H

National
Qualifications

X847/76/12

Mathematics
Paper 2

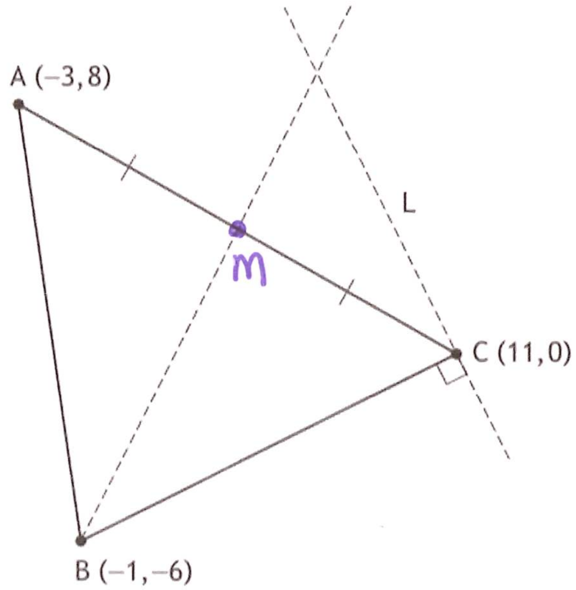
Duration — 1 hour 30 minutes

2024 PAPER 2 – WORKED SOLUTIONS

Total marks — 65

H Wallace

1. Triangle ABC has vertices A(−3, 8), B(−1, −6) and C(11, 0).



(a) Find the equation of the median through B. **3**

Midpoint $m\left(\frac{-3+11}{2}, \frac{8+0}{2}\right)$

$m(4, 4)$

Gradient BM

$m_{BM} = \frac{4-(-6)}{4-(-1)} = \frac{10}{5}$

$m_{BM} = 2$

Equation of BM

$y - b = m(x - a)$

$y - 4 = 2(x - 4)$

$y - 4 = 2x - 8$

$y = 2x - 4.$

(b) Find the equation of L, the line perpendicular to BC passing through C. **3**

Gradient BC

$m_{BC} = \frac{0-(-6)}{11-(-1)} = \frac{6}{12} = \frac{1}{2}$

Gradient of L

$m_{L} = -2$

since $\frac{1}{2} \times -2 = -1$

Equation of L

$y - b = m(x - a)$

$y - 0 = -2(x - 11)$

$y = -2x + 22$

(c) Determine the coordinates of the point of intersection of the median through B and the line L. **2**

Point of intersection

Let $y = y$

$2x - 4 = -2x + 22$

$4x = 26$

$x = \frac{26}{4}$

$x = 6.5.$

$y = 2x - 4$

$y = 2(6.5) - 4$

$y = 13 - 4$

$y = 9$

Point $(6.5, 9)$

2. A curve has equation $y = \frac{8}{x^3}$, $x > 0$.

Find the equation of the tangent to this curve at the point where $x = 2$. **5**

$y = 8x^{-3}$

$\frac{dy}{dx} = -24x^{-4}$

$\frac{dy}{dx} = -\frac{24}{x^4}$

Gradient at $x = 2$

$m = -\frac{24}{2^4}$

$m = -\frac{24}{16}$

$m = -\frac{3}{2}$

Point of tangency

$x = 2$

$y = \frac{8}{2^3} = \frac{8}{8} = 1$

Point $(2, 1)$

Equation of tangent

$y - b = m(x - a)$

$y - 1 = -\frac{3}{2}(x - 2)$

$2y - 2 = -3x + 6$

$2y = -3x + 8$

3. The coordinates of points D, E and F are given by D (2, −3, 4), E (1, 1, −2) and F (3, 2, 1).

(a) Express $\overrightarrow{ED}$ and $\overrightarrow{EF}$ in component form. 2

$\overrightarrow{ED} = d - e$

$= \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\overrightarrow{ED} = \begin{pmatrix} 1 \\ -4 \\ 6 \end{pmatrix}$

$\overrightarrow{EF} = f - e$

$= \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$\overrightarrow{EF} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$

