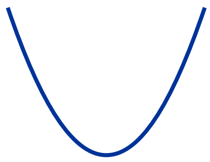
1 Quadratics

A **quadratic** has the form $ax^2 + bx + c$ where a, b, and c are any real numbers, provided $a \neq 0$.

You should already be familiar with the following.

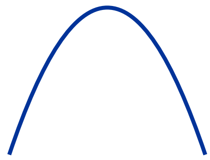
The graph of a quadratic is called a **parabola**. There are two possible shapes:

concave up (if $a > 0$)



This has a minimum turning point

concave down (if $a < 0$)



This has a maximum turning point

To find the roots (i.e. solutions) of the quadratic equation $ax^2 + bx + c = 0$, we can use:

- factorisation;
- completing the square (see Section 3);
- the quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (this is *not* given in the exam).

EXAMPLES

1. Find the roots of $x^2 - 2x - 3 = 0$.

$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$x + 1 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -1 \qquad\qquad x = 3.$$

2. Solve $x^2 + 8x + 16 = 0$.

$$x^2 + 8x + 16 = 0$$
$$(x+4)(x+4) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = -4 \qquad\qquad x = -4.$$

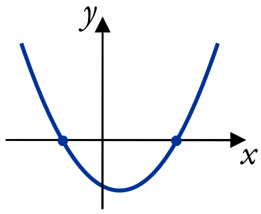
3. Find the roots of $x^2 + 4x - 1 = 0$.

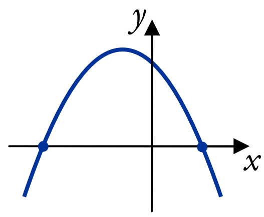
We cannot factorise $x^2 + 4x - 1$, but we can use the quadratic formula:

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{4^2 - 4 \times 1 \times (-1)}}{2 \times 1} \\ &= \frac{-4 \pm \sqrt{16 + 4}}{2} \\ &= \frac{-4 \pm \sqrt{20}}{2} \\ &= -\tfrac{4}{2} \pm \tfrac{\sqrt{4}\sqrt{5}}{2} \\ &= -2 \pm \sqrt{5}. \end{aligned}$$

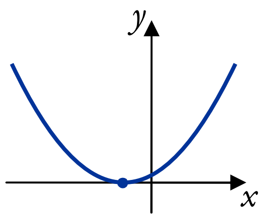
Note

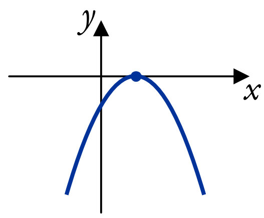
- If there are two distinct solutions, the curve intersects the x-axis twice.



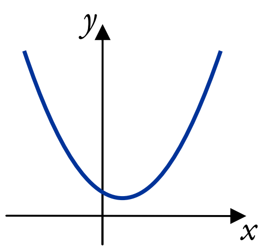


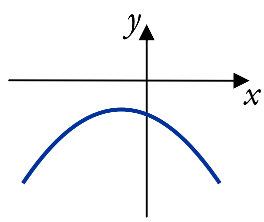
- If there is one repeated solution, the turning point lies on the x-axis.





- If $b^2 - 4ac < 0$ when using the quadratic formula, there are no points where the curve intersects the x-axis.



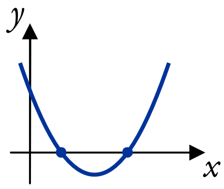


2 The Discriminant

Given $ax^2 + bx + c$, we call $b^2 - 4ac$ the **discriminant**.

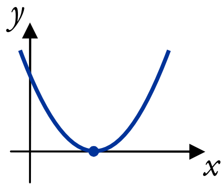
This is the part of the quadratic formula which determines the number of real roots of the equation $ax^2 + bx + c = 0$.

- If $b^2 - 4ac > 0$, the roots are real and unequal (distinct).



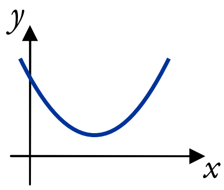
two roots

- If $b^2 - 4ac = 0$, the roots are real and equal (i.e. a repeated root).



one root

- If $b^2 - 4ac < 0$, the roots are not real; the parabola does not cross the x-axis.



no real roots

EXAMPLE

1. Find the nature of the roots of $9x^2 + 24x + 16 = 0$.

$a = 9$ $\quad b^2 - 4ac = 24^2 - 4 \times 9 \times 16$

$b = 24$ $\quad = 576 - 576$

$c = 16$ $\quad = 0$

Since $b^2 - 4ac = 0$, the roots are real and equal.

2. Find the values of q such that $6x^2 + 12x + q = 0$ has real roots.

Since $6x^2 + 12x + q = 0$ has real roots, $b^2 - 4ac \geq 0$:

$a = 6$ $\quad b^2 - 4ac \geq 0$

$b = 12$ $\quad 12^2 - 4 \times 6 \times q \geq 0$

$c = q$ $\quad 144 - 24q \geq 0$

$$144 \geq 24q$$
$$24q \leq 144$$
$$q \leq 6.$$

3. Find the range of values of k for which the equation $kx^2 + 2x - 7 = 0$ has no real roots.

For no real roots, we need $b^2 - 4ac < 0$:

$a = k$

$b = 2$

$c = -7$

$$\begin{aligned} b^2 - 4ac &< 0 \\ 2^2 - 4 \times k \times (-7) &< 0 \\ 4 + 28k &< 0 \\ 28k &< -4 \\ k &< -\frac{4}{28} \\ k &< -\tfrac{1}{7}. \end{aligned}$$

4. Show that $(2k+4)x^2 + (3k+2)x + (k-2) = 0$ has real roots for all real values of k.

$a = 2k + 4$

$b = 3k + 2$

$c = k - 2$

$$\begin{aligned} & b^2 - 4ac \\ &= (3k+2)^2 - 4(2k+4)(k-2) \\ &= 9k^2 + 12k + 4 - (2k+4)(4k-8) \\ &= 9k^2 + 12k + 4 - 8k^2 + 32 \\ &= k^2 + 12k + 36 \\ &= (k+6)^2. \end{aligned}$$

Since $b^2 - 4ac = (k+6)^2 \geq 0$, the roots are always real.

3 Completing the Square

The process of writing $y = ax^2 + bx + c$ in the form $y = a(x+p)^2 + q$ is called **completing the square**.

Once in "completed square" form we can determine the turning point of any parabola, including those with no real roots.

The axis of symmetry is $x = -p$ and the turning point is $(-p, q)$.

The process relies on the fact that $(x+p)^2 = x^2 + 2px + p^2$. For example, we can write the expression $x^2 + 4x$ using the bracket $(x+2)^2$ since when multiplied out this gives the terms we want – with an extra constant term.

This means we can rewrite the expression $x^2 + kx$ using $\left(x + \frac{k}{2}\right)^2$ since this gives us the correct x^2 and x terms, with an extra constant.

We will use this to help complete the square for $y = 3x^2 + 12x - 3$.

Step 1

Make sure the equation is in the form $y = ax^2 + bx + c$.

$y = 3x^2 + 12x - 3.$

Step 2

Take out the x^2-coefficient as a factor of the x^2 and x terms.

$y = 3\left(x^2 + 4x\right) - 3.$

Step 3

Replace the $x^2 + kx$ expression and compensate for the extra constant.

$$\begin{aligned} y &= 3\left((x+2)^2 - 4\right) - 3 \\ &= 3(x+2)^2 - 12 - 3. \end{aligned}$$

Step 4

Collect together the constant terms.

$y = 3(x+2)^2 - 15.$

Now that we have completed the square, we can see that the parabola with equation $y = 3x^2 + 12x - 3$ has turning point $(-2, -15)$.

EXAMPLES

1. Write $y = x^2 + 6x - 5$ in the form $y = (x + p)^2 + q$.

$$\begin{aligned} y &= x^2 + 6x - 5 \\ &= (x+3)^2 - 9 - 5 \\ &= (x+3)^2 - 14. \end{aligned}$$

Note

You can always check your answer by expanding the brackets.

2. Write $x^2 + 3x - 4$ in the form $(x + p)^2 + q$.

$$\begin{aligned} & x^2 + 3x - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{9}{4} - 4 \\ &= \left(x + \tfrac{3}{2}\right)^2 - \tfrac{25}{4}. \end{aligned}$$

3. Write $y = x^2 + 8x - 3$ in the form $y = (x + a)^2 + b$ and then state:
   (i) the axis of symmetry, and
   (ii) the minimum turning point of the parabola with this equation.

$$\begin{aligned} y &= x^2 + 8x - 3 \\ &= (x+4)^2 - 16 - 3 \\ &= (x+4)^2 - 19. \end{aligned}$$

(i) The axis of symmetry is $x = -4$.

(ii) The minimum turning point is $(-4, -19)$.

4. A parabola has equation $y = 4x^2 - 12x + 7$.
   (a) Express the equation in the form $y = (x + a)^2 + b$.
   (b) State the turning point of the parabola and its nature.

(a)
$$\begin{aligned} y &= 4x^2 - 12x + 7 \\ &= 4\left(x^2 - 3x\right) + 7 \\ &= 4\left(\left(x - \tfrac{3}{2}\right)^2 - \tfrac{9}{4}\right) + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 9 + 7 \\ &= 4\left(x - \tfrac{3}{2}\right)^2 - 2. \end{aligned}$$

(b) The turning point is $\left(\tfrac{3}{2}, -2\right)$ and is a minimum.

Remember
If the coefficient of x^2 is positive then the parabola is concave up.

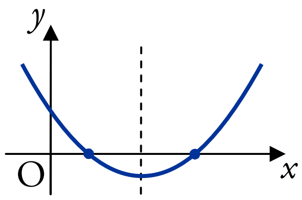
4 Sketching Parabolas

The method used to sketch the curve with equation $y = ax^2 + bx + c$ depends on how many times the curve intersects the x-axis.

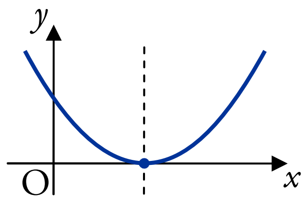
We have met curve sketching before. However, when sketching parabolas, we *do not* need to use calculus. We know there is only one turning point, and we have methods for finding it.

Parabolas with one or two roots

- Find the x-axis intercepts by factorising or using the quadratic formula.
- Find the y-axis intercept (i.e. where $x = 0$).
- The turning point is on the axis of symmetry:



The axis of symmetry is halfway between two distinct roots.



A repeated root lies on the axis of symmetry.

Parabolas with no real roots

- There are no x-axis intercepts.
- Find the y-axis intercept (i.e. where $x = 0$).
- Find the turning point by completing the square.

EXAMPLES

1. Sketch the graph of $y = x^2 - 8x + 7$.

Since $b^2 - 4ac = (-8)^2 - 4 \times 1 \times 7 > 0$, the parabola crosses the x-axis twice.

The y-axis intercept $(x = 0)$:

$$y = (0)^2 - 8(0) + 7$$
$$= 7$$
$$(0, 7).$$

The x-axis intercepts $(y = 0)$:

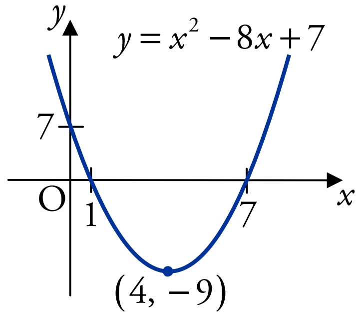
$$x^2 - 8x + 7 = 0$$
$$(x-1)(x-7) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x - 7 = 0$$
$$x = 1 \qquad x = 7$$
$$(1, 0) \qquad (7, 0).$$

The axis of symmetry lies halfway between $x = 1$ and $x = 7$, i.e. $x = 4$, so the x-coordinate of the turning point is 4.

We can now find the y-coordinate:

$$\begin{aligned} y &= (4)^2 - 8(4) + 7 \\ &= 16 - 32 + 7 \\ &= -9. \end{aligned}$$

So the turning point is $(4, -9)$.



2. Sketch the parabola with equation $y = -x^2 - 6x - 9$.

Since $b^2 - 4ac = (-6)^2 - 4\times(-1)\times(-9) = 0$, there is a repeated root.

The y-axis intercept $(x = 0)$:

$$\begin{aligned} y &= -(0)^2 - 6(0) - 9 \\ &= -9 \end{aligned}$$

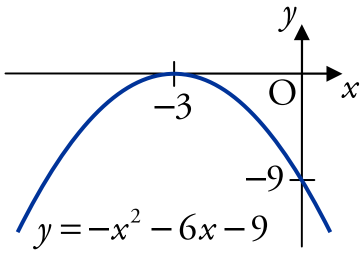
$(0, -9)$.

The x-axis intercept $(y = 0)$:

$$\begin{aligned} -x^2 - 6x - 9 &= 0 \\ -\left(x^2 + 6x + 9\right) &= 0 \\ (x+3)(x+3) &= 0 \\ x + 3 &= 0 \\ x &= -3 \end{aligned}$$

$(-3, 0)$.

Since there is a repeated root, $(-3, 0)$ is the turning point.



3. Sketch the curve with equation $y = 2x^2 - 8x + 13$.

Since $b^2 - 4ac = (-8)^2 - 4\times 2\times 13 < 0$, there are no real roots.

The y-axis intercept $(x = 0)$:

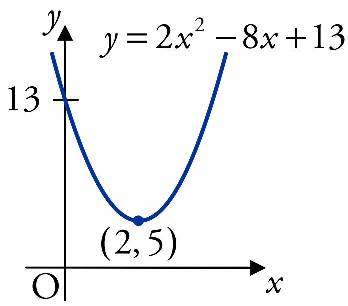
$$\begin{aligned} y &= 2(0)^2 - 8(0) + 13 \\ &= 13 \end{aligned}$$

$(0, 13)$.

Complete the square:

$$\begin{aligned} y &= 2x^2 - 8x + 13 \\ &= 2\left(x^2 - 4x\right) + 13 \\ &= 2(x-2)^2 - 8 + 13 \\ &= 2(x-2)^2 + 5. \end{aligned}$$

So the turning point is $(2, 5)$.



5 Determining the Equation of a Parabola

Given the equation of a parabola, we have seen how to sketch its graph. We will now consider the opposite problem: finding an equation for a parabola based on information about its graph.

We can find the equation given:

- the roots and another point, or
- the turning point and another point.

When we know the roots

If a parabola has roots $x = a$ and $x = b$ then its equation is of the form

$$y = k(x-a)(x-b)$$

where k is some constant.

If we know another point on the parabola, then we can find the value of k.

