Higher 2012 Paper 2 Solutions

(1) $f(x) = x^2 + 3$, $g(x) = x + 4$

ai) $f(g(x))$

$= f(x+4)$

$= (x+4)^2 + 3$

$= \underline{\underline{x^2 + 8x + 19}}$

ii) $g(f(x))$

$= g(x^2+3)$

$= x^2 + 3 + 4$

$= \underline{\underline{x^2 + 7}}$

b) $f(g(x)) + g(f(x)) = 0$

$\Rightarrow x^2 + 8x + 19 + x^2 + 7 = 0$

$\Rightarrow 2x^2 + 8x + 26 = 0$

$a = 2$, $b = 8$, $c = 26$

$b^2 - 4ac = 8^2 - 4(2)(26) = 64 - 208 = -144$

$\underline{\underline{b^2 - 4ac < 0 \quad (-144 < 0) \text{ so there are no real roots.}}}$

(2) $x^2 + y^2 - 6x - 2y - 30 = 0$, $y = 2x + 5$

$x^2 + (2x+5)^2 - 6x - 2(2x+5) - 30 = 0$

$x^2 + 4x^2 + 20x + 25 - 6x - 4x - 10 - 30 = 0$

$5x^2 + 10x - 15 = 0$

$5(x^2 + 2x - 3) = 0$

$5(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

when $x = -3$, $y = 2(-3) + 5 = -1$ so P is $(-3, -1)$

when $x = 1$, $y = 2(1) + 5 = 7$ so Q is $(1, 7)$

2b) Circle from (a): $x^2+y^2-6x-2y-30=0$

$$x^2+y^2+2gx+2fy+c=0$$

$2g=-6$ $\quad 2f=-2 \quad c=-30$

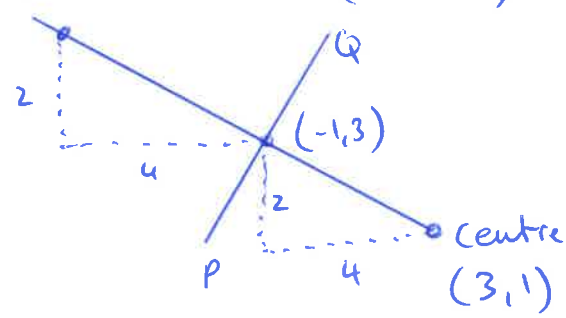
$g=-3 \quad f=-1$

centre $(3,1)$

$$r=\sqrt{g^2+f^2-c}$$
$$=\sqrt{9+1+30}$$
$$=\sqrt{40}=2\sqrt{10}$$

midpoint of PQ is $\left(\frac{-3+1}{2}, \frac{-1+7}{2}\right)$

$=(-1,3)$



so centre of 2nd circle is $(-5,5)$

$r=\sqrt{40}$

$$(x-a)^2+(y-b)^2=r^2$$

$$(x+5)^2+(y-5)^2=40$$

(3) $f(x)=x^3-2x^2-4x+6 \quad 0 \leq x \leq 3$

Closed Interval Question

$f'(x)=3x^2-4x-4=0$ for stationary points

$$(3x+2)(x-2)=0$$

$$x=-\frac{2}{3} \text{ or } x=2$$

Find $f(0)$, $f(3)$ and $f(2)$ $\quad$ [$f(-\frac{2}{3})$ is outside range]

$f(0)=6$

$f(2)=2^3-2(2)^2-4(2)+6=8-8-8+6=-2$

$f(3)=3^3-2(3)^2-4(3)+6=27-18-12+6=3$

max value $=6$

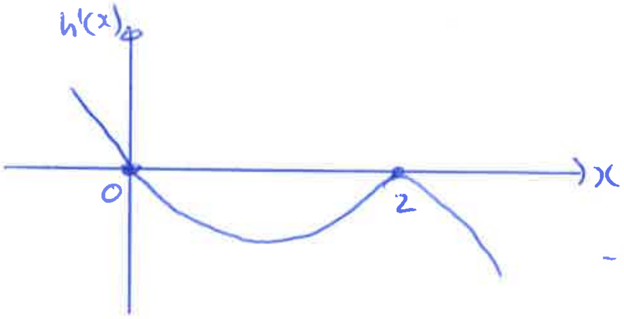
min value $=-2$

③

④ a)

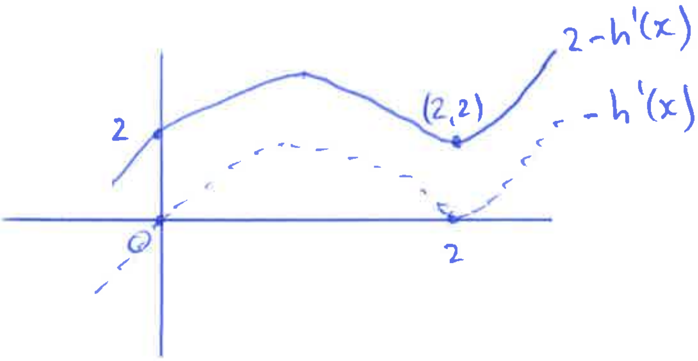
x	$x<0$	$x=0$ (S.P)	$0<x<2$	$x=2$ (S.P)	$x>2$
$h'(x)$	$+$	0	$-$	0	$-$

← gradient of $h(x)$



b) $2 - h'(x) = -h'(x) + 2$

reflect in x axis then move up 2



⑤ ai) $\overrightarrow{BA} = a - b = \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}}$