(b) (i) Calculate $\overrightarrow{ED}.\overrightarrow{EF}$. 1

(ii) Hence, or otherwise, calculate the size of angle DEF. 4

(i) $\overrightarrow{ED}.\overrightarrow{EF} = (1)(2) + (-4)(1) + (6)(3)$

$= 2 - 4 + 18$

$\overrightarrow{ED}.\overrightarrow{EF} = 16$

(ii) $\cos DEF = \dfrac{\overrightarrow{ED}.\overrightarrow{EF}}{|\overrightarrow{ED}||\overrightarrow{EF}|}$

$|\overrightarrow{ED}| = \sqrt{1^2 + (-4)^2 + 6^2}$

$= \sqrt{53}$

$\cos DEF = \dfrac{16}{\sqrt{53}\sqrt{14}}$

$|\overrightarrow{EF}| = \sqrt{2^2 + 1^2 + 3^2}$

$= \sqrt{14}$

$DEF = \cos^{-1}\left(\dfrac{16}{\sqrt{53}\sqrt{14}}\right)$

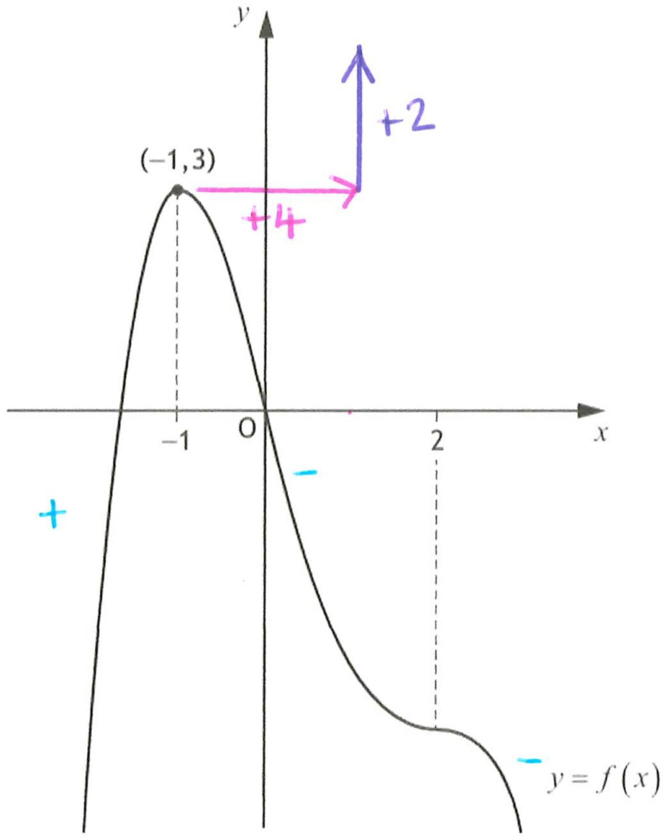
$= 54.0288...$

Angle DEF $= 54.0°$ (1dp)

4. The diagram shows the graph of a quartic function $y = f(x)$.

A maximum turning point occurs at $(-1, 3)$.

The graph of $y = f(x)$ also has a point of inflection at $x = 2$.



(a) Determine the coordinates of the maximum turning point on the graph of $y = f(x-4)+2$. 2

$y = f(x-4) + 2$

shift right 4 units.