EXAMPLES

1. A parabola passes through the points $(1, 0)$, $(5, 0)$ and $(0, 3)$.
   Find the equation of the parabola.

   Since the parabola cuts the x-axis where $x = 1$ and $x = 5$, the equation is of the form:

   $$y = k(x-1)(x-5).$$

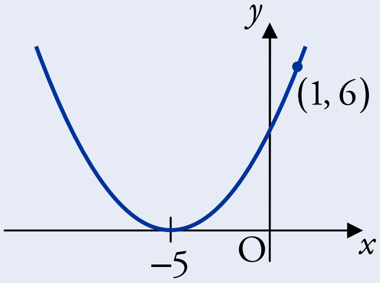
   To find k, we use the point $(0, 3)$:

   $$\begin{aligned} y &= k(x-1)(x-5) \\ 3 &= k(0-1)(0-5) \\ 3 &= 5k \\ k &= \tfrac{3}{5}. \end{aligned}$$

   So the equation of the parabola is:

   $$\begin{aligned} y &= \tfrac{3}{5}(x-1)(x-5) \\ &= \tfrac{3}{5}(x^2 - 6x + 5) \\ &= \tfrac{3}{5}x^2 - \tfrac{18}{5}x + 3. \end{aligned}$$

2. Find the equation of the parabola shown below.



Since there is a repeated root, the equation is of the form:

$$y = k(x+5)(x+5)$$
$$= k(x+5)^2.$$

To find k, we use $(1, 6)$:

$$y = k(x+5)^2$$
$$6 = k(1+5)^2$$
$$k = \tfrac{6}{6^2} = \tfrac{1}{6}.$$

Hence $y = \frac{1}{6}(x+5)^2$.

When we know the turning point

Recall from Completing the Square that a parabola with turning point $(-p, q)$ has an equation of the form

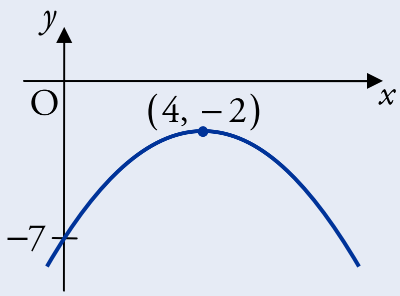
$$y = a(x+p)^2 + q$$

where a is some constant.

If we know another point on the parabola, then we can find the value of a.

EXAMPLE

3. Find the equation of the parabola shown below.



Since the turning point is $(4, -2)$, the equation is of the form:

$$y = a(x-4)^2 - 2.$$

To find a, we use $(0, -7)$:

$$y = a(x-4)^2 - 2$$
$$-7 = a(0-4)^2 - 2$$
$$16a = -5$$
$$a = -\tfrac{5}{16}.$$

Hence $y = -\frac{5}{16}(x-4)^2 - 2$.

6 Solving Quadratic Inequalities

The most efficient way of solving a quadratic inequality is by making a rough sketch of the parabola. To do this we need to know:

- the shape – concave up or concave down,
- the x-axis intercepts.

We can then solve the quadratic inequality by inspection of the sketch.

EXAMPLES

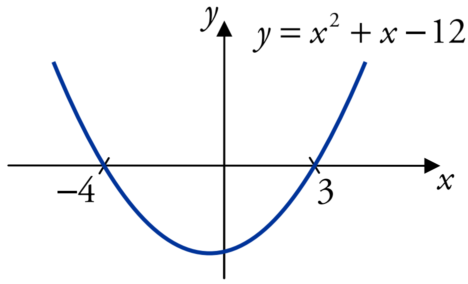
1. Solve $x^2 + x - 12 < 0$.

The parabola with equation $y = x^2 + x - 12$ is concave up.

The x-axis intercepts are given by:

$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x + 4 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -4 \qquad\qquad x = 3.$$

Make a sketch:



So $x^2 + x - 12 < 0$ for $-4 < x < 3$.

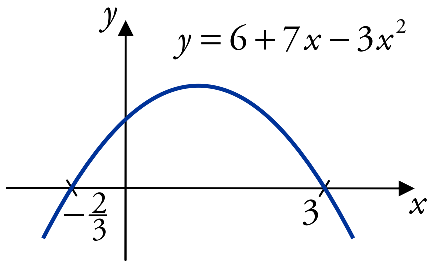
2. Find the values of x for which $6 + 7x - 3x^2 \geq 0$.

The parabola with equation $y = 6 + 7x - 3x^2$ is concave down.

The x-axis intercepts are given by:

$$6 + 7x - 3x^2 = 0$$
$$-\left(3x^2 - 7x - 6\right) = 0$$
$$(3x+2)(x-3) = 0$$
$$3x + 2 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = -\tfrac{2}{3} \qquad\qquad x = 3.$$

Make a sketch:



So $6 + 7x - 3x^2 \geq 0$ for $-\tfrac{2}{3} \leq x \leq 3$.

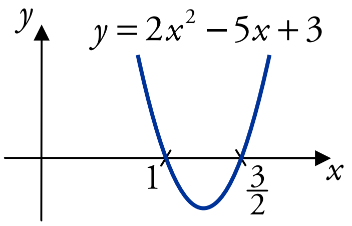
3. Solve $2x^2 - 5x + 3 > 0$.

The parabola with equation $y = 2x^2 - 5x + 3$ is concave up.

The x-axis intercepts are given by:

$$2x^2 - 5x + 3 = 0$$
$$(x-1)(2x-3) = 0$$
$$x - 1 = 0 \quad \text{or} \quad 2x - 3 = 0$$
$$x = 1 \qquad\qquad x = \tfrac{3}{2}.$$

Make a sketch:



So $2x^2 - 5x + 3 > 0$ for $x < 1$ and $x > \frac{3}{2}$.

4. Find the range of values of x for which the curve $y = \frac{1}{3}x^3 + 2x^2 - 5x + 3$ is strictly increasing.

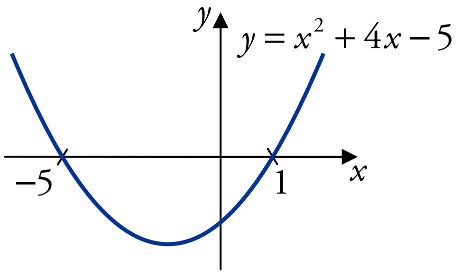
We have $\dfrac{dy}{dx} = x^2 + 4x - 5$.

Remember
Strictly increasing means $\dfrac{dy}{dx} > 0$.

The curve is strictly increasing where $x^2 + 4x - 5 > 0$.

$$x^2 + 4x - 5 = 0$$
$$(x-1)(x+5) = 0$$
$$x - 1 = 0 \quad \text{or} \quad x + 5 = 0$$
$$x = 1 \qquad\qquad x = -5.$$

Make a sketch:



So the curve is strictly increasing for $x < -5$ and $x > 1$.

5. Find the values of q for which $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots.

For no real roots, $b^2 - 4ac < 0$:

$a = 1$

$b = q - 4$

$c = \frac{1}{2}q$

$$\begin{aligned} b^2 - 4ac &= (q-4)^2 - 4(1)\left(\tfrac{1}{2}q\right) \\ &= (q-4)(q-4) - 2q \\ &= q^2 - 8q + 16 - 2q \\ &= q^2 - 10q + 16. \end{aligned}$$

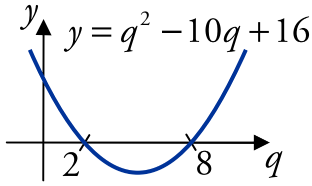
We now need to solve the inequality $q^2 - 10q + 16 < 0$.

The parabola with equation $y = q^2 - 10q + 16$ is concave up.

The x-axis intercepts are given by:

$$q^2 - 10q + 16 = 0$$
$$(q-2)(q-8) = 0$$
$$q - 2 = 0 \quad \text{or} \quad q - 8 = 0$$
$$q = 2 \qquad\qquad q = 8.$$

Make a sketch:

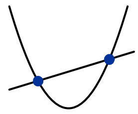


Therefore $b^2 - 4ac < 0$ for $2 < q < 8$, and so $x^2 + (q-4)x + \frac{1}{2}q = 0$ has no real roots when $2 < q < 8$.

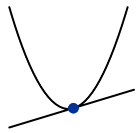
7 Intersections of Lines and Parabolas

To determine how many times a line intersects a parabola, we substitute the equation of the line into the equation of the parabola. We can then use the discriminant, or factorisation, to find the number of intersections.

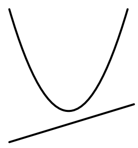
- If $b^2 - 4ac > 0$, the line and curve intersect twice.



- If $b^2 - 4ac = 0$, the line and curve intersect once (i.e. the line is a tangent to the curve).



- If $b^2 - 4ac < 0$, the line and the parabola do not intersect.



EXAMPLES

1. Show that the line $y = 5x - 2$ is a tangent to the parabola $y = 2x^2 + x$ and find the point of contact.

Substitute $y = 5x - 2$ into:

$$y = 2x^2 + x$$
$$5x - 2 = 2x^2 + x$$
$$2x^2 - 4x + 2 = 0$$
$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0.$$

Since there is a repeated root, the line is a tangent at $x = 1$.

To find the y-coordinate, substitute $x = 1$ into the equation of the line:

$$y = 5 \times 1 - 2 = 3.$$

So the point of contact is $(1, 3)$.

2. Find the equation of the tangent to $y = x^2 + 1$ that has gradient 3.

The equation of the tangent is of the form $y = mx + c$, with $m = 3$, i.e. $y = 3x + c$.

Substitute this into $y = x^2 + 1$

$$3x + c = x^2 + 1$$
$$x^2 - 3x + 1 - c = 0.$$

Since the line is a tangent:

$$b^2 - 4ac = 0$$
$$(-3)^2 - 4 \times (1 - c) = 0$$
$$9 - 4 + 4c = 0$$
$$4c = -5$$
$$c = -\tfrac{5}{4}.$$

Therefore the equation of the tangent is:

$$y = 3x - \tfrac{5}{4}$$
$$3x - y - \tfrac{5}{4} = 0.$$

Note
You could also do this question using methods from Differentiation.

8 Polynomials

Polynomials are expressions with one or more terms added together, where each term has a number (called the **coefficient**) followed by a variable (such as *x*) raised to a whole number power. For example:

$$3x^5 + x^3 + 2x^2 - 6 \quad \text{or} \quad 2x^{18} + 10.$$

The **degree** of the polynomial is the value of its highest power, for example:

$3x^5 + x^3 + 2x^2 - 6$ has degree 5 $\qquad$ $2x^{18} + 10$ has degree 18.

Note that quadratics are polynomials of degree two. Also, constants are polynomials of degree zero (e.g. 6 is a polynomial, since $6 = 6x^0$).

9 Synthetic Division

Synthetic division provides a quick way of evaluating polynomials.

For example, consider $f(x) = 2x^3 - 9x^2 + 2x + 1$. Evaluating directly, we find $f(6) = 121$. We can also evaluate this using "synthetic division with detached coefficients".

Step 1

Detach the coefficients, and write them across the top row of the table.

Note that they must be in order of *decreasing* degree. If there is no term of a specific degree, then zero is its coefficient.

	2	–9	2	1

Step 2

Write the number for which you want to evaluate the polynomial (the input number) to the left.

6	2	–9	2	1

Step 3

Bring down the first coefficient.

6	2	–9	2	1
	2			

Step 4

Multiply this by the input number, writing the result underneath the next coefficient.

6	2	–9	2	1
×		12		
	2			

Step 5

Add the numbers in this column.

6	2	–9	2	1
		+ 12		
	2	3		

Repeat Steps 4 and 5 until the last column has been completed.

The number in the lower-right cell is the value of the polynomial for the input value, often referred to as the **remainder**.

6	2	–9	2	1	
		12	18	120	
	2	3	20	121	$= f(6)$

EXAMPLE

1. Given $f(x) = x^3 + x^2 - 22x - 40$, evaluate $f(-2)$ using synthetic division.

–2	1	1	–22	–40

using the above process →

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So $f(-2) = 0$.

Note

In this example, the remainder is zero, so $f(-2) = 0$.

This means $x^3 + x^2 - 22x - 40 = 0$ when $x = -2$, which means that $x = -2$ is a root of the equation. So $x + 2$ must be a factor of the cubic.

We can use this to help with factorisation:

$$f(x) = (x+2)(\ q(x)\ ) \qquad \text{where } q(x) \text{ is a quadratic}$$

Is it possible to find the quadratic $q(x)$ using the table?

Trying the numbers from the bottom row as coefficients, we find:

$$(x+2)(x^2 - x - 20)$$
$$= x^3 - x^2 - 20x + 2x^2 - 2x - 40$$
$$= x^3 - x^2 - 22x - 40$$
$$= f(x).$$

–2	1	1	–22	–40
		–2	2	40
	1	–1	–20	0

So using the numbers from the bottom row as coefficients has given the correct quadratic. In fact, this method *always* gives the correct quadratic, making synthetic division a useful tool for factorising polynomials.

EXAMPLES

2. Show that $x - 4$ is a factor of $2x^4 - 9x^3 + 5x^2 - 3x - 4$.

$x - 4$ is a factor $\Leftrightarrow$ $x = 4$ is a root.

4	2	–9	5	–3	–4
		8	–4	4	4
	2	–1	1	1	0

Since the remainder is zero, $x = 4$ is a root, so $x - 4$ is a factor.

3. Given $f(x)=x^3-37x+84$, show that $x=-7$ is a root of $f(x)=0$, and hence fully factorise $f(x)$.

−7	1	0	−37	84
		−7	49	−84
	1	−7	12	0

Since the remainder is zero, $x=-7$ is a root.

Hence we have
$$\begin{aligned} f(x) &= x^3-37x+84 \\ &= (x+7)(x^2-7x+12) \\ &= (x+7)(x-3)(x-4). \end{aligned}$$

4. Show that $x=-5$ is a root of $2x^3+7x^2-9x+30=0$, and hence fully factorise the cubic.

−5	2	7	−9	30
		−10	15	−30
	2	−3	6	0

Since $x=-5$ is a root, $x+5$ is a factor.

$$2x^3+7x^2-9x+30=(x+5)(2x^2-3x+6)$$

This does not factorise any further since the quadratic has $b^2-4ac<0$.

Using synthetic division to factorise

In the examples above, we have been given a root or factor to help factorise polynomials. However, we can still use synthetic division if we do not know a factor or root.

Provided that the polynomial has an integer root, it will divide the constant term exactly. So by trying synthetic division with all divisors of the constant term, we will eventually find the integer root.

5. Fully factorise $2x^3+5x^2-28x-15$.

Numbers which divide −15: $\pm1, \pm3, \pm5, \pm15$.

Try $x=1$: $2(1)^3+5(1)^2-28(1)-15$
$= 2+5-28-15 \neq 0.$

Try $x=-1$: $2(-1)^3+5(-1)^2-28(-1)-15$
$= -2+5+28-15 \neq 0.$

Note
For ±1, it is simpler just to evaluate the polynomial directly, to see if these values are roots.

Try $x = 3$:

3	2	5	–28	–15
		6	33	15
	2	11	5	0

Since $x = 3$ is a root, $x - 3$ is a factor.

So $2x^3 + 5x^2 - 28x - 15 = (x-3)(2x^2 + 11x + 5)$
$= (x-3)(2x+1)(x+5)$.

Using synthetic division to solve equations

We can also use synthetic division to help solve equations.

EXAMPLE

6. Find the solutions of $2x^3 - 15x^2 + 16x + 12 = 0$.

Numbers which divide 12: $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$.

Try $x = 1$: $2(1)^3 - 15(1)^2 + 16(1) + 12$
$= 2 - 15 + 16 + 12 \neq 0$.

