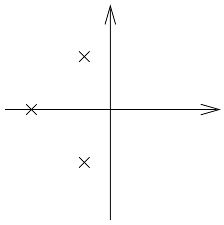
2012 Mathematics

Advanced Higher

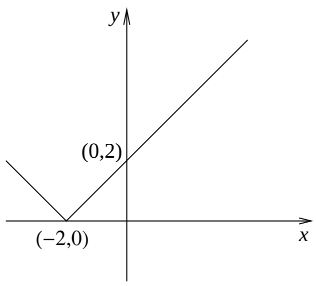
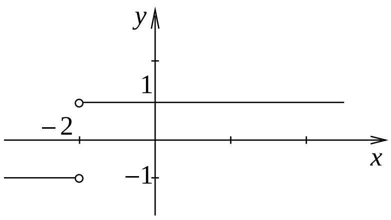
Finalised Marking Instructions

Advanced Higher Mathematics 2012

			Marks awarded for
1. **(3,4)**	(a) $f(x) = \dfrac{3x+1}{x^2+1}$		
	$f'(x) = \dfrac{3(x^2+1) - (3x+1)2x}{(x^2+1)^2}$	**1M**	for quotient rule (or product)
		1	for two correct terms
		1	for third correct term
	$= \dfrac{3x^2 + 3 - 6x^2 - 2x}{(x^2+1)^2}$		
	$= \dfrac{-3x^2 - 2x + 3}{(x^2+1)^2}$		
	(b) $g(x) = \cos^2 x\, e^{\tan x}$		
		1M	product rule
	$g'(x) = 2\cos x(-\sin x)e^{\tan x} + (\cos^2 x)(\sec^2 x)e^{\tan x}$	**1**	first correct term
		1	second correct term
	$= -\sin 2x\, e^{\tan x} + e^{\tan x}$	**1**	simplification
	$= (1 - \sin 2x)e^{\tan x}$		
	(b) *alternative*		
	$g(x) = \cos^2 x \exp(\tan x)$		
	$\ln(g(x)) = \ln(\cos^2 x) + \tan x$	**1M**	
	$= 2\ln(\cos x) + \tan x$		
	Differentiating		
	$\dfrac{g'(x)}{g(x)} = 2\dfrac{(-\sin x)}{\cos x} + \sec^2 x$	**1**	
	$g'(x) = \left(\dfrac{1 - 2\sin x\cos x}{\cos^2 x}\right)\cos^2 x \exp(\tan x)$	**1**	
	$= (1 - \sin 2x)\tan x$	**1**	
2. **(5)**	$a = 2048$ and $ar^3 = 256$	**1M**	valid approach
	$\Rightarrow r^3 = \frac{1}{8}$		
	$\Rightarrow r = \frac{1}{2}$.	**1**	correct answer only, 2 marks
	$S_n = \dfrac{a(1 - r^n)}{1 - r}$		
	$\Rightarrow \dfrac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = \dfrac{4088}{2048}$	**1M**	for sum formula
	$= \dfrac{511}{256}$		
	$\Rightarrow 1 - \left(\dfrac{1}{2}\right)^n = \dfrac{511}{256} \times \dfrac{1}{2} = \dfrac{511}{512}$		
	$\dfrac{1}{2^n} = 1 - \dfrac{511}{512} = \dfrac{1}{512}$	**1**	
	$\Rightarrow 2^n = 512 \Rightarrow n = 9$	**1**	any valid method

