H

National Qualifications

X847/76/11

Mathematics
Paper 1 (Non-calculator)

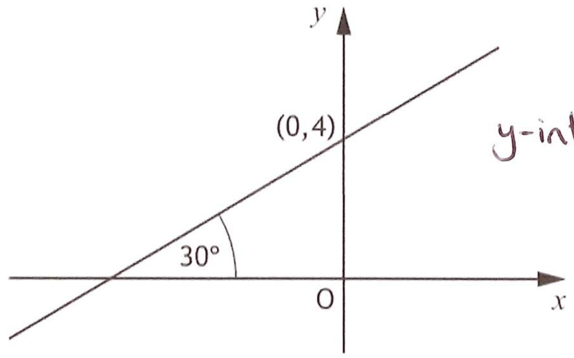
Duration — 1 hour 15 minutes

2024 PAPER 1 – WORKED SOLUTIONS

Total marks — 55

HWallace.

1. A line passes through the point $(0, 4)$ and makes an angle of $30°$ with the positive direction of the x-axis as shown in the diagram.

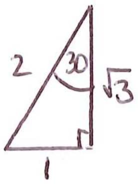


y-intercept, $c = 4$

Determine the equation of the line. **3**

Gradient, $m = \tan\theta$

$m = \tan 30°$



$m = \frac{1}{\sqrt{3}}$

Equation

$y = mx + c$

$y = \frac{1}{\sqrt{3}}x + 4$

2. A sequence is defined by the recurrence relation $u_{n+1} = \frac{1}{5}u_n + 12$ with $u_1 = 20$.

(a) Calculate the value of u_2. **1**

(b) (i) Explain why this sequence approaches a limit as $n \to \infty$. **1**

(ii) Calculate this limit. **2**

a) $U_2 = \frac{1}{5}U_1 + 12$

$U_2 = \frac{1}{5}(20) + 12$

$U_2 = 4 + 12$

$U_2 = 16$

b) (i) A limit exists since $-1 < \frac{1}{5} < 1$

(ii) Limit $= \frac{12}{1 - \frac{1}{5}} = \frac{12}{\frac{4}{5}}$

$= 12 \times \frac{5}{4}$

Limit $= 15$

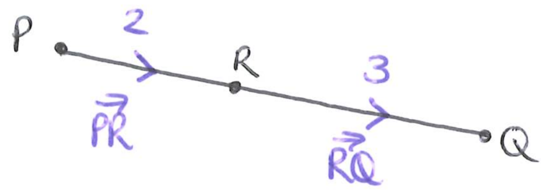
3. Given that $y = (5x^2 + 3)^7$, find $\frac{dy}{dx}$. **2**

$$\frac{dy}{dx} = 7(5x^2 + 3)^6 \times 10x$$

$$\frac{dy}{dx} = 70x\,(5x^2 + 3)^6.$$

4. P and Q have coordinates $(-6, 1, 2)$ and $(-1, 11, -8)$ respectively.

Find the coordinates of the point R which divides PQ in the ratio 2:3. **2**



$$\frac{\overrightarrow{PR}}{\overrightarrow{RQ}} = \frac{2}{3}$$

$$3\overrightarrow{PR} = 2\overrightarrow{RQ}$$

$$3(r - p) = 2(q - r)$$

$$3r - 3p = 2q - 2r$$

$$5r = 3p + 2q$$

$$5r = 3\begin{pmatrix}-6\\1\\2\end{pmatrix} + 2\begin{pmatrix}-1\\11\\-8\end{pmatrix}$$

$$5r = \begin{pmatrix}-18\\3\\6\end{pmatrix} + \begin{pmatrix}-2\\22\\-16\end{pmatrix}$$

$$5r = \begin{pmatrix}-20\\25\\-10\end{pmatrix}$$

$$r = \begin{pmatrix}-4\\5\\-2\end{pmatrix} \quad \therefore \quad \text{Point } R\,(-4, 5, -2)$$

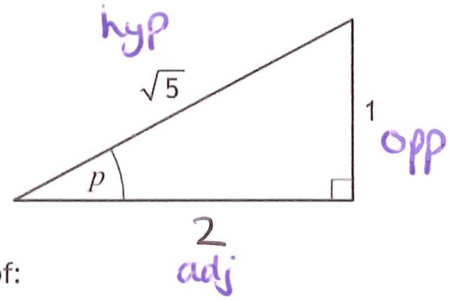
5. A function, h, is defined by $h(x) = 2x^3 - 7$ where $x \in \mathbb{R}$.