Try $x = -1$: $2(-1)^3 - 15(-1)^2 + 16(-1) + 12$
$= -2 - 15 - 16 + 12 \neq 0$.

Try $x = 2$:

2	2	–15	16	12
		4	–22	–12
	2	–11	–6	0

Since $x = 2$ is a root, $x - 2$ is a factor:

$$2x^3 - 15x^2 + 16x + 12 = 0$$
$$(x-2)(2x^2 - 11x - 6) = 0$$
$$(x-2)(2x+1)(x-6) = 0$$

$x - 2 = 0$ or $2x + 1 = 0$ or $x - 6 = 0$
$x = 2$ $\quad x = -\frac{1}{2}$ $\quad x = 6$.

The Factor Theorem and Remainder Theorem

For a polynomial $f(x)$:

If $f(x)$ is divided by $x - h$ then the remainder is $f(h)$, and
$f(h) = 0 \Leftrightarrow x - h$ is a factor of $f(x)$.

As we saw, synthetic division helps us to write $f(x)$ in the form

$$(x-h)q(x) + f(h)$$

where $q(x)$ is called the **quotient** and $f(h)$ the **remainder**.

EXAMPLE

7. Find the quotient and remainder when $f(x)=4x^3+x^2-x-1$ is divided by $x+1$, and express $f(x)$ as $(x+1)q(x)+f(h)$.

–1	4	1	–1	–1
		–4	3	–2
	4	–3	2	–3

The quotient is $4x^2-3x+2$ and the remainder is –3, so

$$f(x)=(x+1)(4x^2-3x+2)-3.$$

10 Finding Unknown Coefficients

Consider a polynomial with some unknown coefficients, such as x^3+2px^2-px+4, where p is a constant.

If we divide the polynomial by $x-h$, then we will obtain an expression for the remainder in terms of the unknown constants. If we already know the value of the remainder, we can solve for the unknown constants.

EXAMPLES

1. Given that $x-3$ is a factor of $x^3-x^2+px+24$, find the value of p.

$x-3$ is a factor $\Leftrightarrow$ $x=3$ is a root.

3	1	–1	p	24
		3	6	$18+3p$
	1	2	$6+p$	$42+3p$

Note
This is just the same synthetic division procedure we are used to.

Since $x=3$ is a root, the remainder is zero:

$$42+3p=0$$
$$3p=-42$$
$$p=-14.$$

2. When $f(x) = px^3 + qx^2 - 17x + 4q$ is divided by $x-2$, the remainder is 6, and $x-1$ is a factor of $f(x)$.

   Find the values of p and q.

When $f(x)$ is divided by $x-2$, the remainder is 6.

$$\begin{array}{c|cccc} 2 & p & q & -17 & 4q \\ & & 2p & 4p+2q & 8p+4q-34 \\ \hline & p & 2p+q & 4p+2q-17 & \boxed{8p+8q-34} \end{array}$$

Since the remainder is 6, we have:

$$\begin{aligned} 8p+8q-34 &= 6 \\ 8p+8q &= 40 \\ p+q &= 5. \quad ① \end{aligned}$$

Since $x-1$ is a factor, $f(1)=0$:

$$\begin{aligned} f(1) &= p(1)^3 + q(1)^2 - 17(1) + 4q \\ &= p+q-17+4q \\ &= p+5q-17 \end{aligned}$$

i.e. $p+5q=17$. ②

Note
There is no need to use synthetic division here, but you could if you wish.

Solving ① and ② simultaneously, we obtain:

②−①: $4q = 12$

$q = 3.$

Put $q=3$ into ①: $p+3=5$

$p=2.$

Hence $p=2$ and $q=3$.

11 Finding Intersections of Curves

We have already met intersections of lines and parabolas in this outcome, but we were mainly interested in finding equations of tangents

We will now look at how to find the actual points of intersection – and not just for lines and parabolas; the technique works for any polynomials.

EXAMPLES

1. Find the points of intersection of the line $y = 4x - 4$ and the parabola $y = 2x^2 - 2x - 12$.

To find intersections, equate:

$$\begin{aligned} 2x^2 - 2x - 12 &= 4x - 4 \\ 2x^2 - 6x - 8 &= 0 \\ x^2 - 3x - 4 &= 0 \\ (x+1)(x-4) &= 0 \\ x = -1 \quad &\text{or} \quad x = 4. \end{aligned}$$

Find the y-coordinates by putting the x-values into one of the equations:

when $x = -1$, $y = 4 \times (-1) - 4 = -4 - 4 = -8$,

when $x = 4$, $y = 4 \times 4 - 4 = 16 - 4 = 12$.

So the points of intersection are $(-1,-8)$ and $(4,12)$.

2. Find the coordinates of the points of intersection of the cubic $y = x^3 - 9x^2 + 20x - 10$ and the line $y = -3x + 5$.

To find intersections, equate:

$$\begin{aligned} x^3 - 9x^2 + 20x - 10 &= -3x + 5 \\ x^3 - 9x^2 + 23x - 15 &= 0 \\ (x-1)(x^2 - 8x + 15) &= 0 \\ (x-1)(x-3)(x-5) &= 0 \\ x = 1 \quad \text{or} \quad x = 3 \quad &\text{or} \quad x = 5. \end{aligned}$$

Remember
You can use synthetic division to help with factorising.

Find the y-coordinates by putting the x-values into one of the equations:

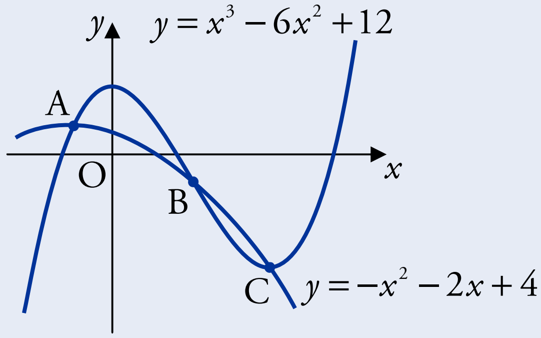
when $x = 1$, $y = -3 \times 1 + 5 = -3 + 5 = 2$,

when $x = 3$, $y = -3 \times 3 + 5 = -9 + 5 = -4$,

when $x = 5$, $y = -3 \times 5 + 5 = -15 + 5 = -10$.

So the points of intersection are $(1,2)$, $(3,-4)$ and $(5,-10)$.

3. The curves $y=-x^2-2x+4$ and $y=x^3-6x^2+12$ are shown below.



Find the x-coordinates of A, B and C, where the curves intersect.

To find intersections, equate:

$$-x^2-2x+4=x^3-6x^2+12$$
$$x^3-5x^2+2x+8=0$$
$$(x+1)(x^2-6x+8)=0$$
$$(x+1)(x-2)(x-4)=0$$
$$x=-1 \quad \text{or} \quad x=2 \quad \text{or} \quad x=4.$$

Remember
You can use synthetic division to help with factorising.

So at A, $x=-1$; at B, $x=2$; and at C, $x=4$.

4. Find the x-coordinates of the points where the curves $y=2x^3-3x^2-10$ and $y=3x^3-10x^2+7x+5$ intersect.

To find intersections, equate:

$$2x^3-3x^2-10=3x^3-10x^2+7x+5$$
$$x^3-7x^2+7x+15=0$$
$$(x+1)(x^2-8x+15)=0$$
$$(x+1)(x-3)(x-5)=0$$
$$x=-1 \quad \text{or} \quad x=3 \quad \text{or} \quad x=5.$$

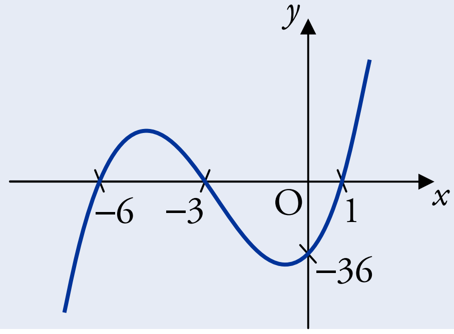
So the curves intersect where $x=-1,3,5$.

12 Determining the Equation of a Curve

Given the roots, and at least one other point lying on the curve, we can establish its equation using a process similar to that used when finding the equation of a parabola.

EXAMPLE

1. Find the equation of the cubic shown in the diagram below.



Step 1

Write out the roots, then rearrange to get the factors.

$x=-6 \qquad x=-3 \qquad x=1$

$x+6=0 \qquad x+3=0 \qquad x-1=0.$

Step 2

The equation then has these factors multiplied together with a constant, k.

$y=k(x+6)(x+3)(x-1).$

Step 3

Substitute the coordinates of a known point into this equation to find the value of k.

Using $(0,\,-36)$:

$$\begin{aligned} k(0+6)(0+3)(0-1) &= -36 \\ k(3)(-1)(6) &= -36 \\ -18k &= -36 \\ k &= 2. \end{aligned}$$

Step 4

Replace k with this value in the equation.

$$\begin{aligned} y &= 2(x+6)(x+3)(x-1) \\ &= 2(x+3)\left(x^2+5x-6\right) \\ &= 2\left(x^3+5x^2-6x+3x^2+15x-18\right) \\ &= 2x^3+16x^2+18x-36. \end{aligned}$$

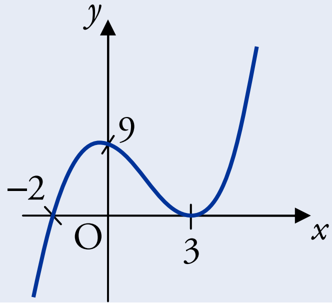
Repeated Roots

If a repeated root exists, then a stationary point lies on the x-axis.

Recall that a repeated root exists when two roots, and hence two factors, are equal.

EXAMPLE

2. Find the equation of the cubic shown in the diagram below.



$$x=-2 \qquad x=3 \qquad x=3$$
$$x+2=0 \qquad x-3=0 \qquad x-3=0.$$

So $y=k(x+2)(x-3)^2$.

Use $(0,9)$ to find k:

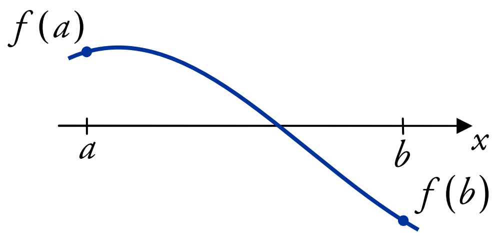
$$9=k(0+2)(0-3)^2$$
$$9=k\times2\times9$$
$$k=\tfrac{1}{2}.$$

So
$$\begin{aligned} y &= \tfrac{1}{2}(x+2)(x-3)^2 \\ &= \tfrac{1}{2}(x+2)\left(x^2-6x+9\right) \\ &= \tfrac{1}{2}\left(x^3-6x^2+9x+2x^2-12x+18\right) \\ &= \tfrac{1}{2}x^3-2x^2-\tfrac{3}{2}x+9. \end{aligned}$$

Note
$x=3$ is a repeated root, so the factor $(x-3)$ appears twice in the equation.

13 Approximating Roots

Polynomials have the special property that if $f(a)$ is positive and $f(b)$ is negative then f must have a root between a and b.



We can use this property to find approximations for roots of polynomials to any degree of accuracy by repeatedly "zooming in" on the root.

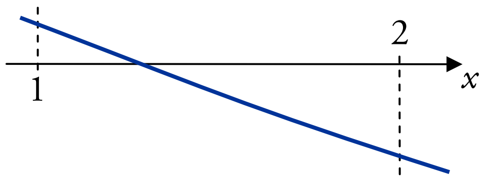
EXAMPLE

Given $f(x)=x^3-4x^2-2x+7$, show that there is a real root between $x=1$ and $x=2$. Find this root correct to two decimal places.

Evaluate $f(x)$ at $x=1$ and $x=2$:

$$f(1)=1^3-4(1)^2-2(1)+7=2$$

$$f(2)=2^3-4(2)^2-2(2)+7=-5$$



Since $f(1)>0$ and $f(2)<0$, $f(x)$ has a root between these values.

Start halfway between $x=1$ and $x=2$, then take little steps to find a change in sign:

$$f(1{\cdot}5)=-1{\cdot}625<0$$

$$f(1{\cdot}4)=-0{\cdot}896<0$$

$$f(1{\cdot}3)=-0{\cdot}163<0$$

$$f(1{\cdot}2)=0{\cdot}568>0.$$

Since $f(1{\cdot}2)>0$ and $f(1{\cdot}3)<0$, the root is between $x=1{\cdot}2$ and $x=1{\cdot}3$.

Start halfway between $x=1{\cdot}2$ and $x=1{\cdot}3$:

$$f(1{\cdot}25)=0{\cdot}203125>0$$

$$f(1{\cdot}26)=0{\cdot}129976>0$$

$$f(1{\cdot}27)=0{\cdot}056783>0$$

$$f(1{\cdot}28)=-0{\cdot}016448<0.$$

Since $f(1{\cdot}27)>0$ and $f(1{\cdot}28)<0$, the root is between these values.

Finally, $f(1.275)=0.020171875>0$. Since $f(1{\cdot}275)>0$ and $f(1{\cdot}28)<0$, the root is between $x=1{\cdot}275$ and $x=1{\cdot}28$.

Therefore the root is $x=1{\cdot}28$ to 2 d.p.

OUTCOME 2

Integration

1 Indefinite Integrals

In **integration**, our aim is to "undo" the process of differentiation. Later we will see that integration is a useful tool for evaluating areas and solving a special type of equation.

We have already seen how to differentiate polynomials, so we will now look at how to undo this process. The basic technique is:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1, \ c \text{ is the constant of integration.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add the constant of integration (c).

EXAMPLES

1. Find $\int x^2 \, dx$.

$$\int x^2 \, dx = \frac{x^3}{3} + c = \tfrac{1}{3}x^3 + c.$$

2. Find $\int x^{-3} \, dx$.

$$\int x^{-3} \, dx = \frac{x^{-2}}{-2} + c = -\frac{1}{2x^2} + c.$$

3. Find $\int x^{\frac{5}{4}} \, dx$.

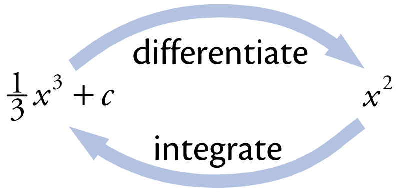
$$\int x^{\frac{5}{4}} \, dx = \frac{x^{\frac{9}{4}}}{\frac{9}{4}} + c = \tfrac{4}{9}x^{\frac{9}{4}} + c.$$

- We use the symbol $\int$ for integration.
- The $\int$ must be used with "dx" in the examples above, to indicate that we are integrating with respect to x.
- The constant of integration is included to represent any constant term in the original expression, since this would have been zeroed by differentiation.
- Integrals with a constant of integration are called **indefinite integrals**.

Checking the answer

Since integration and differentiation are reverse processes, if we differentiate our answer we should get back to what we started with.

For example, if we differentiate our answer to Example 1 above, we do get back to the expression we started with.



