H

National
Qualifications

X847/76/12

Mathematics
Paper 2

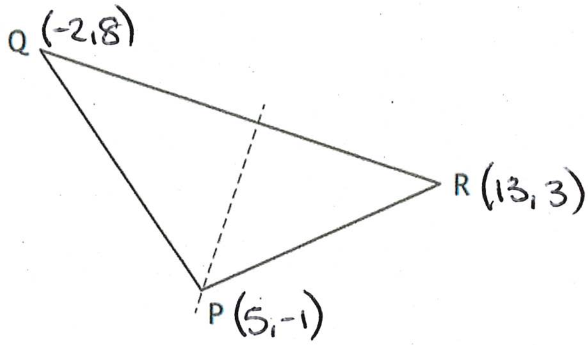
Duration — 1 hour 30 minutes

2023 PAPER 2 - WORKED SOLUTIONS

Total marks — 65

H Wallace

1. Triangle PQR has vertices P(5, −1), Q(−2, 8) and R(13, 3).



(a) Find the equation of the altitude from P. 3

(b) Calculate the angle that the side PR makes with the positive direction of the x-axis. 2

a) Gradient

$m_{QR} = \frac{3-8}{13+2} = \frac{-5}{15} = -\frac{1}{3}$ ✓

$m_{\perp} = 3$ ✓ since $m_{QR} \times m_{\perp} = -1$

$-\frac{1}{3} \times 3 = -1$

Equation

$y+1 = 3(x-5)$

$y+1 = 3x - 15$

$y = 3x - 16$ ✓

b) $m_{PR} = \frac{3-(-1)}{13-5} = \frac{4}{8} = \frac{1}{2}$ ✓

$m = \tan\theta$

$\tan\theta = \frac{1}{2}$

$\theta = \tan^{-1}(\frac{1}{2})$

$\theta = 26.6^{\circ}$ ✓

2. Find the equation of the tangent to the curve with equation $y = 2x^5 - 3x$ at the point where $x = 1$. **4**

P.O.T.