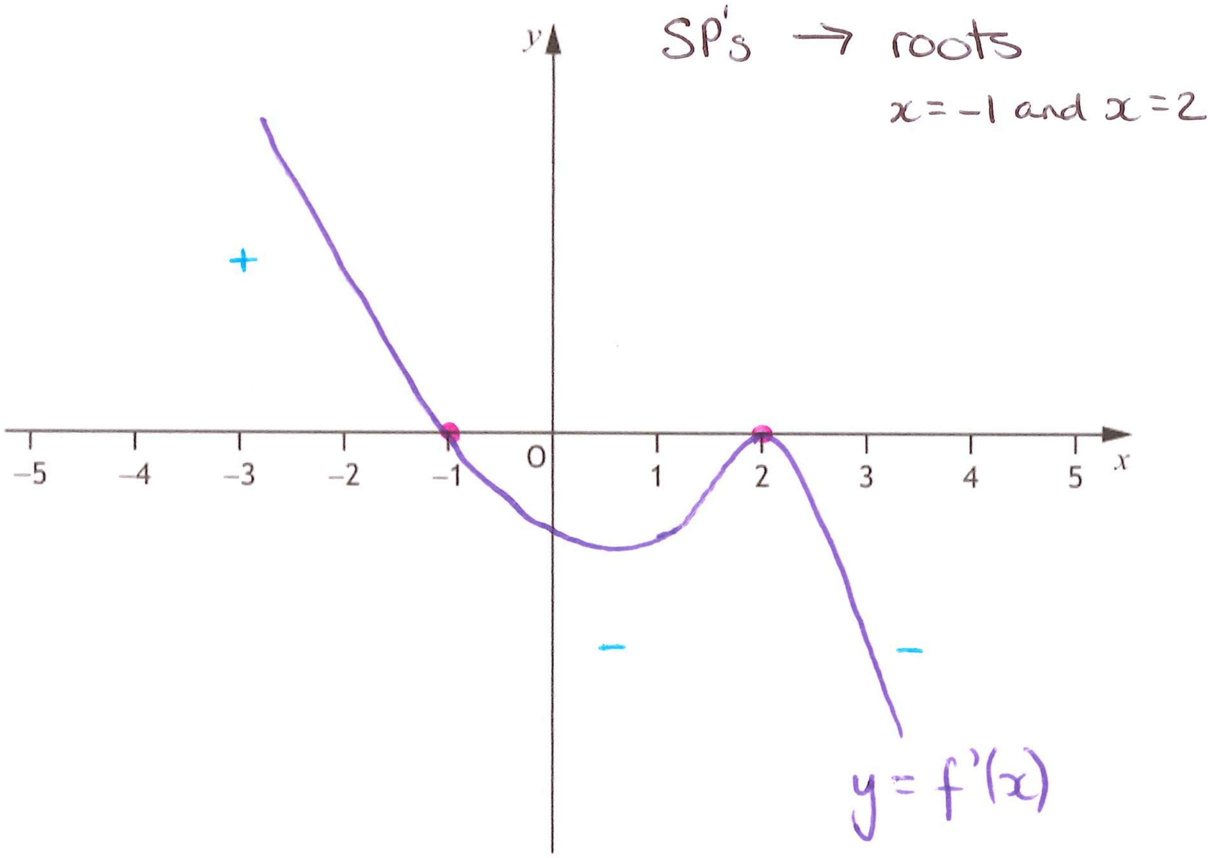
shift up 2 units.

Max TP $(-1, 3)$ on $y = f(x)$

+4 +2

$\therefore$ Max TP $(3, 5)$ on $y = f(x-4)+2$.

(b) On the diagram in your answer booklet, sketch the graph of $y = f'(x)$. **3**



5. Evaluate $\int_0^{\frac{\pi}{7}} \sin 5x \, dx$. **3**

$$= \left[-\frac{1}{5}\cos 5x\right]_0^{\frac{\pi}{7}}$$

$\cos 0 = 1$

$$= \left(-\frac{1}{5}\cos\frac{5\pi}{7}\right) - \left(-\frac{1}{5}\cos 0\right)$$

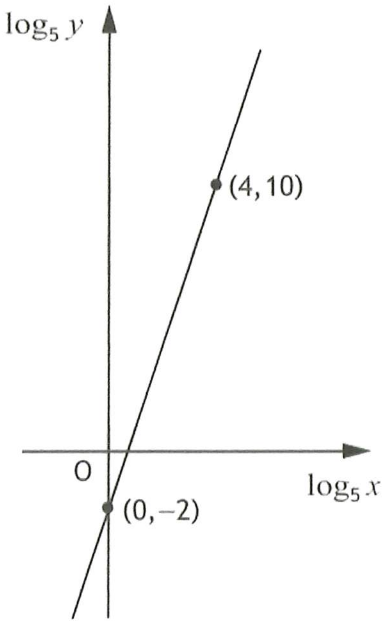
$$= -\frac{1}{5}\cos\frac{5\pi}{7} + \frac{1}{5}$$

$$= 0.32469796$$

$$= \underline{\underline{0.3247}} \quad (4dp)$$

6. Two variables, x and y, are connected by the equation $y = ax^b$.

The graph of $\log_5 y$ against $\log_5 x$ is a straight line as shown.