Find the inverse function, $h^{-1}(x)$. **3**

Let $y = 2x^3 - 7$

$$2x^3 - 7 = y$$
$$2x^3 = y + 7$$
$$x^3 = \frac{y+7}{2}$$
$$x = \sqrt[3]{\frac{y+7}{2}} \quad \therefore \quad h^{-1}(x) = \sqrt[3]{\frac{x+7}{2}}$$

6. The right-angled triangle in the diagram is such that $\sin p = \frac{1}{\sqrt{5}}$ and $0 < p < \frac{\pi}{4}$.



$$(\sqrt{5})^2 - 1^2 = 4$$
$$\sqrt{4} = 2$$

(a) Determine the value of:

(i) $\sin 2p$ **3**

(ii) $\cos 2p$. **1**

(b) Hence determine the value of $\sin 4p$. **1**

$$\sin p = \frac{1}{\sqrt{5}}$$
$$\cos p = \frac{2}{\sqrt{5}}$$

a) $\sin 2p = 2\sin p \cos p$
$$= 2\left(\frac{1}{\sqrt{5}}\right)\left(\frac{2}{\sqrt{5}}\right)$$
$$\sin 2p = \frac{4}{5}$$

$\cos 2p = \cos^2 p - \sin^2 p$
$$= \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{1}{\sqrt{5}}\right)^2$$
$$\cos 2p = \frac{3}{5}$$

b) $\sin 4p = \sin 2(2p) = 2\sin 2p \cos 2p$
$$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right) = \frac{24}{25}.$$

7. The line $y = 2x$ is a tangent to the circle with equation $x^2 + y^2 - 14x - 8y + 45 = 0$.

Determine the coordinates of the point of contact. 4

$$x^2 + (2x)^2 - 14x - 8(2x) + 45 = 0$$

$$x^2 + 4x^2 - 14x - 16x + 45 = 0$$

$$5x^2 - 30x + 45 = 0$$

$$5(x^2 - 6x + 9) = 0$$

$$5(x-3)(x-3) = 0$$

$$5(x-3)^2 = 0$$

$$\therefore x - 3 = 0$$

$$x = 3$$

If $y = 2x$

$y = 2(3) = 6$

Point of tangency $(3,6)$.

8. The equation $x^2 + (m-4)x + (2m-3) = 0$ has no real roots.

Determine the range of values for m. 4

Justify your answer.

$b^2 - 4ac < 0$ for no real roots

$a = 1$

$b = (m-4)$

$c = (2m-3)$

$$(m-4)^2 - 4(1)(2m-3) < 0$$

$$m^2 - 8m + 16 - 8m + 12 < 0$$

$$m^2 - 16m + 28 < 0$$

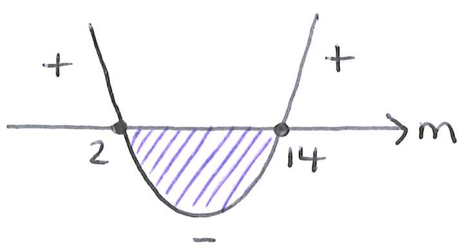
Consider roots

$m^2 - 16m + 28 = 0$

$(m-2)(m-14) = 0$

$m = 2$ and $m = 14$

Sketch:



Solution: $2 < m < 14$

9. Express $\log_a 5 + \log_a 80 - 2\log_a 10$ in the form $\log_a k$ where k is a positive integer. 3

$$\log_a 5 + \log_a 80 - \log_a 10^2$$

$$\log_a 5 + \log_a 80 - \log_a 100$$

$$\log_a \left(\frac{5 \times 80}{100}\right)$$

$$= \underline{\underline{\log_a 4}}$$

10. (a) Show that $(x-1)$ is a factor of $2x^4+3x^3-4x^2-3x+2$. **2**

$$\begin{array}{r|rrrrr} 1 & 2 & 3 & -4 & -3 & 2 \\ & \downarrow & 2 & 5 & 1 & -2 \\ \hline & 2 & 5 & 1 & -2 & \boxed{0} \end{array}$$

Since the remainder is zero, then $x=1$ is a root, and $(x-1)$ is a factor!