$x = 1$

$y = 2(1)^5 - 3(1)$

$y = 2 - 3$

$y = -1$ ✓

P.O.T. $(1, -1)$

Gradient

$y = 2x^5 - 3x$

$\frac{dy}{dx} = 10x^4 - 3$ ✓

$m = 10(1)^4 - 3$

$m = 10 - 3$

$m = 7$ ✓

Equation

$y + 1 = 7(x - 1)$

$y + 1 = 7x - 7$

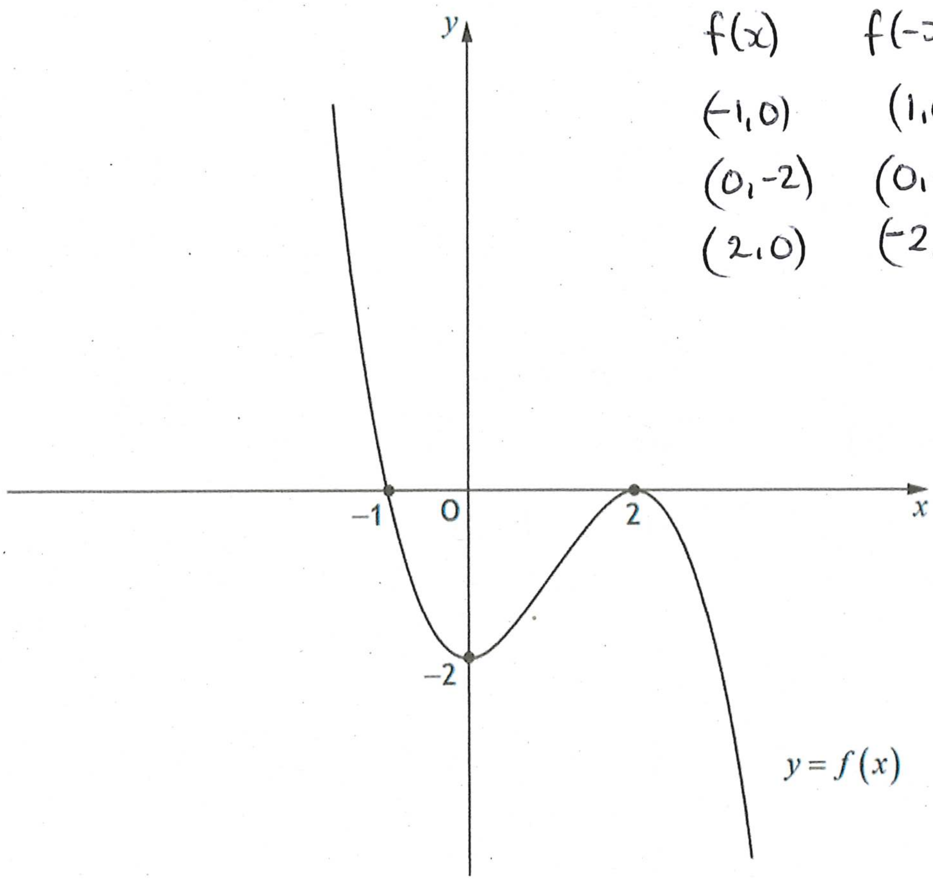
$\underline{y = 7x - 8}$ ✓

3. Find $\int 7\cos\left(4x + \frac{\pi}{3}\right) dx$. **2**

$= 7 \times \frac{1}{4} \sin\left(4x + \frac{\pi}{3}\right) + c$ ✓✓

$= \underline{\frac{7}{4} \sin\left(4x + \frac{\pi}{3}\right) + c}$.

4. The diagram shows the cubic graph of $y = f(x)$, with stationary points at (2, 0) and (0, −2).



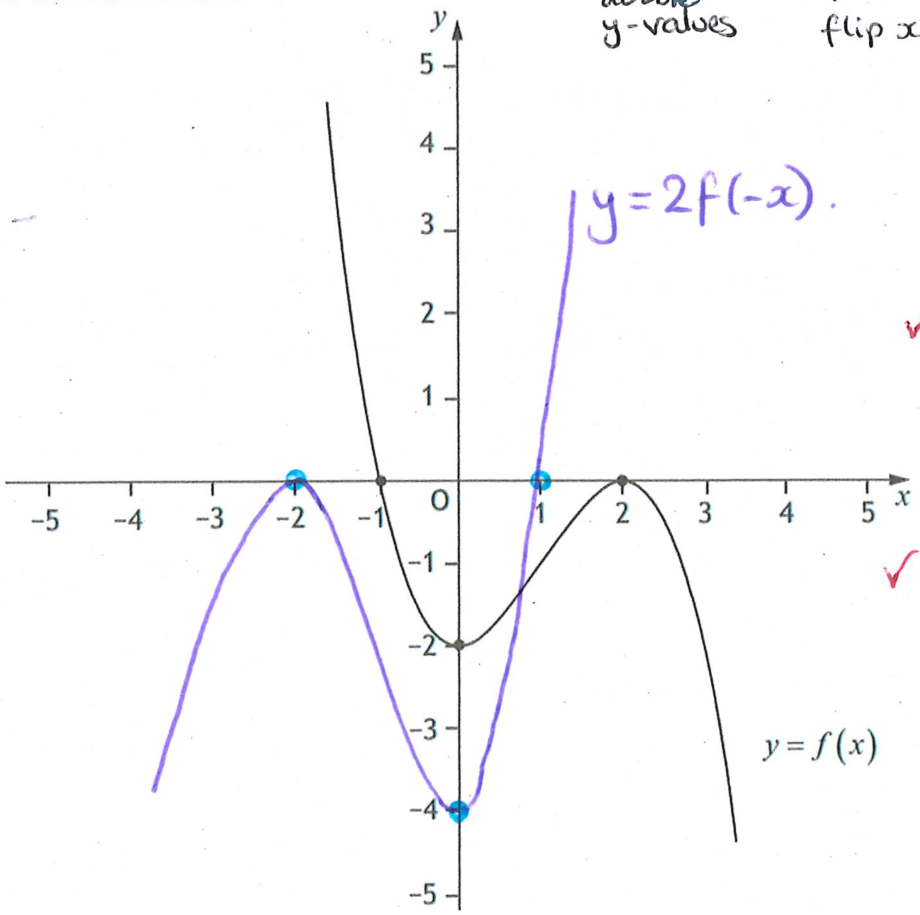
f(x)	f(-x)	2f(-x)
(-1,0)	(1,0)	(1,0)
(0,-2)	(0,-2)	(0,-4)
(2,0)	(-2,0)	(-2,0)