Gradient

$m = \dfrac{10-(-2)}{4-0}$

$m = \dfrac{12}{4}$

$m = 3.$

$c = -2$

Find the values of a and b. 5

let $Y = mX + c$

$Y = 3X - 2$

$\log_5 y = 3\log_5 x - 2$

$\log_5 y = 3\log_5 x - 2\log_5 5$

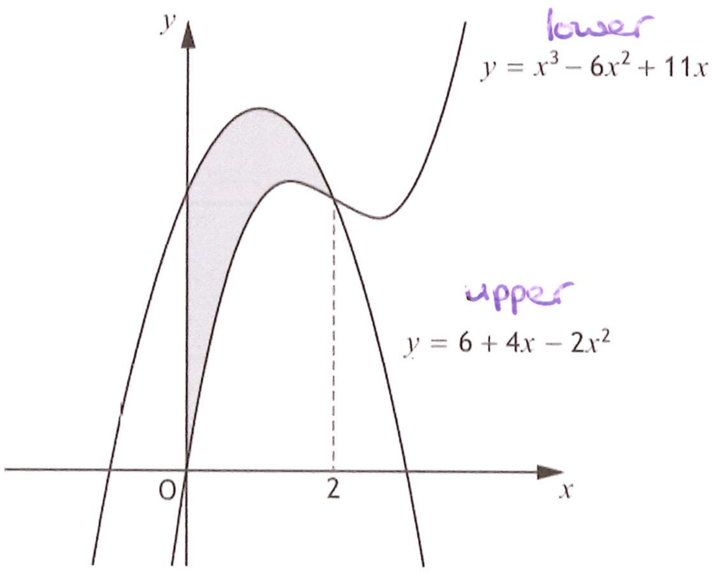
$\log_5 y = \log_5 x^3 - \log_5 5^2$

$\log_5 y = \log_5\left(\dfrac{x^3}{25}\right)$

$y = \dfrac{1}{25}x^3$

↳ $a = \dfrac{1}{25}$ and $b = 3$

7. The diagram shows the curve with equation $y = x^3 - 6x^2 + 11x$ intersecting the curve with equation $y = 6 + 4x - 2x^2$ at $x = 2$.



Calculate the shaded area. **5**

$$\text{Area} = \int_0^2 (6 + 4x - 2x^2) - (x^3 - 6x^2 + 11x)\, dx$$

$$= \int_0^2 (6 + 4x - 2x^2 - x^3 + 6x^2 - 11x)\, dx$$

$$= \int_0^2 (6 - 7x + 4x^2 - x^3)\, dx$$

$$= \left[6x - \frac{7x^2}{2} + \frac{4x^3}{3} - \frac{x^4}{4} \right]_0^2$$

$$= \left(6(2) - \frac{7(2)^2}{2} + \frac{4(2)^3}{3} - \frac{(2)^4}{4} \right) - 0$$

$$= 12 - 14 + \frac{32}{3} - 4$$

$$= \frac{32}{3} - 6$$

$$\text{Area} = \frac{14}{3} = 4\tfrac{2}{3} \text{ units}^2$$

8. Functions f and g are defined on $\mathbb{R}$, the set of real numbers, by:

- $f(x) = 2x^2 - 18$
- $g(x) = x + 1$.