Integrating terms with coefficients

The above technique can be extended to:

$$\int ax^n \, dx = a\int x^n \, dx = \frac{ax^{n+1}}{n+1} + c \qquad n \neq -1, \ a \text{ is a constant.}$$

Stated simply: raise the power (n) by one (giving $n+1$), divide by the new power ($n+1$), and add on c.

EXAMPLES

4. Find $\int 6x^3 \, dx$.

$$\int 6x^3 \, dx = \frac{6x^4}{4} + c$$
$$= \tfrac{3}{2}x^4 + c.$$

5. Find $\int 4x^{-\frac{3}{2}} \, dx$.

$$\int 4x^{-\frac{3}{2}} \, dx = \frac{4x^{-\frac{1}{2}}}{-\frac{1}{2}} + c$$
$$= -8x^{-\frac{1}{2}} + c$$
$$= -\frac{8}{\sqrt{x}} + c.$$

Note

It can be easy to confuse integration and differentiation, so remember:

$$\int x \, dx = \tfrac{1}{2}x^2 + c \qquad\qquad \int k \, dx = kx + c.$$

Other variables

Just as with differentiation, we can integrate with respect to any variable.

EXAMPLES

6. Find $\int 2p^{-5}\,dp$.

$$\int 2p^{-5}\,dp = \frac{2p^{-4}}{-4} + c$$
$$= -\frac{1}{2p^4} + c.$$

Note
dp tells us to integrate with respect to *p*.

7. Find $\int p\,dx$.

$$\int p\,dx$$
$$= px + c.$$

Note
Since we are integrating with respect to *x*, we treat *p* as a constant.

Integrating several terms

The following rule is used to integrate an expression with several terms:

$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

Stated simply: integrate each term separately.

EXAMPLES

8. Find $\int \left(3x^2 - 2x^{\frac{1}{2}}\right)dx$.

$$\int \left(3x^2 - 2x^{\frac{1}{2}}\right)dx = \frac{3x^3}{3} - \frac{2x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$= x^3 - \frac{4x^{\frac{3}{2}}}{3} + c$$
$$= x^3 - \tfrac{4}{3}\sqrt{x^3} + c.$$

9. Find $\int \left(4x^{-\frac{5}{8}} + 3x + 7\right)dx$.

$$\int \left(4x^{-\frac{5}{8}} + 3x + 7\right)dx = \frac{4x^{\frac{3}{8}}}{\frac{3}{8}} + \frac{3x^2}{2} + 7x + c$$
$$= \tfrac{8}{3} \times 4x^{\frac{3}{8}} + \tfrac{3}{2}x^2 + 7x + c$$
$$= \tfrac{32}{3}x^{\frac{3}{8}} + \tfrac{3}{2}x^2 + 7x + c.$$

2 Preparing to Integrate

As with differentiation, it is important that before integrating all brackets are multiplied out and there are no fractions with an x term in the denominator (bottom line), for example:

$$\frac{1}{x^3}=x^{-3} \qquad \frac{3}{x^2}=3x^{-2} \qquad \frac{1}{\sqrt{x}}=x^{-\frac{1}{2}} \qquad \frac{1}{4x^5}=\tfrac{1}{4}x^{-5} \qquad \frac{5}{4\sqrt[3]{x}^2}=\tfrac{5}{4}x^{-\frac{2}{3}}.$$

EXAMPLES

1. Find $\int \frac{dx}{x^2}$ for $x \neq 0$.

$\int \frac{dx}{x^2}$ is just a short way of writing $\int \frac{1}{x^2}\,dx$, so:

$$\begin{aligned}\int \frac{dx}{x^2} = \int \frac{1}{x^2}\,dx = \int x^{-2}dx &= \frac{x^{-1}}{-1}+c \\ &= -\frac{1}{x}+c.\end{aligned}$$

2. Find $\int \frac{dx}{\sqrt{x}}$ for $x > 0$.

$$\begin{aligned}\int \frac{dx}{\sqrt{x}} = \int \frac{1}{x^{\frac{1}{2}}}dx = \int x^{-\frac{1}{2}}dx &= \frac{x^{\frac{1}{2}}}{\frac{1}{2}}+c \\ &= 2\sqrt{x}+c.\end{aligned}$$

3. Find $\int \frac{7}{2p^2}\,dp$ where $p \neq 0$.

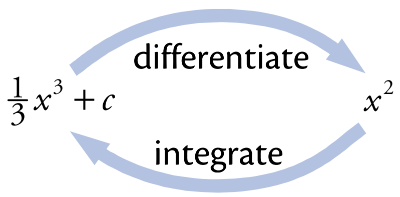
$$\begin{aligned}\int \frac{7}{2p^2}\,dp = \int \tfrac{7}{2}p^{-2}\,dp &= \tfrac{7}{2}\times\frac{p^{-1}}{-1}+c \\ &= -\frac{7}{2p}+c.\end{aligned}$$

4. Find $\int \frac{3x^5 - 5x}{4}\,dx$.

$$\int \frac{3x^5 - 5x}{4}\,dx = \int \left(\tfrac{3}{4}x^5 - \tfrac{5}{4}x\right)dx$$
$$= \frac{3x^6}{4\times 6} - \frac{5x^2}{4\times 2} + c$$
$$= \tfrac{3}{24}x^6 - \tfrac{5}{8}x^2 + c$$
$$= \tfrac{1}{8}x^6 - \tfrac{5}{8}x^2 + c.$$

3 Differential Equations

A **differential equation** is an equation involving derivatives, e.g. $\frac{dy}{dx} = x^2$.

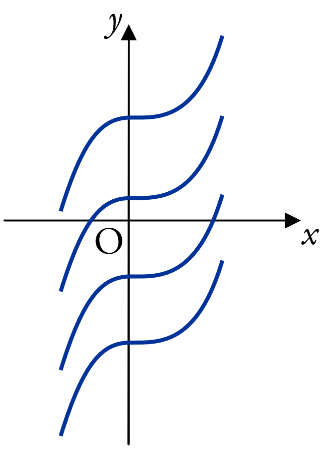


A solution of a differential equation is an expression for the original function; in this case $y = \frac{1}{3}x^3 + c$ is a solution.

In general, we obtain solutions using integration:

$$y = \int \frac{dy}{dx}\,dx \qquad \text{or} \qquad f(x) = \int f'(x)\,dx.$$

This will result in a **general solution** since we can choose the value of c, the constant of integration.



The general solution corresponds to a “family” of curves, each with a different value for c.

The graph to the left illustrates some of the curves $y = \frac{1}{3}x^3 + c$ for particular values of c.

If we have additional information about the function (such as a point its graph passes through), we can find the value of c and obtain a **particular solution**.

EXAMPLES

1. The graph of $y = f(x)$ passes through the point $(3, -4)$.

   If $\frac{dy}{dx} = x^2 - 5$, express y in terms of x.

$$y = \int \frac{dy}{dx}\, dx$$
$$= \int \left(x^2 - 5\right)\, dx$$
$$= \tfrac{1}{3}x^3 - 5x + c.$$

We know that when $x = 3$, $y = -4$ so we can find c:

$$y = \tfrac{1}{3}x^3 - 5x + c$$
$$-4 = \tfrac{1}{3}(3)^3 - 5(3) + c$$
$$-4 = 9 - 15 + c$$
$$c = 2$$

So $y = \tfrac{1}{3}x^3 - 5x + 2$.

2. The function f, defined on a suitable domain, is such that

   $f'(x) = x^2 + \frac{1}{x^2} + \tfrac{2}{3}$.

   Given that $f(1) = 4$, find a formula for $f(x)$ in terms of x.

$$f(x) = \int f'(x)\, dx$$
$$= \int \left(x^2 + \frac{1}{x^2} + \tfrac{2}{3}\right) dx$$
$$= \int \left(x^2 + x^{-2} + \tfrac{2}{3}\right) dx$$
$$= \tfrac{1}{3}x^3 - x^{-1} + \tfrac{2}{3}x + c$$
$$= \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + c.$$

We know that $f(1) = 4$, so we can find c:

$$f(x) = \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + c$$
$$4 = \tfrac{1}{3}(1)^3 - \tfrac{1}{1} + \tfrac{2}{3}(1) + c$$
$$4 = \tfrac{1}{3} - 1 + \tfrac{2}{3} + c$$
$$c = 4.$$

So $f(x) = \tfrac{1}{3}x^3 - \frac{1}{x} + \tfrac{2}{3}x + 4$.

4 Definite Integrals

If $F(x)$ is an integral of $f(x)$, then we define:

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$$

where a and b are called the **limits** of the integral.

Stated simply:

- Work out the integral as normal, leaving out the constant of integration.
- Evaluate the integral for $x = b$ (the upper limit value).
- Evaluate the integral for $x = a$ (the lower limit value).
- Subtract the lower limit value from the upper limit value.

Since there is no constant of integration and we are calculating a numerical value, this is called a **definite integral**.

EXAMPLES

1. Find $\int_1^3 5x^2\,dx$.

$$\begin{aligned}\int_1^3 5x^2\,dx &= \left[\frac{5x^3}{3}\right]_1^3 \\ &= \left(\frac{5(3)^3}{3}\right) - \left(\frac{5(1)^3}{3}\right) \\ &= 5\times 3^2 - \tfrac{5}{3} \\ &= 45 - \tfrac{5}{3} = 43\tfrac{1}{3}.\end{aligned}$$

2. Find $\int_0^2 \left(x^3 + 3x^2\right) dx$.

$$\begin{aligned}\int_0^2 \left(x^3 + 3x^2\right) dx &= \left[\frac{x^4}{4} + \frac{3x^3}{3}\right]_0^2 \\ &= \left[\frac{x^4}{4} + x^3\right]_0^2 \\ &= \left(\frac{2^4}{4} + 2^3\right) - \left(\frac{0^4}{4} + 0^3\right) \\ &= \tfrac{16}{4} + 8 - 0 \\ &= 4 + 8 = 12.\end{aligned}$$

3. Find $\int_{-1}^{4} \frac{4}{x^3}\, dx$.

$$\begin{aligned}
\int_{-1}^{4} \frac{4}{x^3}\, dx &= \int_{-1}^{4} 4x^{-3}\, dx \\
&= \left[\frac{4x^{-2}}{-2}\right]_{-1}^{4} \\
&= \left[-\frac{2}{x^2}\right]_{-1}^{4} \\
&= \left(-\frac{2}{4^2}\right) - \left(-\frac{2}{(-1)^2}\right) \\
&= -\tfrac{2}{16} + 2 = 1\tfrac{7}{8}.
\end{aligned}$$

5 Geometric Interpretation of Integration

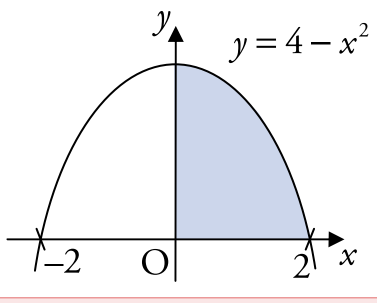
We will now consider the meaning of integration in the context of areas.

Consider $\int_0^2 \left(4 - x^2\right) dx = \left[4x - \tfrac{1}{3}x^3\right]_0^2$

$$= \left(8 - \tfrac{8}{3}\right) - 0$$

$$= 5\tfrac{1}{3}.$$

On the graph of $y = 4 - x^2$:



The shaded area is given by $\int_0^2 \left(4 - x^2\right) dx$.

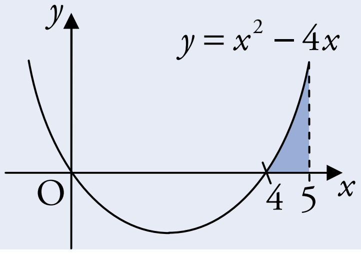
Therefore the shaded area is $5\tfrac{1}{3}$ square units.

In general, the area enclosed by the graph $y = f(x)$ and the x-axis, between $x = a$ and $x = b$, is given by

$$\int_a^b f(x)\, dx.$$

EXAMPLE

1. The graph of $y = x^2 - 4x$ is shown below. Calculate the shaded area.

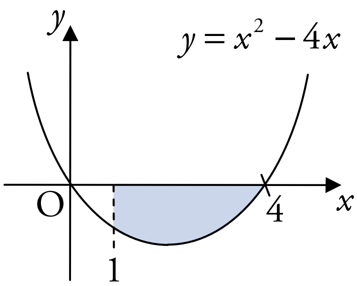


$$\begin{aligned}\int_4^5 \left(x^2 - 4x\right) dx &= \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_4^5 \\ &= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\frac{4^3}{3} - 2(4)^2\right) \\ &= \tfrac{125}{3} - 50 - \tfrac{64}{3} + 32 \\ &= \tfrac{61}{3} - 18 \\ &= 2\tfrac{1}{3}.\end{aligned}$$

So the shaded area is $2\frac{1}{3}$ square units.

Areas below the *x*-axis

Care needs to be taken if part or all of the area lies below the x-axis. For example if we look at the graph of $y = x^2 - 4$:



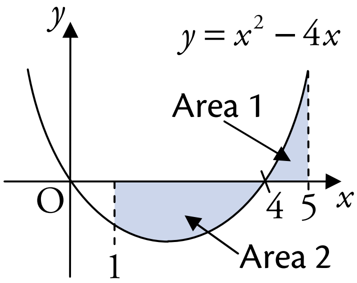
The shaded area is given by

$$\begin{aligned}\int_1^4 \left(x^2 - 4x\right) dx &= \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^4 \\ &= \left(\frac{4^3}{3} - 2(4)^2\right) - \left(\tfrac{1}{3} - 2\right) \\ &= \tfrac{64}{3} - 32 - \tfrac{1}{3} + 2 \\ &= \tfrac{63}{3} - 30 = 21 - 30 = -9.\end{aligned}$$

In this case, the negative indicates that the area is below the x-axis, as can be seen from the diagram. The area is therefore 9 square units.

Areas above and below the *x*-axis

Consider the graph from the example above, with a different shaded area:



From the working above, the total shaded area is:

$$\text{Area 1} + \text{Area 2} = 2\tfrac{1}{3} + 9 = 11\tfrac{1}{3} \text{ square units.}$$

Using the method from above, we might try to calculate the shaded area as follows:

$$\begin{aligned}\int_1^5 \left(x^2 - 4x\right) dx &= \left[\frac{x^3}{3} - \frac{4x^2}{2}\right]_1^5 \\ &= \left(\frac{5^3}{3} - 2(5)^2\right) - \left(\tfrac{1}{3} - 2\right) \\ &= \tfrac{125}{3} - 50 - \tfrac{1}{3} + 2 \\ &= \tfrac{124}{3} - 48 = -6\tfrac{2}{3}.\end{aligned}$$

Clearly this shaded area is not $6\frac{2}{3}$ square units since we already found it to be $11\frac{1}{3}$ square units. This problem arises because Area 1 is above the *x*-axis, while Area 2 is below.

To find the true area, we needed to evaluate two integrals:

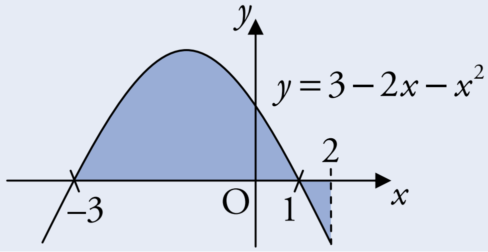
$$\int_1^4 \left(x^2 - 4x\right) dx \qquad \text{and} \qquad \int_4^5 \left(x^2 - 4x\right) dx.$$

We then found the total shaded area by adding the two areas together.