On the diagram in your answer booklet, sketch the graph of $y = 2f(-x)$. **2**

double y-values

reflect over y, flip x-values.

Diagram from the Answer Booklet:



✓ cubic graph with correct roots reflected

✓ vertical stretch

5. A function, f, is defined by $f(x) = (3-2x)^4$, where $x \in \mathbb{R}$.

Calculate the rate of change of f when $x = 4$. 3

$f(x) = (3-2x)^4$

$f'(x) = 4(3-2x)^3 \times -2$

$f'(x) = -8(3-2x)^3$

$f'(4) = -8(3-2(4))^3$

$= -8(3-8)^3$

$= -8(-5)^3$

$= -8(-125)$

$f'(4) = 1000$

6. A function $f(x)$ is defined by $f(x) = \frac{2}{x} + 3$, $x > 0$.

Find the inverse function, $f^{-1}(x)$. 3

let $y = \frac{2}{x} + 3$

$y - 3 = \frac{2}{x}$

$x(y-3) = 2$

$x = \frac{2}{y-3}$

$\hookrightarrow f^{-1}(x) = \frac{2}{x-3}$

7. Solve the equation $\sin x° + 2 = 3\cos 2x°$ for $0 \le x < 360$. 5

$\sin x° + 2 = 3(1 - 2\sin^2 x°)$

$\sin x° + 2 = 3 - 6\sin^2 x°$

$6\sin^2 x + \sin x - 1 = 0$

consider $6s^2 + s - 1 = 0$

$(3s - 1)(2s + 1) = 0$

↳ $(3\sin x° - 1)(2\sin x° + 1) = 0$

$3\sin x - 1 = 0$

$3\sin x = 1$

$\sin x° = \frac{1}{3}$ S|A T|C

$ra = \sin^{-1}\left(\frac{1}{3}\right) = 19.5°$ (1dp)