(b) Hence, or otherwise, factorise $2x^4+3x^3-4x^2-3x+2$ fully. **4**

$$2x^4+3x^3-4x^2-3x+2$$
$$(x-1)(2x^3+5x^2+x-2)$$

Try $x=-1$

$$\begin{array}{r|rrrr} -1 & 2 & 5 & 1 & -2 \\ & \downarrow & -2 & -3 & 2 \\ \hline & 2 & 3 & -2 & \boxed{0} \end{array}$$

Remainder zero $\therefore$ $(x+1)$ is a factor!

$\hookrightarrow (x-1)(x+1)(2x^2+3x-2)$

$= \underline{\underline{(x-1)(x+1)(2x-1)(x+2)}}$

11. (a) Express $\cos x° + \sqrt{3}\sin x°$ in the form $k\cos(x-a)°$, where $k>0$ and $0<a<360$. **4**

$$k\cos(x-a) = k\cos x\cos a + k\sin x\sin a$$
$$= k\cos a.\cos x + k\sin a.\sin x$$
$$\rightarrow \quad 1.\cos x + \sqrt{3}.\sin x$$

$k\cos a = 1$

$k\sin a = \sqrt{3}$

$k = \sqrt{1^2 + (\sqrt{3})^2}$

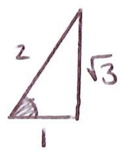
$k = \sqrt{4}$

$k = 2$

$\tan a = \dfrac{k\sin a}{k\cos a}$

$\tan a = \dfrac{\sqrt{3}}{1}$

$a = 60°$



$\hookrightarrow \underline{\cos x° + \sqrt{3}\sin x° = 2\cos(x-60)°}$

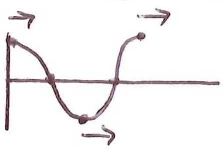
(b) Hence, or otherwise, sketch the graph with equation $y = \cos x° + \sqrt{3}\sin x°$, $0 \le x \le 360$.

Use the diagram provided in your answer booklet. **3**

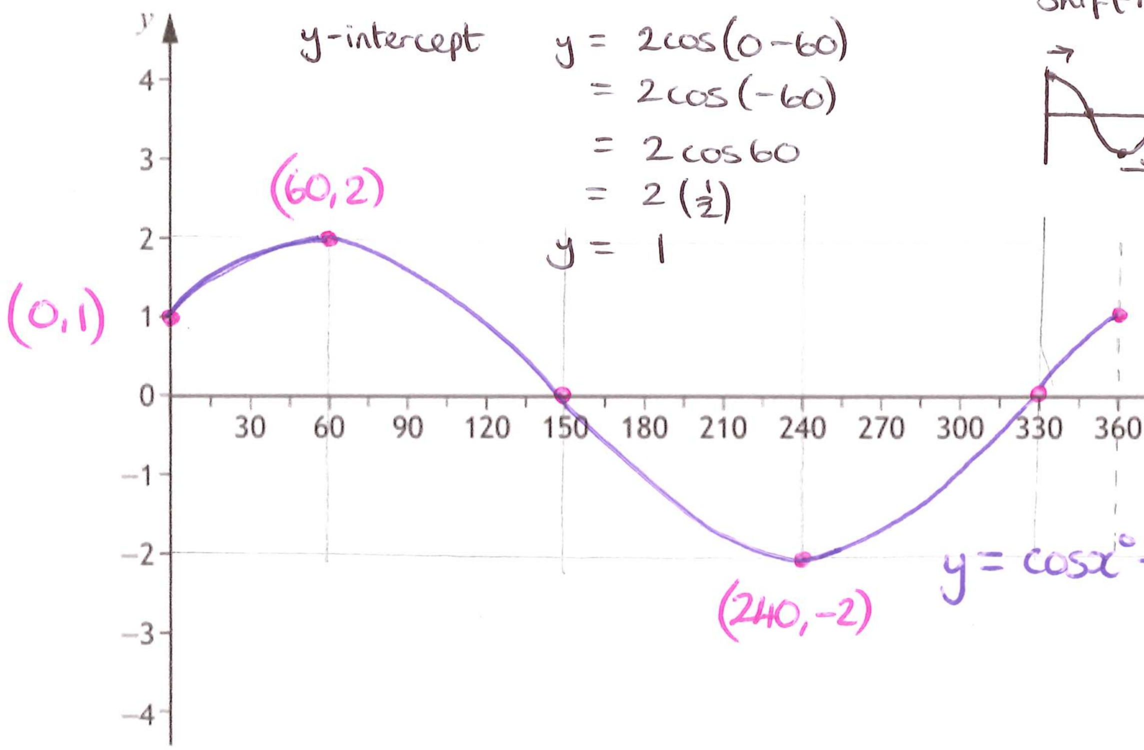
$\hookrightarrow$ sketch $y = 2\cos(x-60)°$