(a) Find an expression for $f(g(x))$. **2**

$$f(g(x)) = f(x+1)$$

$$f(g(x)) = 2(x+1)^2 - 18$$

(b) Find the values of x for which $\dfrac{1}{f(g(x))}$ is undefined. **2**

Consider $\dfrac{1}{2(x+1)^2-18}$

We cannot divide by zero

$\therefore$ let $2(x+1)^2 - 18 = 0$ for undefined

$$2(x+1)^2 = 18$$

$$(x+1)^2 = 9$$

$$x+1 = \pm 3$$

$x + 1 = -3$ $x + 1 = 3$

$x = -4$ $x = 2$

Undefined when $x = -4, 2$

9. (a) Determine the coordinates of the stationary points on the curve with equation $y = \frac{1}{3}x^3 - x^2 - 3x + 1$. 4

$\frac{dy}{dx} = x^2 - 2x - 3$

SP's occur when $\frac{dy}{dx} = 0$

$\therefore \; x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x - 3 = 0 \qquad x + 1 = 0$

$x = 3 \qquad x = -1$

y-coords

$x = -1$

$y = \frac{1}{3}(-1)^3 - (-1)^2 - 3(-1) + 1$

$y = -\frac{1}{3} - 1 + 3 + 1$

$y = \frac{8}{3}$

$x = 3$

$y = \frac{1}{3}(3)^3 - (3)^2 - 3(3) + 1$

$y = \frac{1}{3}(27) - 9 - 9 + 1$

$y = -8$

SP's $(-1, \frac{8}{3})$ and $(3, -8)$

(b) Hence, determine the greatest and least values of y in the interval $-1 \le x \le 6$. 2

Consider y-values at limits

$x = -1$

$y = \frac{8}{3}$ from SP

$x = 6$

$y = \frac{1}{3}(6)^3 - (6)^2 - 3(6) + 1$

$y = 72 - 36 - 18 + 1$

$y = 19$

Greatest value $= 19$ @ $x = 6$ limit

Least value $= -8$ @ $x = 3$ SP.

10. The circle C_1 has equation $x^2 + y^2 + 18x - 2y - 8 = 0$.

(a) Find the centre and radius of C_1. **2**

$x^2 + y^2 + 18x - 2y - 8 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

$2g = 18$ $\quad 2f = -2$ $\quad c = -8$

$g = 9$ $\quad f = -1$

Radius $= \sqrt{g^2 + f^2 - c}$

$= \sqrt{9^2 + (-1)^2 - (-8)}$

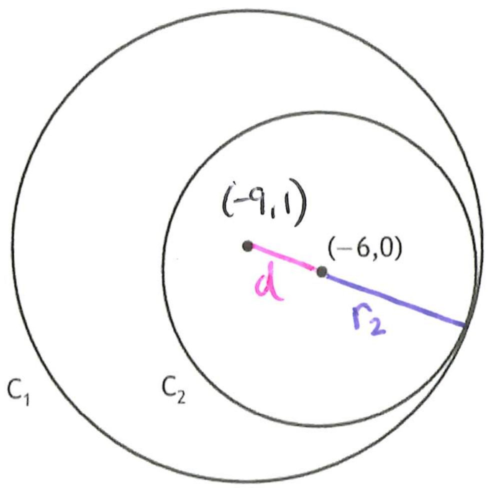
Centre $(-g, -f)$

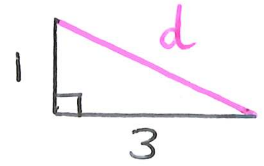
Centre $(-9, 1)$

Radius $= \sqrt{90}$ units.

A second circle, C_2, touches C_1 internally.

The centre of C_2 is $(-6, 0)$.





(b) Determine the equation of C_2. **2**

Distance between centres

$d = \sqrt{1^2 + 3^2}$

$d = \sqrt{10}$

Radius of C_2

$r_2 = r_1 - d$

$= \sqrt{90} - \sqrt{10}$

$= 3\sqrt{10} - \sqrt{10}$

$r_2 = 2\sqrt{10}$

$r_2 = \sqrt{40}$ units.

Equation of C_2

$(x-a)^2 + (y-b)^2 = r^2$

$(x+6)^2 + y^2 = 40$

11. The number of electric vehicles worldwide can be modelled by

$$N = 6.8e^{kt}$$

where:

- N is the estimated number of vehicles in millions
- t is the number of years since the end of 2020
- k is a constant.