$x = 19.5,\ 180 - 19.5$

$x = 19.5,\ 160.5$

$2\sin x + 1 = 0$

$2\sin x = -1$

$\sin x° = -\frac{1}{2}$ S|A T|C

$ra = \sin^{-1}\left(\frac{1}{2}\right) = 30°$

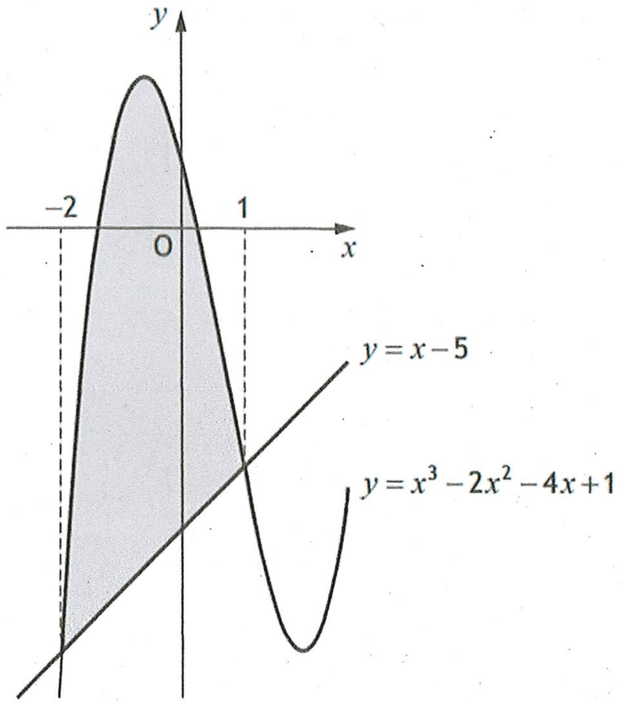
$x = 180 + 30,\ 360 - 30$

$x = 210,\ 330.$

↳ $x° = 19.5°,\ 160.5°,\ 210°,\ 330°.$

8. The diagram shows part of the curve with equation $y = x^3 - 2x^2 - 4x + 1$ and the line with equation $y = x - 5$.

The curve and the line intersect at the points where $x = -2$ and $x = 1$.



Calculate the shaded area. **5**

$$\text{Area} = \int_{-2}^{1} \text{curve} - \text{line} \; dx$$

$$= \int_{-2}^{1} (x^3 - 2x^2 - 4x + 1) - (x - 5) \, . dx$$

$$= \int_{-2}^{1} x^3 - 2x^2 - 4x + 1 - x + 5 \, . \, dx$$

$$= \int_{-2}^{1} x^3 - 2x^2 - 5x + 6 \, . \, dx$$

$$= \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x \right]_{-2}^{1}$$

$$= \left(\frac{1^4}{4} - \frac{2(1)^3}{3} - \frac{5(1)^2}{2} + 6(1) \right) - \left(\frac{(-2)^4}{4} - \frac{2(-2)^3}{3} - \frac{5(-2)^2}{2} + 6(-2) \right)$$

$$\text{Area} = \frac{63}{4} = 15\tfrac{3}{4} \text{ units}^2.$$

9. (a) Express $7\cos x° - 3\sin x°$ in the form $k\sin(x+a)°$ where $k > 0$, $0 < a < 360$. 4

(b) Hence, or otherwise, find:

(i) the maximum value of $14\cos x° - 6\sin x°$ 1

(ii) the value of x for which it occurs where $0 \le x < 360$. 2

a) $k\sin(x+a)° = k\sin x\cos a + k\cos x\sin a$

$= k\sin a.\cos x + k\cos a.\sin x$

$\rightarrow 7.\cos x° - 3.\sin x°$

$k\sin a = 7$

$k\cos a = -3$

$k = \sqrt{7^2+(-3)^2}$

$= \sqrt{49+9}$

$k = \sqrt{58}$

$\tan a = \frac{k\sin a}{k\cos a}$

$\tan a = \frac{7}{-3}$

S	A
T	C

$ra = \tan^{-1}(7/3) = 66.8°$

$a = 180 - 66.8$

$a = 113.2°$

$7\cos x° - 3\sin x° = \sqrt{58}\sin(x+113.2)°$

b) $14\cos x° - 6\sin x° = 2\sqrt{58}\sin(x+113.2)°$

(i) Maximum value is $2\sqrt{58}$.

(ii) $x = 90° - 113.2° = -23.2°$

$+ 360°$

Max value occurs at $x = 336.8°$.

10. Determine the range of values of x for which the function $f(x) = 2x^3 + 9x^2 - 24x + 6$ is strictly decreasing. 4

$$f(x) = 2x^3 + 9x^2 - 24x + 6$$

$$f'(x) = 6x^2 + 18x - 24$$ ✓

SP's occur when $f'(x) = 0$

$$\therefore \quad 6x^2 + 18x - 24 = 0$$

$$6(x^2 + 3x - 4) = 0$$

$$6(x+4)(x-1) = 0$$

$x + 4 = 0$ $x - 1 = 0$

$x = -4$ $x = 1$ ✓

Nature Table

x	$\rightarrow$	-4	$\rightarrow$	1	$\rightarrow$
$f'(x)$	$+$	0	$-$	0	$+$
slope	/	—	\	—	/

✓

Function $f(x)$ is decreasing when $-4 < x < 1$. ✓

11. Circle C_1 has equation $(x-4)^2+(y+2)^2=37$.

Circle C_2 has equation $x^2+y^2+2x-6y-7=0$.