Amplitude = 2

Shift right by 60°



y-intercept $y = 2\cos(0-60)$
$= 2\cos(-60)$
$= 2\cos 60$
$= 2\left(\tfrac{1}{2}\right)$
$y = 1$



12. The function f is given by $f(x) = 12\sqrt[3]{x}$, $x > 0$.

When $x = a$ the rate of change of f with respect to x is 1.

Determine the value of a. 4

$$f'(a) = 1$$

$$f(x) = 12x^{\frac{1}{3}}$$

$$f'(x) = 4x^{-\frac{2}{3}}$$

$$f'(x) = \frac{4}{\sqrt[3]{x^2}}$$

$$f'(a) = \frac{4}{\sqrt[3]{a^2}} = 1$$

$$4 = \sqrt[3]{a^2}$$

$$4^3 = a^2$$

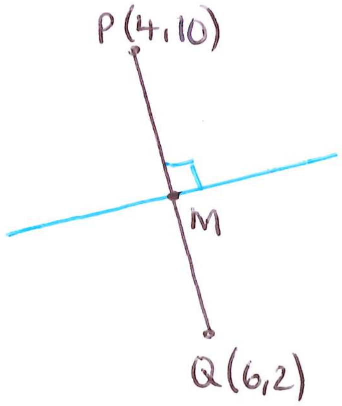
$$64 = a^2$$

$$a = \sqrt{64}$$

$$a = 8 \qquad (a > 0)$$

13. P and Q are the points (4, 10) and (6, 2) respectively.

(a) Find the equation of the perpendicular bisector of PQ. **4**



Midpoint $\left(\frac{4+6}{2}, \frac{10+2}{2}\right)$

Midpoint $(5, 6)$

Gradient PQ.

$m_{PQ} = \frac{2-10}{6-4} = \frac{-8}{2} = -4$

$\therefore m_{\perp} = \frac{1}{4}$

Equation $y - b = m(x-a)$

$y - 6 = \frac{1}{4}(x-5)$

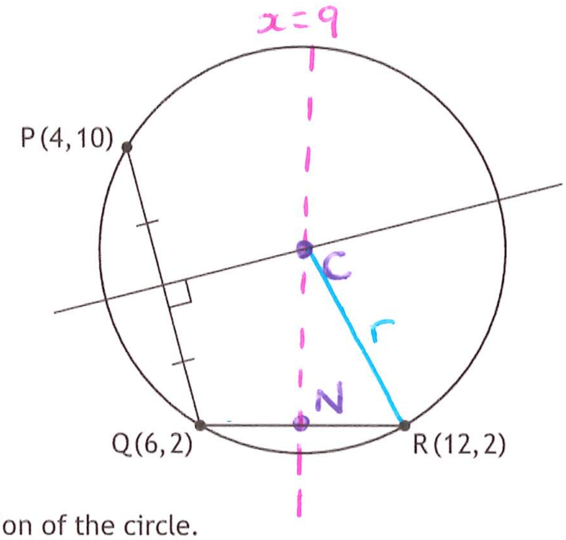
$4y - 24 = x - 5$

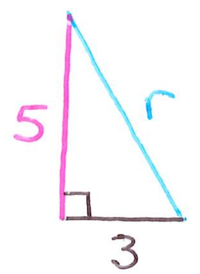
$\underline{4y = x + 19}$

The point R has coordinates (12, 2).

A circle passes through the points P, Q and R.

The chord QR is horizontal.





(b) Find the equation of the circle. **4**

Midpoint N $(9, 2)$

Vertical line $\underline{x = 9}$

Centre C $(9, y)$

$4y = 9 + 19$

$4y = 28$

$y = 7$ $\therefore$ Centre C $(9, 7)$

Radius, $r = \sqrt{3^2 + 5^2}$

$r = \sqrt{34}$

Circle Equation

$(x-a)^2 + (y-b)^2 = r^2$

$\underline{(x-9)^2 + (y-7)^2 = 34}$