			Marks awarded for
3.	Since w is a root, $\overline{w} = -1 - 2i$ is also a root.	**1**	for conjugate
(6)	The corresponding factors are		
	$(z + 1 - 2i)$ and $(z + 1 + 2i)$		
	from which		
	$((z+1) - 2i)((z+1) + 2i) = (z+1)^2 + 4$		
	$= z^2 + 2z + 5$	**1**	
	$z^3 + 5z^2 + 11z + 15 = (z^2 + 2z + 5)(z + 3)$	**1**	evidence needed
	The roots are $(-1 + 2i)$, $(-1 - 2i)$ and -3.	**1**	for stating roots together
		**1**	for two correct points
		1	for third correct point
4.	The general term is given by:		
(5)	$\binom{9}{r}(2x)^{9-r}\left(-\frac{1}{x^2}\right)^r$	**1**	
	$= \binom{9}{r} \times \frac{2^{9-r}x^{9-r}(-1)^r}{x^{2r}}$	**1**	
	$= \binom{9}{r} \times (-1)^r 2^{9-r} x^{9-3r}$	**1**	
	The term independent of x occurs when		
	$9 - 3r = 0$, i.e. when $r = 3$.	**1**	
	The term is: $\frac{9!}{6!\,3!}(-1)^3 2^6$		
	$= -5376.$	**1**	
5.	*Method* 1		
(5)	$\overrightarrow{PQ} = 3\mathbf{i} + \mathbf{j} + 4\mathbf{k}$ and $\overrightarrow{QR} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$	**1**	$\overrightarrow{PR}$ could be used
	A normal to the plane:		
	$\overrightarrow{PQ} \times \overrightarrow{QR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 4 \\ 2 & -2 & -2 \end{vmatrix}$	**1M**	
	$= \mathbf{i}\begin{vmatrix} 1 & 4 \\ -2 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 3 & 1 \\ 2 & -2 \end{vmatrix}$		
	$= 6\mathbf{i} + 14\mathbf{j} - 8\mathbf{k}$	**1**	
	Hence the equation has the form:		
	$6x + 14y - 8z = d.$	**1**	
	The plane passes through $P(-2, 1, -1)$ so		
	$d = -12 + 14 + 8 = 10$		
	which gives an equation $6x + 14y - 8z = 10$	**1**	
	i.e. $3x + 7y - 4z = 5.$		

		Marks awarded for
Method 2		
A plane has an equation of the form		
$ax + by + cz = d$. Using the points P, Q, R we get		
$$\begin{aligned} -2a + b - c &= d \\ a + 2b + 3c &= d \\ 3a + c &= d \end{aligned}$$	**1M**	
Using Gaussian elimination to solve these we have		or other valid method
$$\left\lvert \begin{matrix} -2 & 1 & -1 & d \\ 1 & 2 & 3 & d \\ 3 & 0 & 1 & d \end{matrix} \right\rvert \Rightarrow \left\lvert \begin{matrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \\ 0 & 6 & 8 & 2d \end{matrix} \right\rvert$$	**1**	
$$\Rightarrow \left\lvert \begin{matrix} -2 & 1 & -1 & d \\ 0 & 5 & 5 & 3d \\ 0 & 0 & 2 & -\frac{8}{5}d \end{matrix} \right\rvert$$	**1**	
$$\Rightarrow c = -\frac{4}{5}d, \quad b = \frac{7}{5}d, \quad a = \frac{3}{5}d$$	**1**	
These give the equation		
$$\left(\tfrac{3}{5}d\right)x + \left(\tfrac{7}{5}d\right)y + \left(-\tfrac{4}{5}d\right)z = d$$		
i.e. $3x + 7y - 4z = 5$	**1**	

			Marks awarded for
6. **(5)**	*Method* 1		
	$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$	**1**	
	$(1 + e^x)^2 = 1 + 2e^x + e^{2x}$	**1M**	
	$= 1 + 2\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$ $+\left(1 + 2x + \frac{(2x)^2}{2} + \frac{(2x)^3}{6} + \dots\right)$	**1**	
	$= 1 + 2 + 2x + x^2 + \frac{1}{3}x^3 + 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots$	**1**	
	$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \dots$	**1**	
	Method 2		
	$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$	**1**	
	$(1 + e^x) = 2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$	**1**	
	$(1 + e^x)^2 = \left(2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)\left(2 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots\right)$	**1M**	
	$= 4 + 4x + 3x^2 + \frac{1}{3}x^3 + \frac{1}{2}x^3 + \frac{1}{2}x^3 + \frac{1}{3}x^3 + \dots$	**1**	
	$= 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \dots$	**1**	
	Method 3		
	$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$	**1**	
	$f(x) = (1 + e^x)^2$ $\quad f(0) = 4$		
	$f'(x) = 2e^x(1 + e^x)$ $\quad f'(0) = 4$	**1**	
	$= 2e^x + 2e^{2x}$		can award marks for correct columns.
	$f''(x) = 2e^x + 4e^{2x}$ $\quad f''(0) = 6$	**1**	
	$f'''(x) = 2e^x + 8e^{2x}$ $\quad f'''(0) = 10$	**1**	
	$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{6} + \dots$		
	$(1 + e^x)^2 = 4 + 4x + 3x^2 + \frac{5}{3}x^3 + \dots$	**1**	
7. **(4)**	$y = \lvert x + 2 \rvert$		
	(a) 	**1** **1**	for shape; for coordinates
	(b) 	**1** **1**	for both horizontal lines; for values: 1, −1, −2