We must take care to do this whenever the area is split up in this way.

EXAMPLES

2. Calculate the shaded area shown in the diagram below.



To calculate the area from $x=-3$ to $x=1$:

$$\int_{-3}^{1}\left(3-2x-x^2\right)dx=\left[3x-\frac{2x^2}{2}-\frac{x^3}{3}\right]_{-3}^{1}$$

$$=\left[3x-x^2-\tfrac{1}{3}x^3\right]_{-3}^{1}$$

$$=\left(3(1)-(1)^2-\tfrac{1}{3}(1)^3\right)-\left(3(-3)-(-3)^2-\tfrac{1}{3}(-3)^3\right)$$

$$=\left(3-1-\tfrac{1}{3}\right)-(-9-9+9)$$

$$=3-1-\tfrac{1}{3}+9$$

$$=10\tfrac{2}{3}$$ So the area is $10\tfrac{2}{3}$ square units.

We have already carried out the integration, so we can just substitute in new limits to work out the area from $x=1$ to $x=2$:

$$\int_{1}^{2}\left(3-2x-x^2\right)dx=\left[3x-x^2-\tfrac{1}{3}x^3\right]_{1}^{2}$$

$$=\left(3(2)-(2)^2-\tfrac{1}{3}(2)^3\right)-\left(3(1)-(1)^2-\tfrac{1}{3}(1)^3\right)$$

$$=\left(6-4-\tfrac{8}{3}\right)-\left(3-1-\tfrac{1}{3}\right)$$

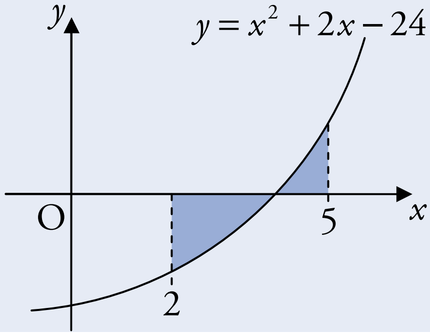
$$=2-\tfrac{8}{3}-2+\tfrac{1}{3}$$

$$=-2\tfrac{1}{3}.$$ So the area is $2\tfrac{1}{3}$ square units.

Remember
The negative sign just indicates that the area lies below the axis.

So the total shaded area is $10\tfrac{2}{3}+2\tfrac{1}{3}=13$ square units.

3. Calculate the shaded area shown in the diagram below.



First, we need to calculate the root between $x = 2$ and $x = 5$:

$$x^2 + 2x - 24 = 0$$
$$(x-4)(x+6) = 0$$
$$x = 4 \quad \text{or} \quad x = -6.$$

So the root is $x = 4$

To calculate the area from $x = 2$ to $x = 4$:

$$\begin{aligned}\int_2^4 \left(x^2 + 2x - 24\right) dx &= \left[\frac{x^3}{3} + \frac{2x^2}{2} - 24x\right]_2^4 \\ &= \left[\tfrac{1}{3}x^3 + x^2 - 24x\right]_2^4 \\ &= \left(\tfrac{1}{3}(4)^3 + (4)^2 - 24(4)\right) - \left(\tfrac{1}{3}(2)^3 + (2)^2 - 24(2)\right) \\ &= \left(\tfrac{64}{3} + 16 - 96\right) - \left(\tfrac{8}{3} + 4 - 48\right) \\ &= \tfrac{56}{3} - 36 \\ &= -17\tfrac{1}{3}\end{aligned}$$

So the area is $17\frac{1}{3}$ square units.

To calculate the area from $x = 4$ to $x = 5$:

$$\begin{aligned}\int_4^5 \left(x^2 + 2x - 24\right) dx &= \left[\tfrac{1}{3}x^3 + x^2 - 24x\right]_4^5 \\ &= \left(\tfrac{1}{3}(5)^3 + (5)^2 - 24(5)\right) - \left(\tfrac{1}{3}(4)^3 + (4)^2 - 24(4)\right) \\ &= \left(\tfrac{125}{3} + 25 - 120\right) - \left(\tfrac{64}{3} + 16 - 96\right) \\ &= \tfrac{61}{3} - 15 \\ &= 5\tfrac{1}{3}\end{aligned}$$

So the area is $5\frac{1}{3}$ square units.

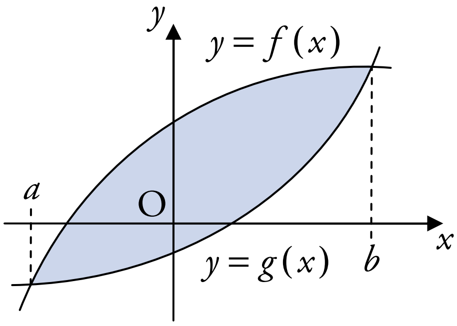
So the total shaded area is $17\frac{1}{3} + 5\frac{1}{3} = 22\frac{2}{3}$ square units.

6 Areas between Curves

The area between two curves between $x = a$ and $x = b$ is calculated as:

$$\int_a^b (\text{upper curve} - \text{lower curve})\, dx \text{ square units.}$$

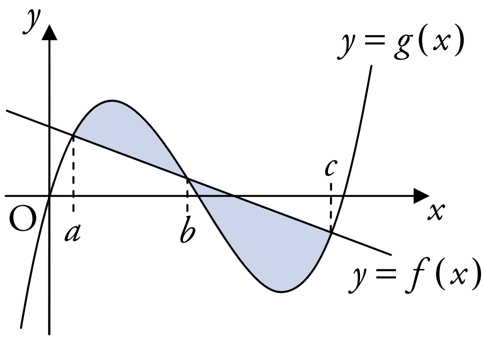
So for the shaded area shown below:



The area is $\int_a^b (f(x) - g(x))\, dx$ square units.

When dealing with areas between curves, areas above and below the *x*-axis do not need to be calculated separately.

However, care must be taken with more complicated curves, as these may give rise to more than one closed area. These areas must be evaluated separately. For example:



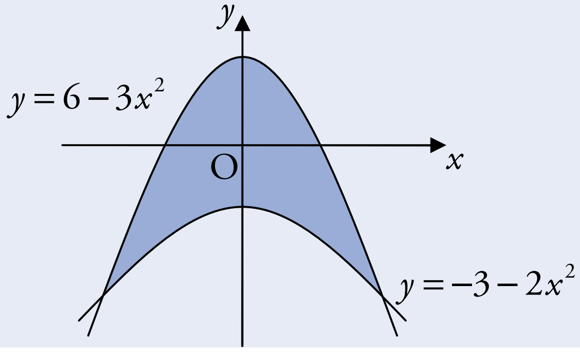
In this case we apply $\int_a^b (\text{upper curve} - \text{lower curve})\, dx$ to each area.

So the shaded area is given by:

$$\int_a^b (g(x) - f(x))\, dx + \int_b^c (f(x) - g(x))\, dx.$$

EXAMPLES

1. Calculate the shaded area enclosed by the curves with equations $y = 6 - 3x^2$ and $y = -3 - 2x^2$.



To work out the points of intersection, equate the curves:

$$\begin{aligned} 6 - 3x^2 &= -3 - 2x^2 \\ 6 + 3 - 3x^2 + 2x^2 &= 0 \\ 9 - x^2 &= 0 \\ (3+x)(3-x) &= 0 \\ x = -3 \quad &\text{or} \quad x = 3. \end{aligned}$$

Set up the integral and simplify:

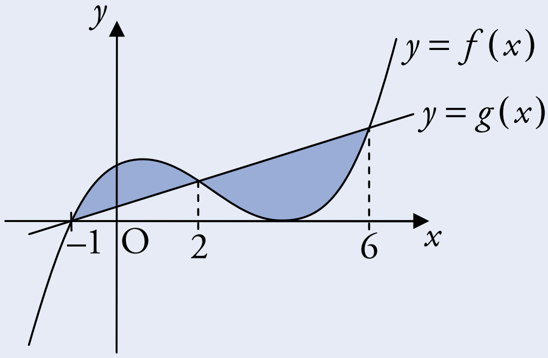
$$\begin{aligned} &\int_{-3}^{3} (\text{upper curve} - \text{lower curve})\, dx \\ &= \int_{-3}^{3} \left((6 - 3x^2) - (-3 - 2x^2) \right) dx \\ &= \int_{-3}^{3} (6 - 3x^2 + 3 + 2x^2)\, dx \\ &= \int_{-3}^{3} (9 - x^2)\, dx. \end{aligned}$$

Carry out integration:

$$\begin{aligned} \int_{-3}^{3} (9 - x^2)\, dx &= \left[9x - \frac{x^3}{3} \right]_{-3}^{3} \\ &= \left(9(3) - \frac{(3)^3}{3} \right) - \left(9(-3) - \frac{(-3)^3}{3} \right) \\ &= \left(27 - \tfrac{27}{3} \right) - \left(-27 + \tfrac{27}{3} \right) \\ &= 27 - 9 + 27 - 9 \\ &= 36. \end{aligned}$$

Therefore the shaded area is 36 square units.

2. Two functions are defined for $x \in \mathbb{R}$ by $f(x) = x^3 - 7x^2 + 8x + 16$ and $g(x) = 4x + 4$. The graphs of $y = f(x)$ and $y = g(x)$ are shown below.



Calculate the shaded area.

Since the shaded area is in two parts, we apply $\int_a^b (\text{upper} - \text{lower})\, dx$ twice.

Area from $x = -1$ to $x = 2$:

$$\int_{-1}^{2} (\text{upper} - \text{lower})\, dx$$
$$= \int_{-1}^{2} \left(x^3 - 7x^2 + 8x + 16 - (4x + 4)\right) dx$$
$$= \int_{-1}^{2} \left(x^3 - 7x^2 + 4x + 12\right) dx$$
$$= \left[\frac{x^4}{4} - \frac{7x^3}{3} + \frac{4x^2}{2} + 12x\right]_{-1}^{2}$$
$$= \left(\frac{2^4}{4} - \frac{7\times 2^3}{3} + 2\times 2^2 + 12\times 2\right) - \left(\frac{(-1)^4}{4} - \frac{7(-1)^3}{3} + 2(-1)^2 + 12(-1)\right)$$
$$= \left(4 - \tfrac{56}{3} + 8 + 24\right) - \left(\tfrac{1}{4} + \tfrac{7}{3} + 2 - 12\right)$$
$$= \frac{99}{4}$$
$$= 24\tfrac{3}{4}.$$

Note
The curve is at the top of this area.

So the first area is $24\frac{3}{4}$ square units.

Area from $x = 2$ to $x = 6$:

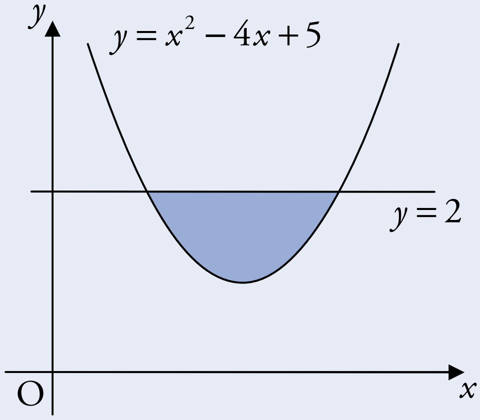
$$\int_2^6 (\text{upper} - \text{lower})\, dx$$
$$= \int_2^6 \left(4x - 4 - \left(x^3 - 7x^2 + 8x + 16\right)\right) dx$$
$$= \int_2^6 \left(-x^3 + 7x^2 - 4x - 12\right) dx$$
$$= \left[-\frac{x^4}{4} + \frac{7x^3}{3} - \frac{4x^2}{2} - 12x\right]_2^6$$
$$= \left(-\frac{6^4}{4} + \frac{7\times 6^3}{3} - \frac{4\times 6^2}{2} - 12\times 6\right) - \left(-\frac{2^4}{4} + \frac{7\times 2^3}{3} - \frac{4\times 2^2}{2} - 12\times 2\right)$$
$$= (-324 + 504 - 72 - 72) - \left(-4 + \tfrac{56}{3} - 8 - 24\right)$$
$$= \tfrac{160}{3}$$
$$= 53\tfrac{1}{3}.$$

Note
The straight line is at the top of this area.

So the second area is $53\frac{1}{3}$ square units.

So the total shaded area is $24\frac{3}{4} + 53\frac{1}{3} = 78\frac{1}{12}$ square units.

3. A trough is 2 metres long. A cross-section of the trough is shown below.



The cross-section is part of the parabola with equation $y = x^2 - 4x + 5$.

Find the volume of the trough.

To work out the points of intersection, equate the curve and the line:

$$x^2 - 4x + 5 = 2$$
$$x^2 - 4x + 3 = 0$$
$$(x-1)(x-3) = 0 \quad \text{so} \quad x = 1 \text{ or } x = 3.$$

Set up the integral and integrate:

$$\begin{aligned}
\int_1^3 (\text{upper} - \text{lower})\, dx &= \int_1^3 \left(2 - \left(x^2 - 4x + 5\right)\right) dx \\
&= \int_1^3 \left(-x^2 + 4x - 3\right) dx \\
&= \left[-\frac{x^3}{3} + \frac{4x^2}{2} - 3x\right]_1^3 \\
&= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right) - \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right) \\
&= (-9 + 18 - 9) - \left(-\tfrac{1}{3} + 2 - 3\right) \\
&= 0 + \tfrac{1}{3} - 2 + 3 \\
&= \tfrac{4}{3} \\
&= 1\tfrac{1}{3}.
\end{aligned}$$

Therefore the shaded area is $1\frac{1}{3}$ square units.

$$\begin{aligned}
\text{Volume} &= \text{cross-sectional area} \times \text{length} \\
&= \tfrac{4}{3} \times 2 \\
&= \tfrac{8}{3} \quad = 2\tfrac{2}{3}.
\end{aligned}$$

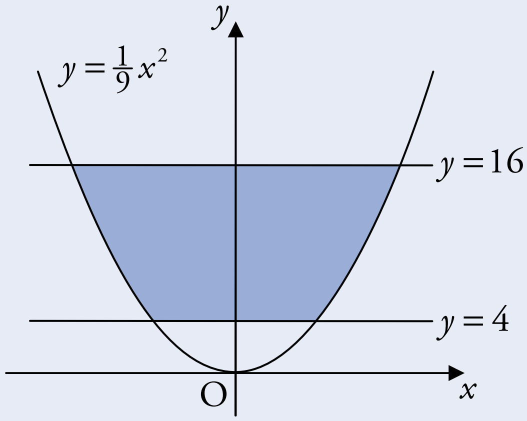
Therefore the volume of the trough is $2\frac{2}{3}$ cubic units.

7 Integrating along the *y*-axis

For some problems, it may be easier to find a shaded area by integrating with respect to y rather than x.