(a) Calculate the distance between the centres of C_1 and C_2. **3**

(b) Hence, show that C_1 and C_2 intersect at two distinct points. **3**

a) Circle C_1

$(x-4)^2+(y+2)^2=37$

$(x-a)^2+(y-2)^2=r^2$

Centre $C_1\ (4,-2)$ ✓

Circle C_2

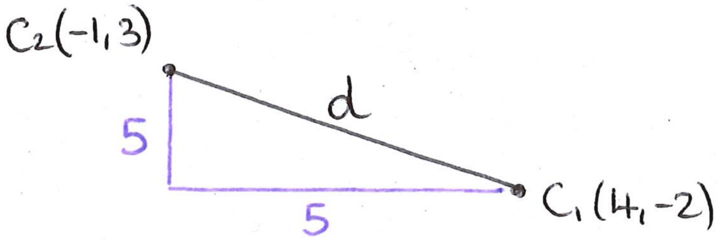
$x^2+y^2+2x-6y-7=0$

$x^2+y^2+2gx+2fy+c=0$

$2g=2 \qquad 2f=-6$

$g=1 \qquad f=-3$

Centre $C_2\ (-1,3)$ ✓



$d^2=5^2+5^2$

$d=\sqrt{50}$

$d=5\sqrt{2}$ units. ✓

b) Radius

$r_1=\sqrt{37}$ ✓

$r_2=\sqrt{f^2+g^2-c}$

$=\sqrt{(-3)^2+(1)^2-(-7)}$

$r_2=\sqrt{17}$ ✓

$r_1+r_2=\sqrt{37}+\sqrt{17}=10.2058\ldots$

$d=5\sqrt{2}=7.0710\ldots$

Since $r_1+r_2>d$, then the circles intersect at two distinct points. ✓

12. A curve, for which $\frac{dy}{dx} = 8x^3 + 3$, passes through the point $(-1, 3)$.

Express y in terms of x. **4**

$$\frac{dy}{dx} = 8x^3 + 3$$

$$y = \int 8x^3 + 3 \, dx$$

$$y = \frac{8x^4}{4} + 3x + c$$

$$y = 2x^4 + 3x + c$$

At point $(-1, 3)$
$\quad\quad\quad\quad\ \ x \quad y$

$$3 = 2(-1)^4 + 3(-1) + c$$

$$3 = 2 - 3 + c$$

$$c = 4$$

$$\hookrightarrow \quad y = 2x^4 + 3x + 4$$

13. A patient is given a dose of medicine.

The concentration of the medicine in the patient's blood is modelled by

$$C_t = 11e^{-0.0053\,t}$$

where:

- t is the time, in minutes, since the dose of medicine was given
- C_t is the concentration of the medicine, in mg/l, at time t.

(a) Calculate the concentration of the medicine 30 minutes after the dose was given. **1**

The dose of medicine becomes ineffective when its concentration falls to 0.66 mg/l.

(b) Calculate the time taken for this dose of the medicine to become ineffective. **3**

