2003

Section A (Mathematics 1 and 2)

All candidates should attempt this Section.

Answer all the questions.

Marks

A1. (a) Given $f(x) = x(1+x)^{10}$, obtain $f'(x)$ and simplify your answer. 3

(b) Given $y = 3^x$, use logarithmic differentiation to obtain $\frac{dy}{dx}$ in terms of x. 3

A2. Given that $u_k = 11 - 2k$, $(k \geq 1)$, obtain a formula for $S_n = \sum_{k=1}^{n} u_k$. 3

Find the values of n for which $S_n = 21$. 2

A3. The equation $y^3 + 3xy = 3x^2 - 5$ defines a curve passing through the point $A(2, 1)$. Obtain an equation for the tangent to the curve at A. 4

A4. Identify the locus in the complex plane given by $|z + i| = 2$. 3

A5. Use the substitution $x = 1 + \sin\theta$ to evaluate $\int_0^{\pi/2} \frac{\cos\theta}{(1+\sin\theta)^3}\,d\theta$. 5

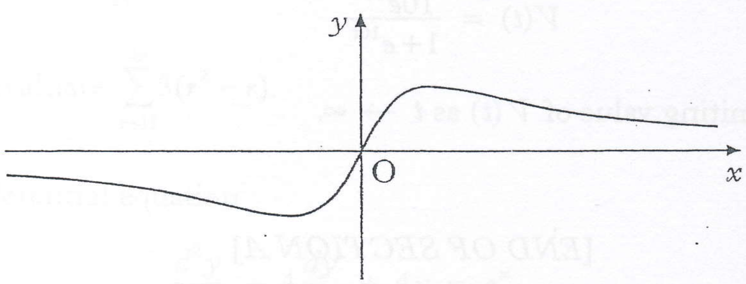
A6. Use elementary row operations to reduce the following system of equations to upper triangular form

$$\begin{aligned} x + y + 3z &= 1 \\ 3x + ay + z &= 1 \\ x + y + z &= -1. \end{aligned}$$ 2

Hence express x, y and z in terms of the parameter a. 2

Explain what happens when $a = 3$. 2

A7.



The diagram shows the shape of the graph of $y = \frac{x}{1+x^2}$. Obtain the stationary points of the graph. 4

Sketch the graph of $y = \left|\frac{x}{1+x^2}\right|$ and identify its three critical points. 3

Marks

A8. Given that $p(n) = n^2 + n$, where n is a positive integer, consider the statements:

A $p(n)$ is always even
B $p(n)$ is always a multiple of 3.

For each statement, prove it if it is true or, otherwise, disprove it. 4

A9. Given that $w = \cos\theta + i\sin\theta$, show that $\frac{1}{w} = \cos\theta - i\sin\theta$. 1

Use de Moivre's theorem to prove $w^k + w^{-k} = 2\cos k\theta$, where k is a natural number. 3

Expand $(w + w^{-1})^4$ by the binomial theorem and hence show that

$$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}.$$ 5

A10. Define $I_n = \int_0^1 x^n e^{-x}\,dx$ for $n \geq 1$.

(*a*) Use integration by parts to obtain the value of $I_1 = \int_0^1 xe^{-x}\,dx$. 3

(*b*) Similarly, show that $I_n = nI_{n-1} - e^{-1}$ for $n \geq 2$. 4

(*c*) Evaluate I_3. 3

A11. The volume $V(t)$ of a cell at time t changes according to the law

$$\frac{dV}{dt} = V(10 - V) \qquad \text{for } 0 < V < 10.$$

Show that

$$\frac{1}{10}\ln V - \frac{1}{10}\ln(10 - V) = t + C$$

for some constant C. 4

Given that $V(0) = 5$, show that

$$V(t) = \frac{10e^{10t}}{1 + e^{10t}}.$$ 3

Obtain the limiting value of $V(t)$ as $t \to \infty$. 2

[*END OF SECTION A*]

Candidates should now attempt ONE of the following

Section B (Mathematics 3) on *Page four*
Section C (Statistics 1) on *Pages five and six*
Section D (Numerical Analysis 1) on *Pages seven and eight*
Section E (Mechanics 1) on *Pages nine, ten and eleven.*

Section B (Mathematics 3)

Marks

ONLY candidates doing the course Mathematics 1, 2 and 3 should attempt this Section.

Answer all the questions.

Answer these questions in a separate answer book, showing clearly the section chosen.

B1. Find the point of intersection of the line

$$\frac{x-3}{4} = \frac{y-2}{-1} = \frac{z+1}{2}$$

and the plane with equation $2x + y - z = 4$. 4

B2. The matrix A is such that $A^2 = 4A - 3I$ where I is the corresponding identity matrix. Find integers p and q such that

$$A^4 = pA + qI.$$ 4

B3. A recurrence relation is defined by the formula

$$x_{n+1} = \frac{1}{2}\left\{x_n + \frac{7}{x_n}\right\}.$$

Find the fixed points of this recurrence relation. 3

B4. Obtain the Maclaurin series for $f(x) = \sin^2 x$ up to the term in x^4. 4
Hence write down a series for $\cos^2 x$ up to the term in x^4. 1

B5. (*a*) Prove by induction that for all natural numbers $n \geq 1$

$$\sum_{r=1}^{n} 3(r^2 - r) = (n-1)n(n+1).$$ 4

(*b*) Hence evaluate $\sum_{r=11}^{40} 3(r^2 - r)$. 2

B6. Solve the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^x,$$

given that $y = 2$ and $\frac{dy}{dx} = 1$, when $x = 0$. 10

[END OF SECTION B]