(a) Use the model to estimate the number of electric vehicles worldwide at the end of 2020. 1

Initially, $t = 0$

$N = 6.8e^{k(0)}$

$N = 6.8e^{0} = 6.8$ million vehicles.

At the end of 2030, it is estimated there will be 125 million electric vehicles worldwide.

(b) Determine the value of k. 4

By 2030, $t = 10$ years.

$N = 6.8e^{kt}$

$125 = 6.8e^{10k}$

$\frac{125}{6.8} = e^{10k}$

$\ln\left(\frac{125}{6.8}\right) = \ln e^{10k}$

$\ln\left(\frac{125}{6.8}\right) = 10k \cancel{\ln e}$

$10k = 2.911391125$

$k = 0.291139112 5.$

12. Solve the equation $2\sin 2x° - \sin^2 x° = 0$, $0 \le x < 360$. 5

$$2\sin 2x - \sin^2 x = 0$$

$$2(2\sin x\cos x) - \sin^2 x = 0$$

$$4\sin x\cos x - \sin^2 x = 0$$

$$\sin x(4\cos x - \sin x) = 0$$

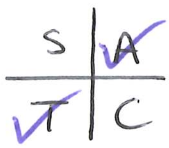
$$\sin x = 0$$

$$x = 0°, 180°$$

$$4\cos x - \sin x = 0$$

$$4\cos x = \sin x$$

$$4 = \frac{\sin x}{\cos x}$$



$$\tan x = 4$$

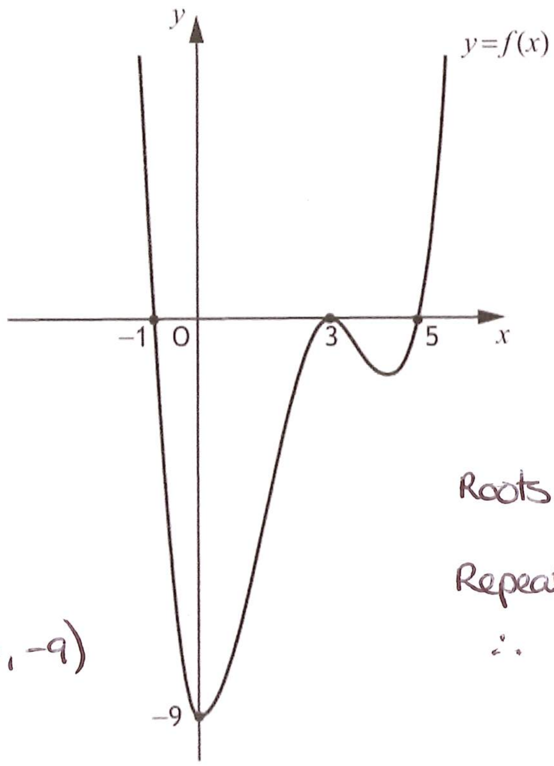
$$ra = \tan^{-1}(4) = 76.0°$$

$$x = 76, 180+76$$

$$x = 76°, 256°.$$

↳ Solutions $x = 0°, 76°, 180°, 256°.$

13. The diagram shows the graph of $y = f(x)$, where $f(x)$ is a quartic function.



Roots at $x = -1, 3, 5$

Repeated root at $x = 3$.

$\therefore a = -3$.

$b = 1$

$c = -5$

y-intercept $(0, -9)$

Express $f(x)$ in the form $f(x) = k(x+a)^2(x+b)(x+c)$. 3

$$f(x) = k(x-3)^2(x+1)(x-5)$$

Sub in $(0, -9)$

$$-9 = k(0-3)^2(0+1)(0-5)$$

$$-9 = k(-3)^2(1)(-5)$$

$$-9 = k(9)(-5)$$

$$-9 = -45k$$

$$45k = 9$$

$$k = \frac{9}{45}$$

$$k = \frac{1}{5}.$$

$\hookrightarrow$ $f(x) = \frac{1}{5}(x-3)^2(x+1)(x-5)$