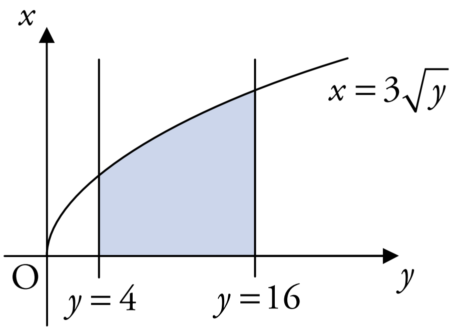
EXAMPLE

The curve with equation $y = \frac{1}{9}x^2$ is shown in the diagram below.



Calculate the shaded area which lies between $y = 4$ and $y = 16$.

We have
$$y = \tfrac{1}{9}x^2$$
$$9y = x^2$$
$$x = \pm\sqrt{9y}$$
$$x = \pm 3\sqrt{y}.$$



The shaded area in the diagram to the right is given by:

$$\begin{aligned}
\int_4^{16} 3\sqrt{y}\, dy &= \int_4^{16} 3y^{\frac{1}{2}}\, dy \\
&= \left[\frac{3y^{\frac{3}{2}}}{\frac{3}{2}}\right]_4^{16} \\
&= \left[2\sqrt{y}^3\right]_4^{16} \\
&= 2\sqrt{16}^3 - 2\sqrt{4}^3 \\
&= 2\times 64 - 2\times 8 \\
&= 112.
\end{aligned}$$

Since this is half of the required area, the total shaded area is 224 square units.

OUTCOME 3

Trigonometry

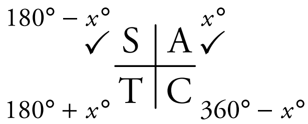
1 Solving Trigonometric Equations

You should already be familiar with solving some trigonometric equations.

EXAMPLES

1. Solve $\sin x° = \frac{1}{2}$ for $0 < x < 360$.

$\sin x° = \frac{1}{2}$



Since $\sin x°$ is positive

First quadrant solution:

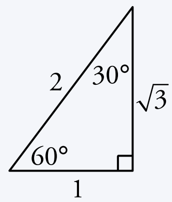
$$x = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= 30.$$

$x = 30$ or $180 - 30$

$x = 30$ or 150.

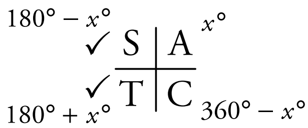
Remember

The exact value triangle:



2. Solve $\cos x° = -\frac{1}{\sqrt{5}}$ for $0 < x < 360$.

$\cos x° = -\frac{1}{\sqrt{5}}$



Since $\cos x°$ is negative

$$x = \cos^{-1}\left(\tfrac{1}{\sqrt{5}}\right)$$
$$= 63{\cdot}435 \quad \text{(to 3 d.p.)}.$$

$x = 180 - 63{\cdot}435$ or $180 + 63{\cdot}435$

$x = 116{\cdot}565$ or $243{\cdot}435$.

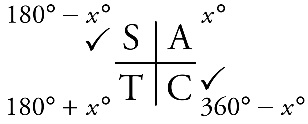
3. Solve $\sin x° = 3$ for $0 < x < 360$.

There are no solutions since $-1 \leq \sin x° \leq 1$.

Note that $-1 \leq \cos x° \leq 1$, so $\cos x° = 3$ also has no solutions.

4. Solve $\tan x° = -5$ for $0 < x < 360$.

$\tan x° = -5$



Since $\tan x°$ is negative

$$x = \tan^{-1}(5)$$
$$= 78{\cdot}690 \quad \text{(to 3 d.p.)}.$$

$$x = 180 - 78{\cdot}690 \quad \text{or} \quad 360 - 78{\cdot}690$$
$$x = 101{\cdot}310 \quad \text{or} \quad 281{\cdot}310.$$

Note

All trigonometric equations we will meet can be reduced to problems like those above. The only differences are:

- the solutions could be required in radians – in this case, the question will not have a degree symbol, e.g. "Solve $3\tan x = 1$" rather than "$3\tan x° = 1$";
- exact value solutions could be required in the non-calculator paper – you will be expected to know the exact values for 0, 30, 45, 60 and 90 degrees.

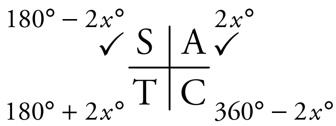
Questions can be worked through in degrees or radians, but make sure the final answer is given in the units asked for in the question.

EXAMPLES

5. Solve $2\sin 2x° - 1 = 0$ where $0 \le x \le 360$.

$2\sin 2x° = 1$

$\sin 2x° = \frac{1}{2}$



$0 \le x \le 360$

$0 \le 2x \le 720$

$$2x = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= 30.$$

$2x = 30 \quad \text{or} \quad 180 - 30$

$\quad \text{or} \quad 360 + 30 \quad \text{or} \quad 360 + 180 - 30$

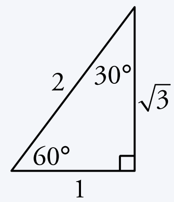
$\quad \text{or} \quad$ ~~$360 + 360 + 30$~~

$2x = 30 \quad \text{or} \quad 150 \quad \text{or} \quad 390 \quad \text{or} \quad 510$

$x = 15 \quad \text{or} \quad 75 \quad \text{or} \quad 195 \quad \text{or} \quad 255.$

Remember

The exact value triangle:



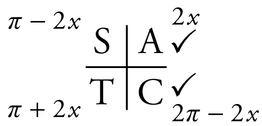
Note

There are more solutions every 360°, since $\sin(30°) = \sin(30° + 360°) = \ldots$

So keep adding 360 until $2x > 720$.

6. Solve $\sqrt{2}\cos 2x = 1$ where $0 \le x \le \pi$.

$\cos 2x = \frac{1}{\sqrt{2}}$

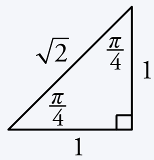


$0 \le x \le \pi$

$0 \le 2x \le 2\pi$

Remember

The exact value triangle:



$$2x = \cos^{-1}\left(\tfrac{1}{\sqrt{2}}\right) = \tfrac{\pi}{4}.$$

$2x = \frac{\pi}{4}$ or $2\pi - \frac{\pi}{4}$

or ~~$2\pi + \frac{\pi}{4}$~~

$2x = \frac{\pi}{4}$ or $\frac{7\pi}{4}$

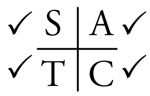
$x = \frac{\pi}{8}$ or $\frac{7\pi}{8}$.

7. Solve $4\cos^2 x = 3$ where $0 < x < 2\pi$.

$(\cos x)^2 = \frac{3}{4}$

$\cos x = \pm\sqrt{\frac{3}{4}}$

$\cos x = \pm\frac{\sqrt{3}}{2}$

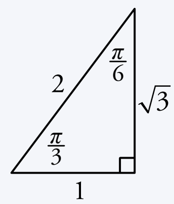


Since $\cos x$ can be positive or negative

$$x = \cos^{-1}\left(\tfrac{\sqrt{3}}{2}\right) = \tfrac{\pi}{6}.$$

Remember

The exact value triangle:



$x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6}$ or $\pi + \frac{\pi}{6}$ or $2\pi - \frac{\pi}{6}$

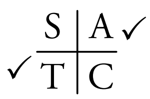
or ~~$2\pi + \frac{\pi}{6}$~~

$x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{7\pi}{6}$ or $\frac{11\pi}{6}$.

8. Solve $3\tan(3x° - 20°) = 5$ where $0 \le x \le 360$.

$3\tan(3x° - 20°) = 5$

$\tan(3x° - 20°) = \frac{5}{3}$



$0 \le x \le 360$

$0 \le 3x \le 1080$

$-20 \le 3x - 20 \le 1060$

$$3x - 20 = \tan^{-1}\left(\tfrac{5}{3}\right) = 59{\cdot}036 \text{ (to 3 d.p.).}$$

$3x - 20 = 59{\cdot}036$ or $180 + 59{\cdot}036$

or $360 + 59{\cdot}036$ or $360 + 180 + 59{\cdot}036$

or $360 + 360 + 59{\cdot}036$ or $360 + 360 + 180 + 59{\cdot}036$

or ~~$360 + 360 + 360 + 59{\cdot}036$~~

$$3x - 20 = 59{\cdot}036 \text{ or } 239{\cdot}036 \text{ or } 419{\cdot}036$$
$$\text{or } 599{\cdot}036 \text{ or } 779{\cdot}036 \text{ or } 959{\cdot}036$$
$$3x = 79{\cdot}036 \text{ or } 259{\cdot}036 \text{ or } 439{\cdot}036$$
$$\text{or } 619{\cdot}036 \text{ or } 799{\cdot}036 \text{ or } 979{\cdot}036$$
$$x = 26{\cdot}35 \text{ or } 86{\cdot}35 \text{ or } 146{\cdot}35 \text{ or } 206{\cdot}35 \text{ or } 266{\cdot}35 \text{ or } 326{\cdot}35.$$

9. Solve $\cos\left(2x + \frac{\pi}{3}\right) = 0{\cdot}812$ for $0 < x < 2\pi$.

$$\cos\left(2x + \tfrac{\pi}{3}\right) = 0{\cdot}812$$

S	A ✓
T	C ✓

$$0 < x < 2\pi$$
$$0 < 2x < 4\pi$$
$$\tfrac{\pi}{3} < 2x + \tfrac{\pi}{3} < 4\pi + \tfrac{\pi}{3}$$
$$1{\cdot}047 < 2x + \tfrac{\pi}{3} < 13{\cdot}614 \quad \text{(to 3 d.p.)}$$

$$2x + \tfrac{\pi}{3} = \cos^{-1}(0{\cdot}812)$$
$$= 0{\cdot}623 \quad \text{(to 3 d.p.)}.$$

Remember
Make sure your calculator uses radians.

$2x + \frac{\pi}{3} =$ ~~$0{\cdot}623$~~ or $2\pi - 0{\cdot}623$

or $2\pi + 0{\cdot}623$ or $2\pi + 2\pi - 0{\cdot}623$

or $2\pi + 2\pi + 0{\cdot}623$ or ~~$2\pi + 2\pi + 2\pi - 0{\cdot}623$~~

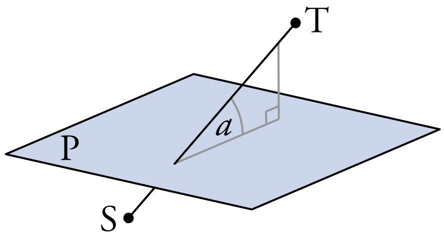
$$2x + \tfrac{\pi}{3} = 5{\cdot}660 \text{ or } 6{\cdot}906 \text{ or } 11{\cdot}943 \text{ or } 13{\cdot}189$$
$$2x = 4{\cdot}613 \text{ or } 5{\cdot}859 \text{ or } 10{\cdot}896 \text{ or } 12{\cdot}142$$
$$x = 2{\cdot}307 \text{ or } 2{\cdot}930 \text{ or } 5{\cdot}448 \text{ or } 6{\cdot}071.$$

2 Trigonometry in Three Dimensions

It is possible to solve trigonometric problems in three dimensions using techniques we already know from two dimensions. The use of sketches is often helpful.

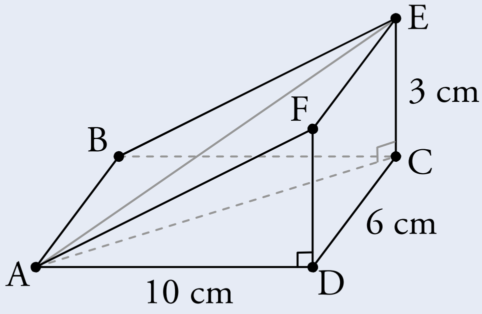
The angle between a line and a plane

The angle a between the plane P and the line ST is calculated by adding a line perpendicular to the plane and then using basic trigonometry.



EXAMPLE

1. The triangular prism ABCDEF is shown below.

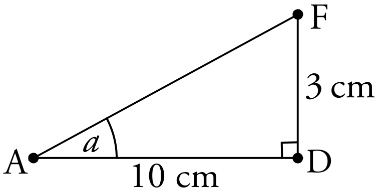


Calculate the acute angle between:

(a) The line AF and the plane ABCD.

(b) AE and ABCD.

(a) Start with a sketch:



$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{10}$$

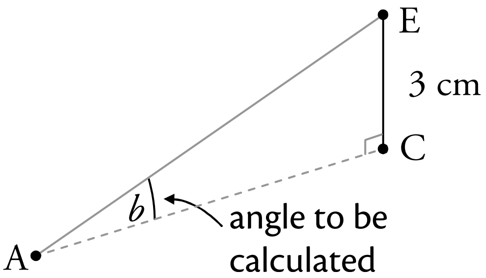
$$a = \tan^{-1}\left(\tfrac{3}{10}\right)$$

$$= 16{\cdot}699° \quad \text{(or } 0{\cdot}291 \text{ radians)} \quad \text{(to 3 d.p.)}.$$

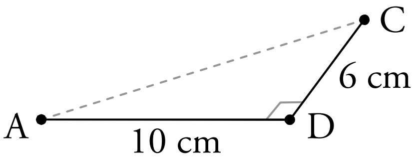
Note

Since the angle is in a right-angled triangle, it must be acute so there is no need for a CAST diagram.

(b) Again, make a sketch:



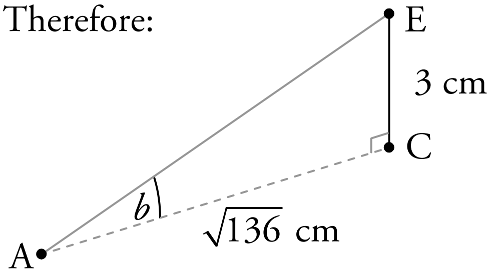
We need to calculate the length of AC first using Pythagoras's Theorem:



$$AC = \sqrt{10^2 + 6^2}$$

$$= \sqrt{136}.$$

Therefore:



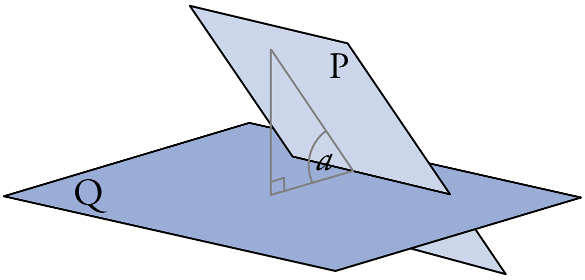
$$\tan b = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{3}{\sqrt{136}}$$

$$b = \tan^{-1}\left(\frac{3}{\sqrt{136}}\right)$$

$$= 14{\cdot}426° \quad \text{(or } 0{\cdot}252 \text{ radians)} \quad \text{(to 3 d.p.)}.$$

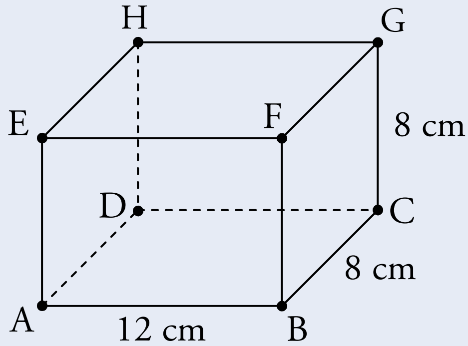
The angle between two planes

The angle a between planes P and Q is calculated by adding a line perpendicular to Q and then using basic trigonometry.