			Marks awarded for
8. (6)	$x = 4\sin\theta \Rightarrow dx = 4\cos\theta\, d\theta$	**1**	
	$\left.\begin{array}{l} x = 0 \Rightarrow \theta = 0 \\ x = 2 \Rightarrow \theta = \frac{\pi}{6}\end{array}\right\}$	**1**	
	$\int_0^2 \sqrt{16 - x^2}\, dx$		
	$= \int_0^{\pi/6} \sqrt{16 - (4\sin\theta)^2}\, .\, 4\cos\theta\, d\theta$	**1**	
	$= \int_0^{\pi/6} \sqrt{16(1 - \sin^2\theta)}\, .\, 4\cos\theta\, d\theta$		
	$= \int_0^{\pi/6} \sqrt{16\cos^2\theta}\, .\, 4\cos\theta\, d\theta$		
	$= \int_0^{\pi/6} 16\cos^2\theta\, d\theta$	**1**	
	$= 8\int_0^{\pi/6} (1 + \cos 2\theta)\, d\theta$		
	$= 8\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/6}$	**1**	for applying trig. identity and integrating
	$= \frac{8\pi}{6} + 4\sin\frac{\pi}{3}$		
	$= \frac{4\pi}{3} + 2\sqrt{3}\,(\approx 7{\cdot}65)$	**1**	numerical approx. allowed
9. (4)	*Method* 1		
	$A + A^{-1} = I$		
	$A^2 + I = A$	**1**	for multiplying by A
	Hence $A^2 + I = I - A^{-1}$	**1**	for rearranging $A + A^{-1} = I$
	$A^2 = -A^{-1}$	**1**	for subtracting I
	$A^3 = -I$, i.e. $k = -1$	**1**	for multiplying by A
	Method 2		
	$A + A^{-1} = I$		
	$A = I - A^{-1}$	**1**	for rearranging
	$A^2 = I - 2A^{-1} + (A^{-1})^2$	**1**	for squaring
	$A^3 = A - 2I + A^{-1}$	**1**	for multiplying by A
	$A^3 = (A + A^{-1}) - 2I = I - 2I$		
	Hence $A^3 = -I$, i.e. $k = -1$	**1**	
	Method 3		
	$A + A^{-1} = I$		
	$A = I - A^{-1}$	**1**	for rearranging
	$A^3 = A^2 - A$	**1**	for multiplying by A^2
	$A^3 = (A - I) - A$	**1**	using $A^2 = A - I$
	$= -I$, i.e. $k = -1$	**1**	
	Plus other valid methods.		

			Marks awarded for
10. (3)	*Method* 1		
	$1234 = 7 \times 176 + 2$	**1**	
	$176 = 7 \times 25 + 1$		
	$25 = 7 \times 3 + 4$	**1**	
	Hence		
	$1234_{10} = 3412_7$	**1**	answer only, 1 of 3
	Method 2		
	$1234 = 7 \times 176 + 2$	**1**	
	$= 7 \times (7 \times 25 + 1) + 2$		
	$= 7 \times (7 \times (7 \times 3 + 4) + 1) + 2$	**1**	
	$= 3 \times 7^3 + 4 \times 7^2 + 1 \times 7 + 2$		
	Hence		
	$1234_{10} = 3412_7$	**1**	answer only, 1 of 3
11. (1,4)	(a) $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}$	**1**	
	(b) $\int \sin^{-1} x . \frac{x}{\sqrt{1-x^2}} \, dx =$		
	$\sin^{-1} x \int \frac{x}{\sqrt{1-x^2}} \, dx - \int \left(\frac{d}{dx}(\sin^{-1} x) \int \frac{x}{\sqrt{1-x^2}} \, dx \right) dx$	**1**	
	$= \sin^{-1} x \int \frac{x}{\sqrt{1-x^2}} \, dx - \int \left(\frac{1}{\sqrt{1-x^2}} \int \frac{x}{\sqrt{1-x^2}} \, dx \right) dx$		
	$= \sin^{-1} x \left(-\sqrt{1-x^2} \right) - \int \left(\frac{1}{\sqrt{1-x^2}} \left(-\sqrt{1-x^2} \right) \right) dx$	**1**	for $\int \frac{x}{\sqrt{1-x^2}} \, dx = -\sqrt{1-x^2}$
	$= \sin^{-1} x \left(-\sqrt{1 - x^2} \right) - \int (-1) \, dx$	**1**	
	$= x - \sin^{-1} x . \sqrt{1 - x^2} + c$	**1**	
12. (5)	$\frac{dr}{dt} = -0.02; \quad \frac{dh}{dt} = 0.01$	**1**	
	$V = \pi r^2 h$		
	$\frac{dV}{dt} = \pi \left(2r \frac{dr}{dt} \right) h + \pi r^2 \frac{dh}{dt}$	**1M** **1**	for implicit differentiation for accuracy
	$= \pi (2 \times 0.6 \times (-0.02) \times 2 + 0.36 \times 0.01)$	**1**	
	$= \pi (-0.048 + 0.0036)$		
	$= -0.0444\pi \ (\approx -0.14)$		
	The rate of change in the volume is $-0.0444\pi \ \mathrm{m^3 \ s^{-1}}$.	**1**	units required