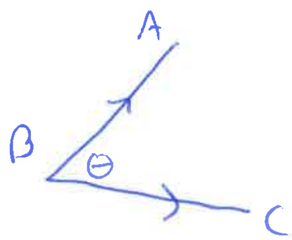
$|\overrightarrow{BA}| = \sqrt{1^2 + 0^2 + (-1)^2}$
$= \sqrt{2}$

$\overrightarrow{BC} = c - b = \begin{pmatrix} 4 \\ k \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ k+3 \\ -1 \end{pmatrix}}}$

$|\overrightarrow{BC}| = \sqrt{2^2 + (k+3)^2 + (-1)^2}$
$= \sqrt{4 + k^2 + 6k + 9 + 1}$
$= \sqrt{k^2 + 6k + 14}$

aii) $a \cdot b = |a||b| \cos\theta$

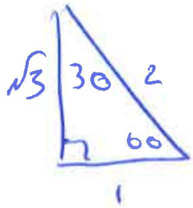
$\cos\theta = \dfrac{a \cdot b}{|a||b|} = \dfrac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}||\overrightarrow{BC}|}$



$\cos ABC = \dfrac{(1 \times 2) + (0 \times (k+3)) + (-1 \times -1)}{\sqrt{2}\sqrt{k^2 + 6k + 14}}$

$$\cos ABC = \frac{3}{\sqrt{2(k^2+6k+14)}}$$ as required.

b) $\angle ABC = 30^\circ \Rightarrow \cos 30^\circ = \frac{\sqrt{3}}{2}$



so $\frac{\sqrt{3}}{2} = \frac{3}{\sqrt{2(k^2+6k+14}}$ (cross multiply $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$)

$\sqrt{6(k^2+6k+14)} = 6$

$6(k^2+6k+14) = 36$ square both sides

$k^2+6k+14 = 6$

$k^2+6k+8=0$

$(k+4)(k+2)=0$

$\underline{\underline{k=-4 \text{ or } k=-2}}$

6) $U_{n+1} = (\sin x)U_n + \cos 2x$

a) For $0 < x < \frac{\pi}{2}$, $\sin x$ is between 0 and 1

hence limit exists as $\underline{\underline{-1 < \sin x < 1}}$

b) $L = (\sin x)L + \cos 2x$

$L - L(\sin x) = \cos 2x$

$L(1-\sin x) = \cos 2x$

$$L = \frac{\cos 2x}{1-\sin x}$$

$L = \frac{1}{2}\sin x \Rightarrow \dfrac{\cos 2x}{1-\sin x} = \frac{1}{2}\sin x$

(cross multiply)

$\cos 2x = \frac{1}{2}\sin x(1-\sin x)$

$\cos 2x = \frac{1}{2}\sin x - \frac{1}{2}\sin^2 x$ $\quad(\cos 2x = 1 - 2\sin^2 x)$

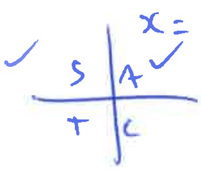
$1 - 2\sin^2 x - \frac{1}{2}\sin x + \frac{1}{2}\sin^2 x = 0$

$-\frac{3}{2}\sin^2 x - \frac{1}{2}\sin x + 1 = 0$

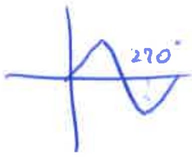
$-\frac{1}{2}(3\sin^2 x + \sin x - 2) = 0$

$-\frac{1}{2}(3\sin x - 2)(\sin x + 1) = 0$

$\sin x = \frac{2}{3}$ or $\sin x = -1$



$x = 41.8°$ or $108.2°$ | $x = 270°$



For $0 < x < \frac{\pi}{2}$, $x = 41.8°$ is only answer

$\Rightarrow x = \dfrac{41.8}{180}\pi = \underline{\underline{0.73 \text{ radians.}}}$

7 a) $4^x = 3^{2-x}$ equate

2 methods :

$$\log_a 4^x = \log_a 3^{2-x}$$
$$x \log_a 4 = (2-x)(\log_a 3)$$
$$x \log_a 4 = 2\log_a 3 - x\log_a 3$$
$$x\log_a 4 + x\log_a 3 = \log_a 3^2$$
$$x(\log_a 4 + \log_a 3) = \log_a 9$$
$$x(\log_a 12) = \log_a 9$$
$$x = \frac{\log_a 9}{\log_a 12}$$

or

$$4^x = 3^{2-x}$$
$$4^x = 3^2 3^{-x}$$
$$4^x = 9(3^{-x})$$
$$4^x = \frac{9}{3^x}$$
$$4^x 3^x = 9$$
$$12^x = 9$$
$$\log_a 12^x = \log_a 9$$
$$x \log_a 12 = \log_a 9$$
$$x = \frac{\log_a 9}{\log_a 12}$$

$\left(\begin{array}{l} 4^1 3^1 = 12 \\ 4^2 3^2 = 144 \\ 4^3 3^3 = 1728 \end{array} \right)$

You might not have seen $4^x 3^x = 12^x$ before but it is true.

$x=1 \quad 4^1 3^1 = 12^1$

$x=2 \quad 4^2 3^2 = 12^2$

$x=3 \quad 4^3 3^3 = 12^3$

(let $a=10$)

b) $x = \frac{\log_{10} 9}{\log_{10} 12} = 0.884$

$$y = 4^x = 4^{0.884} = \underline{\underline{3.41}}$$