EXAMPLE

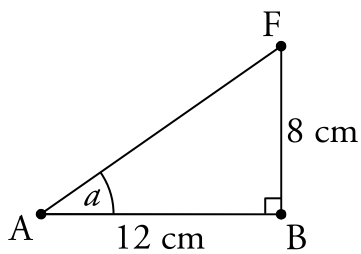
2. ABCDEFGH is a cuboid with dimensions $12 \times 8 \times 8$ cm as shown below.



(a) Calculate the size of the angle between the planes AFGD and ABCD.

(b) Calculate the size of the acute angle between the diagonal planes AFGD and BCHE.

(a) Start with a sketch:



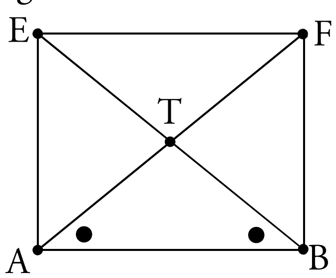
$$\tan a = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{8}{12}$$

$$a = \tan^{-1}\left(\tfrac{2}{3}\right)$$

$$= 33 \cdot 690° \quad (\text{or } 0 \cdot 588 \text{ radians}) \quad (\text{to 3 d.p.}).$$

Note
Angle GDC is the same size as angle FAB.

(b) Again, make a sketch:



Let AF and BE intersect at T.

$\triangle ATB$ is isosceles, so $T\hat{A}B = T\hat{B}A = 33 \cdot 690°$.

$$A\hat{T}B = 180° - (33 \cdot 690° + 33 \cdot 690°)$$
$$= 112 \cdot 620°.$$

So the acute angle is:

$$B\hat{T}F = A\hat{T}E = 180° - 112 \cdot 620°$$
$$= 67 \cdot 380° \quad (\text{or } 1 \cdot 176 \text{ radians}) \quad (\text{to 3 d.p.}).$$

Note
The angle could also have been calculated using rectangle DCGH.

3 Compound Angles

When we add or subtract angles, the result is called a **compound angle**.

For example, $45° + 30°$ is a compound angle. Using a calculator, we find:

- $\sin(45° + 30°) = \sin(75°) = 0{\cdot}966$;
- $\sin(45°) + \sin(30°) = 1{\cdot}207$ (both to 3 d.p.).

This shows that $\sin(A + B)$ is *not* equal to $\sin A + \sin B$. Instead, we can use the following identities:

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B.$$

These are given in the exam in a condensed form:

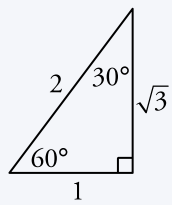
$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B.$$

EXAMPLES

1. Expand and simplify $\cos(x° + 60°)$.

$$\cos(x° + 60°) = \cos x°\cos 60° - \sin x°\sin 60°$$
$$= \tfrac{1}{2}\cos x° - \tfrac{\sqrt{3}}{2}\sin x°.$$

Remember
The exact value triangle:



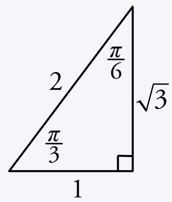
2. Show that $\sin(a + b) = \sin a\cos b + \cos a\sin b$ for $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$.

$$\begin{aligned}\text{LHS} &= \sin(a + b)\\ &= \sin\left(\tfrac{\pi}{6} + \tfrac{\pi}{3}\right)\\ &= \sin\left(\tfrac{\pi}{2}\right)\\ &= 1.\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \sin a\cos b + \cos a\sin b\\ &= \sin\tfrac{\pi}{6}\cos\tfrac{\pi}{3} + \cos\tfrac{\pi}{6}\sin\tfrac{\pi}{3}\\ &= \left(\tfrac{1}{2} \times \tfrac{1}{2}\right) + \left(\tfrac{\sqrt{3}}{2} \times \tfrac{\sqrt{3}}{2}\right)\\ &= \tfrac{1}{4} + \tfrac{3}{4} = 1.\end{aligned}$$

Since $\text{LHS} = \text{RHS}$, the claim is true for $a = \frac{\pi}{6}$ and $b = \frac{\pi}{3}$.

Remember
The exact value triangle:

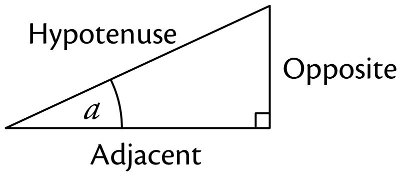


3. Find the exact value of $\sin 75°$.

$$\begin{aligned}\sin 75° &= \sin(45° + 30°)\\ &= \sin 45° \cos 30° + \cos 45° \sin 30°\\ &= \left(\tfrac{1}{\sqrt{2}} \times \tfrac{\sqrt{3}}{2}\right) + \left(\tfrac{1}{\sqrt{2}} \times \tfrac{1}{2}\right)\\ &= \tfrac{\sqrt{3}}{2\sqrt{2}} + \tfrac{1}{2\sqrt{2}}\\ &= \tfrac{\sqrt{3}+1}{2\sqrt{2}}\\ &= \tfrac{\sqrt{6}+\sqrt{2}}{4}.\end{aligned}$$

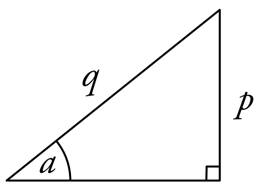
Finding Trigonometric Ratios

You should already be familiar with the following formulae (SOH CAH TOA).



$$\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} \qquad \cos a = \frac{\text{Adjacent}}{\text{Hypotenuse}} \qquad \tan a = \frac{\text{Opposite}}{\text{Adjacent}}.$$

If we have $\sin a = \frac{p}{q}$ where $0 < a < \frac{\pi}{2}$, then we can form a right-angled triangle to represent this ratio.



Since $\sin a = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{p}{q}$ then:

- the side opposite a has length p;
- the hypotenuse has length q.

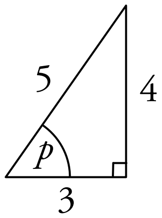
The length of the unknown side can be found using Pythagoras's Theorem.

Once the length of each side is known, we can find $\cos a$ and $\tan a$ using SOH CAH TOA.

The method is similar if we know $\cos a$ and want to find $\sin a$ or $\tan a$.

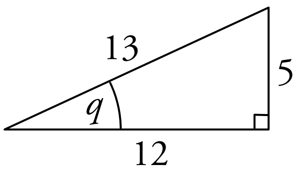
EXAMPLES

4. Acute angles p and q are such that $\sin p = \frac{4}{5}$ and $\sin q = \frac{5}{13}$. Show that $\sin(p+q) = \frac{63}{65}$.



$\sin p = \frac{4}{5}$

$\cos p = \frac{3}{5}$



$\sin q = \frac{5}{13}$

$\cos q = \frac{12}{13}$

$$\begin{aligned}\sin(p+q) &= \sin p \cos q + \cos p \sin q \\ &= \left(\tfrac{4}{5} \times \tfrac{12}{13}\right) + \left(\tfrac{3}{5} \times \tfrac{5}{13}\right) \\ &= \tfrac{48}{65} + \tfrac{15}{65} \\ &= \tfrac{63}{65}.\end{aligned}$$

Note
Since "Show that" is used in the question, all of this working is required.

Confirming Identities

EXAMPLES

5. Show that $\sin\left(x - \frac{\pi}{2}\right) = -\cos x$.

$$\begin{aligned}&\sin\left(x - \tfrac{\pi}{2}\right) \\ &= \sin x \cos \tfrac{\pi}{2} - \cos x \sin \tfrac{\pi}{2} \\ &= \sin x \times 0 - \cos x \times 1 \\ &= -\cos x.\end{aligned}$$

6. Show that $\dfrac{\sin(s+t)}{\cos s \cos t} = \tan s + \tan t$ for $\cos s \neq 0$ and $\cos t \neq 0$.

$$\begin{aligned}\frac{\sin(s+t)}{\cos s \cos t} &= \frac{\sin s \cos t + \cos s \sin t}{\cos s \cos t} \\ &= \frac{\sin s \cos t}{\cos s \cos t} + \frac{\cos s \sin t}{\cos s \cos t} \\ &= \frac{\sin s}{\cos s} + \frac{\sin t}{\cos t} \\ &= \tan s + \tan t.\end{aligned}$$

Remember
$\dfrac{\sin x}{\cos x} = \tan x.$

4 Double-Angle Formulae

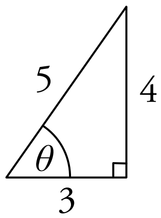
Using the compound angle identities with $A = B$, we obtain expressions for $\sin 2A$ and $\cos 2A$. These are called **double-angle formulae**.

$$\sin 2A = 2\sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A$$
$$= 2\cos^2 A - 1$$
$$= 1 - 2\sin^2 A.$$

Note that these are given in the exam.

EXAMPLES

1. Given that $\tan\theta = \frac{4}{3}$, where $0 < \theta < \frac{\pi}{2}$, find the exact value of $\sin 2\theta$ and $\cos 2\theta$.



$$\sin\theta = \frac{4}{3}$$
$$\cos\theta = \frac{3}{5}$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$= 2 \times \frac{4}{5} \times \frac{3}{5}$$
$$= \frac{24}{25}.$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$= \left(\frac{3}{5}\right)^2 - \left(\frac{4}{5}\right)^2$$
$$= \frac{9}{25} - \frac{16}{25}$$
$$= -\frac{7}{25}.$$

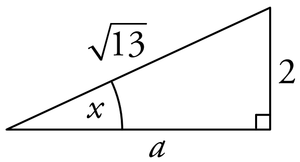
Note
Any of the $\cos 2A$ formulae could have been used here.

2. Given that $\cos 2x = \frac{5}{13}$, where $0 < x < \pi$, find the exact values of $\sin x$ and $\cos x$.

Since $\cos 2x = 1 - 2\sin^2 x$,

$$1 - 2\sin^2 x = \frac{5}{13}$$
$$2\sin^2 x = \frac{8}{13}$$
$$\sin^2 x = \frac{8}{26}$$
$$= \frac{4}{13}$$
$$\sin x = \pm\frac{2}{\sqrt{13}}.$$

We are told that $0 < x < \pi$, so only $\sin x = \frac{2}{\sqrt{13}}$ is possible.



$$a = \sqrt{\sqrt{13}^2 - 2^2} = \sqrt{13 - 4} = \sqrt{9} = 3.$$

So $\cos x = \frac{3}{\sqrt{13}}$.

5 Further Trigonometric Equations

We will now consider trigonometric equations where double-angle formulae can be used to find solutions. These equations will involve:

- $\sin 2x$ and either $\sin x$ or $\cos x$;
- $\cos 2x$ and $\cos x$;
- $\cos 2x$ and $\sin x$.

Remember
The double-angle formulae are given in the exam.

Solving equations involving sin2*x* and either sin*x* or cos*x*

EXAMPLE

1. Solve $\sin 2x° = -\sin x°$ for $0 \leq x < 360$.

$$2\sin x° \cos x° = -\sin x°$$
$$2\sin x° \cos x° + \sin x° = 0$$
$$\sin x°(2\cos x° + 1) = 0$$

- Replace $\sin 2x$ using the double angle formula
- Take all terms to one side, making the equation equal to zero
- Factorise the expression and solve

$$\sin x° = 0$$
$$x = 0 \text{ or } 180 \text{ or } \cancel{360}$$

$$2\cos x° + 1 = 0$$
$$\cos x° = -\tfrac{1}{2}$$
$$x = 180 - 60 \quad\text{or}\quad 180 + 60$$
$$= 120 \quad\text{or}\quad 240.$$

✓ S | A
✓ T | C

$$x = \cos^{-1}\left(\tfrac{1}{2}\right)$$
$$= 60$$

So $x = 0$ or 120 or 180 or 240.

Solving equations involving cos2*x* and cos*x*

EXAMPLE

2. Solve $\cos 2x = \cos x$ for $0 \leq x \leq 2\pi$.

$$\cos 2x = \cos x$$
$$2\cos^2 x - 1 = \cos x$$
$$2\cos^2 x - \cos x - 1 = 0$$
$$(2\cos x + 1)(\cos x - 1) = 0$$

- Replace $\cos 2x$ by $2\cos^2 x - 1$
- Take all terms to one side, making a quadratic equation in $\cos x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$$2\cos x + 1 = 0$$
$$\cos x = -\tfrac{1}{2}$$
$$x = \pi - \tfrac{\pi}{3} \quad\text{or}\quad \pi + \tfrac{\pi}{3}$$
$$= \tfrac{2\pi}{3} \quad\text{or}\quad \tfrac{4\pi}{3}$$

✓ S | A
✓ T | C

$$x = \cos^{-1}\left(\tfrac{1}{2}\right)$$
$$= \tfrac{\pi}{3}$$

$$\cos x - 1 = 0$$
$$\cos x = 1$$
$$x = 0 \text{ or } 2\pi.$$

So $x = 0$ or $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$ or 2π.

Solving equations involving cos2*x* and sin*x*

EXAMPLE

3. Solve $\cos 2x = \sin x$ for $0 < x < 2\pi$.

$$\cos 2x = \sin x$$
$$1 - 2\sin^2 x = \sin x$$
$$2\sin^2 x + \sin x - 1 = 0$$
$$(2\sin x - 1)(\sin x + 1) = 0$$

- Replace $\cos 2x$ by $1 - 2\sin^2 x$
- Take all terms to one side, making a quadratic equation in $\sin x$
- Solve the quadratic equation (using factorisation or the quadratic formula)

$$2\sin x - 1 = 0$$
$$\sin x = \tfrac{1}{2}$$
$$x = \tfrac{\pi}{6} \quad \text{or} \quad \pi - \tfrac{\pi}{6}$$
$$= \tfrac{\pi}{6} \quad \text{or} \quad \tfrac{5\pi}{6}$$

✓ S	A ✓
T	C

$$x = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= \tfrac{\pi}{6}$$

$$\sin x + 1 = 0$$
$$\sin x = -1$$
$$x = \tfrac{3\pi}{2}.$$

So $x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{3\pi}{2}$.

OUTCOME 4

Circles

1 Representing a Circle

The equation of a circle with centre (a, b) and radius r units is

$$(x-a)^2+(y-b)^2=r^2.$$

This is given in the exam.

For example, the circle with centre $(2, -1)$ and radius 4 units has equation:

$$(x-2)^2+(y+1)^2=4^2$$
$$(x-2)^2+(y+1)^2=16.$$

Note that the equation of a circle with centre $(0,0)$ is of the form $x^2+y^2=r^2$, where r is the radius of the circle.

EXAMPLES

1. Find the equation of the circle with centre $(1, -3)$ and radius $\sqrt{3}$ units.

$$(x-a)^2+(y-b)^2=r^2$$
$$(x-1)^2+(y-(-3))^2=\left(\sqrt{3}\right)^2$$
$$(x-1)^2+(y+3)^2=3.$$

2. A is the point $(-3,1)$ and $B(5,3)$.

   Find the equation of the circle which has AB as a diameter.