			Marks awarded for
13. (10)	$x = 2t + \frac{1}{2}t^2 \quad \Rightarrow \quad \frac{dx}{dt} = 2 + t$	**1**	
	$y = \frac{1}{3}t^3 - 3t \quad \Rightarrow \quad \frac{dy}{dt} = t^2 - 3$	**1**	
	$\frac{dy}{dx} = \frac{t^2 - 3}{2 + t}$	**1**	
	$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{2t(2+t) - \left(t^2 - 3\right)}{(2+t)^2} = \frac{t^2 + 4t + 3}{(2+t)^2}$	**1**	
	$\frac{d^2y}{dx^2} = \frac{t^2 + 4t + 3}{(2+t)^2} \times \frac{1}{2+t} = \frac{t^2 + 4t + 3}{(2+t)^3}$	**1**	
	Stationary points when $\frac{dy}{dx} = 0$, i.e.		
	$t^2 - 3 = 0 \Rightarrow t = \pm\sqrt{3}$	**1**	
	When $t = \sqrt{3}$, $\frac{d^2y}{dx^2} = \frac{3 + 4\sqrt{3} + 3}{(2 + \sqrt{3})^3} > 0$ which gives a minimum.	**1**	no marks for using a nature table
	When $t = -\sqrt{3}$, $\frac{d^2y}{dx^2} = \frac{3 - 4\sqrt{3} + 3}{(2 - \sqrt{3})^3} < 0$ which gives a maximum.	**1**	no marks for using a nature table
	At a point of inflexion, $\frac{d^2y}{dx^2} = 0$.	**1**	
	In this case, that means $t^2 + 4t + 3 = (t+1)(t+3) = 0$		
	and this has exactly two roots.	**1**	need to show 2 values exist
	Note that this is a slimmed-down version of the complete story of points of inflexion.		

			Marks	*Marks awarded for*
14. (5, 1,3)	(a)	$\left(\begin{array}{ccc\|c} 4 & 0 & 6 & 1 \\ 2 & -2 & 4 & -1 \\ -1 & 1 & \lambda & 2 \end{array}\right)$	**1**	for augmented matrix
		$\left(\begin{array}{ccc\|c} 4 & 0 & 6 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & 4 & 6+4\lambda & 9 \end{array}\right)$	**1**	
		$\left(\begin{array}{ccc\|c} 4 & 0 & 6 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & 0 & 8+4\lambda & 6 \end{array}\right)$	**1**	triangular form needed
		$z = \dfrac{6}{8 + 4\lambda} = \dfrac{3}{2(2 + \lambda)}$	**1**	first root
		$4y = 3 + 2z \Rightarrow 4y = \dfrac{18 + 6\lambda}{4 + 2\lambda}$ $\Rightarrow y = \dfrac{3\lambda + 9}{4(2 + \lambda)}$ $4x = 1 - 6z \Rightarrow 4x = \dfrac{2\lambda - 14}{4 + 2\lambda}$ $\Rightarrow x = \dfrac{\lambda - 7}{4(2 + \lambda)}$	**1**	other two roots
	(b)	When $\lambda = -2$, the final row gives $0 = 6$ which is inconsistent. **There are no solutions**.	**1**	
	(c)	$\lambda = -2.1$; $x = 22.75$; $y = -6.75$; $z = -15$	**1,1**	1 for first 2 values; 1 for third
		Although the values of λ are close, the values of x, y and z are quite different. The system is **ill-conditioned** near $\lambda = -2$.	**1**	