The centre of the circle is the midpoint of AB;

$$C=\text{midpoint}_{AB}=\left(\frac{5-3}{2},\frac{3+1}{2}\right)=(1,2).$$

The radius r is the distance between A and C:

$$\begin{aligned} r^2 &= (1-(-3))^2+(2-1)^2 \\ &= 4^2+1^2 \\ &= 17. \end{aligned}$$

Note
You could also use the distance between B and C, or half the distance between A and B.

So the equation of the circle is $(x-1)^2+(y-2)^2=17$.

2 Testing a Point

Given a circle with centre (a, b) and radius r units, we can determine whether a point (p, q) lies within, outwith or on the circumference using the following rules:

$(p-a)^2+(q-b)^2<r^2 \Leftrightarrow$ the point lies within the circle

$(p-a)^2+(q-b)^2=r^2 \Leftrightarrow$ the point lies on the circumference of the circle

$(p-a)^2+(q-b)^2>r^2 \Leftrightarrow$ the point lies outwith the circle.

EXAMPLE

A circle has the equation $(x-2)^2+(y+5)^2=29$.

Determine whether the points $(2,1)$, $(7,-3)$ and $(3,-4)$ lie within, outwith or on the circumference of the circle.

Point $(2,1)$:

$$\begin{aligned}&(x-2)^2+(y+3)^2\\&=(2-2)^2+(1+5)^2\\&=0^2+6^2\\&=36>29\end{aligned}$$

So outwith the circle.

Point $(7,-3)$:

$$\begin{aligned}&(x-2)^2+(y+3)^2\\&=(7-2)^2+(-3+5)^2\\&=5^2+2^2\\&=29\end{aligned}$$

So on the circumference.

Point $(3,-4)$:

$$\begin{aligned}&(x-2)^2+(y+3)^2\\&=(3-2)^2+(-4+5)^2\\&=1^2+1^2\\&=2<29\end{aligned}$$

So within the circle.

3 The General Equation of a Circle

The equation of any circle can be written in the form

$$x^2+y^2+2gx+2fy+c=0$$

where the centre is $(-g,-f)$ and the radius is $\sqrt{g^2+f^2-c}$ units.

This is given in the exam.

Note that the above equation only represents a circle if $g^2+f^2-c>0$, since:

- if $g^2+f^2-c<0$ then we cannot obtain a real value for the radius, since we would have to square root a negative;
- if $g^2+f^2-c=0$ then the radius is zero – the equation represents a point rather than a circle.

EXAMPLES

1. Find the radius and centre of the circle with equation $x^2 + y^2 + 4x - 8y + 7 = 0$.

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = 4$ so $g = 2$

$2f = -8$ so $f = -4$

$c = 7$

Centre is $(-g, -f)$
$= (-2, 4)$

Radius is $\sqrt{g^2 + f^2 - c}$
$= \sqrt{2^2 + (-4)^2 - 7}$
$= \sqrt{4 + 16 - 7}$
$= \sqrt{13}$ units.

2. Find the radius and centre of the circle with equation $2x^2 + 2y^2 - 6x + 10y - 2 = 0$.

The equation must be in the form $x^2 + y^2 + 2gx + 2fy + c = 0$, so divide each term by 2:

$$x^2 + y^2 - 3x + 5y - 1 = 0$$

Now compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -3$ so $g = -\frac{3}{2}$

$2f = 5$ so $f = \frac{5}{2}$

$c = -1$

Centre is $(-g, -f)$
$= \left(\frac{3}{2}, -\frac{5}{2}\right)$

Radius is $\sqrt{g^2 + f^2 - c}$
$= \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{5}{2}\right)^2 + 1}$
$= \sqrt{\frac{9}{4} + \frac{25}{4} + \frac{4}{4}}$
$= \sqrt{\frac{38}{4}}$
$= \frac{\sqrt{38}}{2}$ units.

3. Explain why $x^2 + y^2 + 4x - 8y + 29 = 0$ is not the equation of a circle.

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = 4$ so $g = 2$

$2f = -8$ so $f = -4$

$c = 29$

$g^2 + f^2 - c = 2^2 + (-4)^2 - 29$
$= -9 < 0.$

The equation does not represent a circle since $g^2 + f^2 - c > 0$ is not satisfied.

4. For which values of k does $x^2 + y^2 - 2kx - 4y + k^2 + k - 4 = 0$ represent a circle?

Comparing with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -2k$ so $g = -k$

$2f = -4$ so $f = -2$

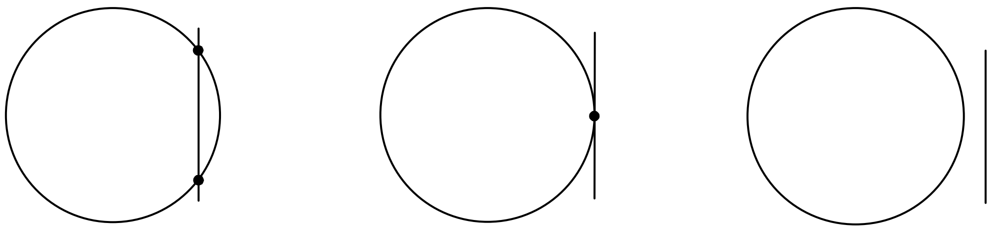
$c = k^2 + k - 4.$

To represent a circle, we need

$$g^2 + f^2 - c > 0$$
$$k^2 + 4 - (k^2 + k - 4) > 0$$
$$-k + 8 > 0$$
$$k < 8.$$

4 Intersection of a Line and a Circle

A straight line and circle can have two, one or no points of intersection:



If a line and a circle only touch at one point, then the line is a **tangent** to the circle at that point.

To find out how many times a line and circle meet, we can use substitution.

EXAMPLES

1. Find the points where the line with equation $y = 3x$ intersects the circle with equation $x^2 + y^2 = 20$.

$$x^2 + y^2 = 20$$
$$x^2 + (3x)^2 = 20$$
$$x^2 + 9x^2 = 20$$
$$10x^2 = 20$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

Remember
$(ab)^m = a^m b^m$.

$x = \sqrt{2}$
$\Rightarrow \quad y = 3(\sqrt{2}) = 3\sqrt{2}$

$x = -\sqrt{2}$
$\Rightarrow \quad y = 3(-\sqrt{2}) = -3\sqrt{2}.$

So the circle and the line meet at $(\sqrt{2}, 3\sqrt{2})$ and $(-\sqrt{2}, -3\sqrt{2})$.

2. Find the points where the line with equation $y = 2x + 6$ and circle with equation $x^2 + y^2 + 2x + 2y - 8 = 0$ intersect.

Substitute $y = 2x + 6$ into the equation of the circle:

$$\begin{aligned} x^2 + (2x+6)^2 + 2x + 2(2x+6) - 8 &= 0 \\ x^2 + (2x+6)(2x+6) + 2x + 4x + 12 - 8 &= 0 \\ x^2 + 4x^2 + 24x + 36 + 2x + 4x + 12 - 8 &= 0 \\ 5x^2 + 30x + 40 &= 0 \\ 5(x^2 + 6x + 8) &= 0 \\ (x+2)(x+4) &= 0 \end{aligned}$$

$$\begin{aligned} x + 2 &= 0 \\ x &= -2 \\ \Rightarrow y &= 2(-2) + 6 = 2 \end{aligned} \qquad \begin{aligned} x + 4 &= 0 \\ x &= -4 \\ \Rightarrow y &= 2(-4) + 6 = -2. \end{aligned}$$

So the line and circle meet at $(-2, 2)$ and $(-4, -2)$.

5 Tangents to Circles

As we have seen, a line is a tangent if it intersects the circle at only one point.

To show that a line is a tangent to a circle, the equation of the line can be substituted into the equation of the circle, and solved – there should only be one solution.

EXAMPLE

Show that the line with equation $x + y = 4$ is a tangent to the circle with equation $x^2 + y^2 + 6x + 2y - 22 = 0$.

Substitute y using the equation of the straight line:

$$\begin{aligned} x^2 + y^2 + 6x + 2y - 22 &= 0 \\ x^2 + (4-x)^2 + 6x + 2(4-x) - 22 &= 0 \\ x^2 + (4-x)(4-x) + 6x + 2(4-x) - 22 &= 0 \\ x^2 + 16 - 8x + x^2 + 6x + 8 - 2x - 22 &= 0 \\ 2x^2 - 4x + 2 &= 0 \\ 2(x^2 - 2x + 1) &= 0 \\ x^2 - 2x + 1 &= 0 \end{aligned}$$

Then (i) factorise or (ii) use the discriminant

(i)

$$x^2 - 2x + 1 = 0$$
$$(x-1)(x-1) = 0$$
$$x - 1 = 0 \qquad x - 1 = 0$$
$$x = 1 \qquad x = 1.$$

Since the solutions are equal, the line is a tangent to the circle.

(ii)

$$x^2 - 2x + 1 = 0$$

$$a = 1 \qquad b = -2 \qquad c = 1$$

$$\begin{aligned} & b^2 - 4ac \\ &= (-2)^2 - 4(1\times 1) \\ &= 4 - 4 \\ &= 0. \end{aligned}$$

Since $b^2 - 4ac = 0$, the line is a tangent to the circle.

Note

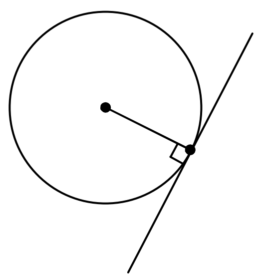
If the point of contact is required then method (i) is more efficient.

To find the point, substitute the value found for x into the equation of the line (or circle) to calculate the corresponding y-coordinate.

6 Equations of Tangents to Circles

If the point of contact between a circle and a tangent is known, then the equation of the tangent can be calculated.

If a line is a tangent to a circle, then a radius will meet the tangent at right angles. The gradient of this radius can be calculated, since the centre and point of contact are known.



Using $m_{\text{radius}} \times m_{\text{tangent}} = -1$, the gradient of the tangent can be found.

The equation can then be found using $y - b = m(x - a)$, since the point is known, and the gradient has just been calculated.

EXAMPLE

Show that $A(1, 3)$ lies on the circle $x^2 + y^2 + 6x + 2y - 22 = 0$ and find the equation of the tangent at A.

Substitute point into equation of circle:

$$\begin{aligned} & x^2 + y^2 + 6x + 2y - 22 \\ &= 1^2 + 3^2 + 6(1) + 2(3) - 22 \\ &= 1 + 9 + 6 + 6 - 22 \\ &= 0. \end{aligned}$$

Since this satisfies the equation of the circle, the point must lie on the circle.

Find the gradient of the radius from $(-3, -1)$ to $(1, 3)$:

$$\begin{aligned} m_{\text{radius}} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{3+1}{1+3} \\ &= 1. \end{aligned}$$

So $m_{\text{tangent}} = -1$ since $m_{\text{radius}} \times m_{\text{tangent}} = -1$.

Find equation of tangent using point $(1, 3)$ and gradient $m = -1$:

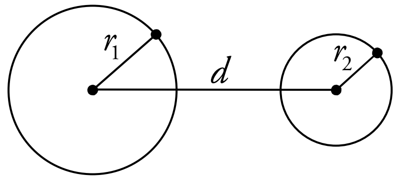
$$\begin{aligned} y - b &= m(x - a) \\ y - 3 &= -(x - 1) \\ y - 3 &= -x + 1 \\ y &= -x + 4 \\ x + y - 4 &= 0. \end{aligned}$$

Therefore the equation of the tangent to the circle at A is $x + y - 4 = 0$.

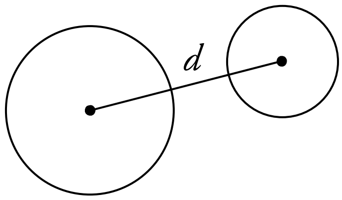
7 Intersection of Circles

Consider two circles with radii r_1 and r_2 with $r_1 > r_2$.

Let d be the distance between the centres of the two circles.

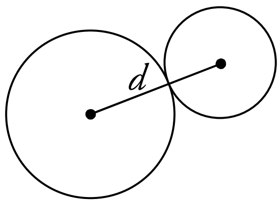


$d > r_1 + r_2$



The circles do not touch.

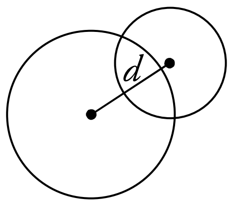
$d = r_1 + r_2$



The circles touch externally.

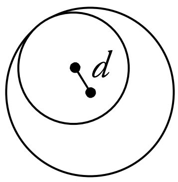
Note
Don't try to memorise this, just try to understand why each one is true.

$r_1 - r_2 < d < r_1 + r_2$



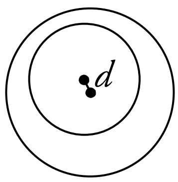
The circles meet at two distinct points.

$d = r_1 - r_2$



The circles touch internally.

$d < r_1 - r_2$



The circles do not touch.

EXAMPLES

1. Circle P has centre $(-4, -1)$ and radius 2 units, circle Q has equation $x^2 + y^2 - 2x + 6y + 1 = 0$. Show that the circles P and Q do not touch.

To find the centre and radius of Q:

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -2$ so $g = -1$
$2f = 6$ so $f = 3$
$c = 1$.

Centre is $(-g, -f)$
$= (1, -3)$.

$$\begin{aligned}\text{Radius } r_Q &= \sqrt{g^2 + f^2 - c} \\ &= \sqrt{1 + 9 - 1} \\ &= \sqrt{9} \\ &= 3 \text{ units.}\end{aligned}$$

We know P has centre $(-4, -1)$ and radius $r_P = 2$ units.

So the distance between the centres $d = \sqrt{(1+4)^2 + (-3+1)^2}$

$= \sqrt{5^2 + (-2)^2}$

$= \sqrt{29} = 5{\cdot}39$ units (to 2 d.p.).

Since $r_P + r_Q = 3 + 2 = 5 < d$, the circles P and Q do not touch.

2. Circle R has equation $x^2 + y^2 - 2x - 4y - 4 = 0$, and circle S has equation $(x-4)^2 + (y-6)^2 = 4$. Show that the circles R and S touch externally.

To find the centre and radius of R:

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$:

$2g = -2$ so $g = -1$
$2f = -4$ so $f = -2$
$c = -4$.

Centre is $(-g, -f) = (1, 2)$.

Radius $r_R = \sqrt{g^2 + f^2 - c}$
$= \sqrt{(-1)^2 + (-2)^2 + 4}$
$= \sqrt{9}$
$= 3$ units.

To find the centre and radius of S:

compare with $(x-a)^2 + (y-b)^2 = r^2$.

$a = 4$
$b = 6$
$r^2 = 4$ so $r = 2$.

Centre is $(a, b) = (4, 6)$.

Radius $r_S = 2$ units.

So the distance between the centres $d = \sqrt{(1-4)^2 + (2-6)^2}$

$= \sqrt{(-3)^2 + (-4)^2}$

$= \sqrt{25}$

$= 5$ units.

Since $r_R + r_S = 3 + 2 = 5 = d$, the circles R and S touch externally.