15. **(4,7)**

(a) $\dfrac{1}{(x-1)(x+2)^2} = \dfrac{A}{x-1} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2}$ **1M**

$1 = A(x+2)^2 + B(x-1)(x+2) + C(x-1)$

$x = 1 \Rightarrow A = \frac{1}{9}$ **1**

$x = -2 \Rightarrow C = -\frac{1}{3}$ **1**

$x = 0 \Rightarrow 1 = \frac{4}{9} - 2B + \frac{1}{3} \Rightarrow B = -\frac{1}{9}$ **1**

$\therefore \dfrac{1}{(x-1)(x+2)^2} = \dfrac{1}{9}\left(\dfrac{1}{x-1} - \dfrac{1}{x+2} - \dfrac{3}{(x+2)^2}\right)$

(b) $(x-1)\dfrac{dy}{dx} - y = \dfrac{x-1}{(x+2)^2}$

$\dfrac{dy}{dx} - \dfrac{1}{x-1}y = \dfrac{1}{(x+2)^2}$ **1M** for rearranging

Integrating factor: $\exp\left(\int -\frac{1}{x-1}dx\right)$ **1**

$= \exp\left(-\ln(x-1)\right) = (x-1)^{-1}$ **1**

$\dfrac{1}{(x-1)}\dfrac{dy}{dx} - \dfrac{1}{(x-1)^2}y = \dfrac{1}{(x-1)(x+2)^2}$

$\dfrac{d}{dx}\left(\dfrac{y}{x-1}\right) = \dfrac{1}{(x-1)(x+2)^2}$ **1**

$= \dfrac{1}{9}\left(\dfrac{1}{x-1} - \dfrac{1}{x+2} - \dfrac{3}{(x+2)^2}\right)$ **1**

$\dfrac{y}{x-1} = \dfrac{1}{9}\left(\ln|x-1| - \ln|x+2| + \dfrac{3}{x+2}\right) + c$ **1** constant of integration needed.

$y = \dfrac{x-1}{9}\left(\ln|x-1| - \ln|x+2| + \dfrac{3}{x+2}\right) + c(x-1)$ **1**

$= \dfrac{x-1}{9}\left(\ln\dfrac{|x-1|}{|x+2|} + \dfrac{3}{x+2}\right) + c(x-1)$

			Marks awarded for
16. **(6,4)**	(a) For $n = 1$, the LHS = $\cos\theta + i\sin\theta$ and the RHS = $\cos\theta + i\sin\theta$. Hence the result is true for $n = 1$.	**1**	
	Assume the result is true for $n = k$, i.e. $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$.	**1**	working with n is penalised.
	Now consider the case when $n = k + 1$: $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$	**1**	
	$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$	**1**	for applying the inductive hypothesis
	$= (\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)$	**1**	multiplying and collecting
	$= \cos(k+1)\theta + i\sin(k+1)\theta$ Thus, if the result is true for $n = k$ the result is true for $n = k + 1$. Since it is true for $n = 1$, the result is true for all $n \geqslant 1$.	**1**	
	(b) $\dfrac{(\cos\frac{\pi}{18} + i\sin\frac{\pi}{18})^{11}}{(\cos\frac{\pi}{36} + i\sin\frac{\pi}{36})^4} = \dfrac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}}$	**1**	using result from above
	$= \dfrac{\cos\frac{11\pi}{18} + i\sin\frac{11\pi}{18}}{\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}} \times \dfrac{\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}}{\cos\frac{\pi}{9} - i\sin\frac{\pi}{9}}$	**1**	
	$= \dfrac{\cos\frac{11\pi}{18}\cos\frac{\pi}{9} + \sin\frac{11\pi}{18}\sin\frac{\pi}{9}}{\cos^2\frac{\pi}{9} + \sin^2\frac{\pi}{9}}$ + imaginary term		
	$= \cos\left(\dfrac{11\pi}{18} - \dfrac{\pi}{9}\right)$ + imaginary term	**1**	
	$= \cos\dfrac{\pi}{2}$ + imaginary term		
	Thus the real part is zero as required.	**1**	or equivalent

END OF SOLUTIONS