a) At $t = 30$

$$C_t = 11e^{-0.0053(30)}$$

$$C_t = 9.38 \text{ mg/L}$$

b) Let $C_t = 0.66$ and find t

$$\therefore \quad 11e^{-0.0053t} = 0.66$$

$$e^{-0.0053t} = \frac{0.66}{11}$$

$$\ln e^{-0.0053t} = \ln 0.06$$

$$-0.0053t(\ln e) = \ln 0.06$$

$$-0.0053t = \ln 0.06$$

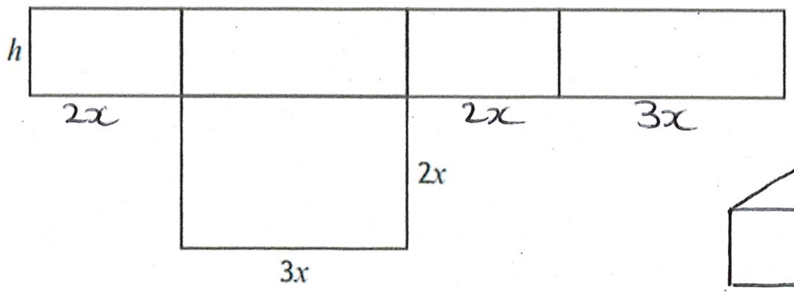
$$t = \frac{\ln 0.06}{-0.0053}$$

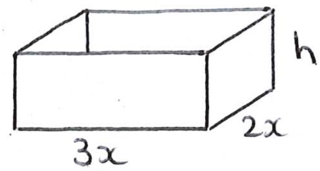
$$t = 530.83\ldots \text{ minutes.}$$

14. A net of an open box is shown.

The box is a cuboid with height h centimetres.

The base is a rectangle measuring $3x$ centimetres by $2x$ centimetres.





(a) (i) Express the area of the net, A cm^2, in terms of h and x. **1**

(ii) Given that $A = 7200$ cm^2, show that the volume of the box, V cm^3, is given by $V = 4320x - \frac{18}{5}x^3$. **2**

(b) Determine the value of x that maximises the volume of the box. **4**

a) (i) Area = base + 2 small sides + 2 larger sides.

$$= (3x)(2x) + 2(2x)(h) + 2(3x)(h)$$

$$= 6x^2 + 4xh + 6xh$$

$$\text{Area} = 6x^2 + 10xh$$

(ii)

$$6x^2 + 10xh = 7200$$

$$10xh = 7200 - 6x^2$$

$$h = \frac{7200 - 6x^2}{10x}$$

$$h = \frac{3600 - 3x^2}{5x} \checkmark$$

Volume = lbh

$$V = (3x)(2x)(h)$$

$$V = 6x^2 h$$

$$V(x) = 6x^2\left(\frac{3600 - 3x^2}{5x}\right) \checkmark$$

$$V(x) = \frac{21600x - 18x^3}{5}$$

$$V(x) = 4320x - \frac{18x^3}{5} \checkmark$$

14b) $V(x) = 4320x - \frac{18}{5}x^3$

$V'(x) = 4320 - \frac{54}{5}x^2$ ✓

SP's occur when $V'(x) = 0$

$0 = 4320 - \frac{54}{5}x^2$ ✓

$\frac{54}{5}x^2 = 4320$

$x^2 = 400$

$x = \sqrt{400}$

$x = \pm 20$

but $x > 0$ for length

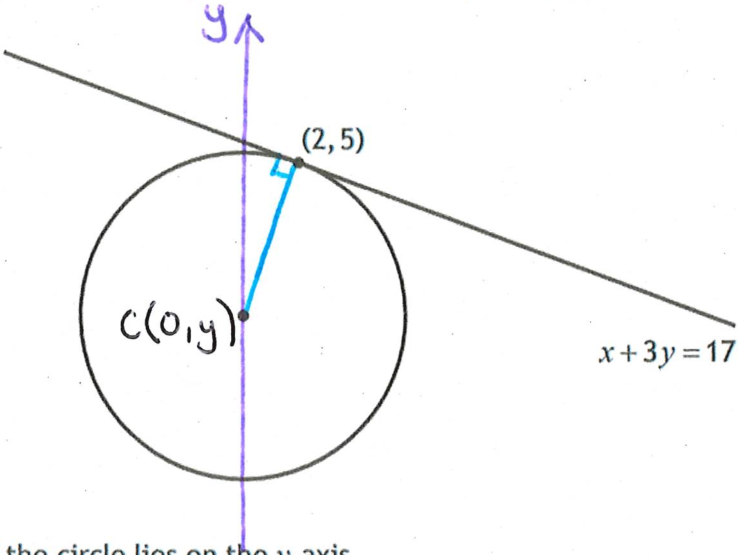
$\therefore$ $x = 20$ cm only ✓

Nature Table

x	$\rightarrow$	20	$\rightarrow$
$V'(x)$	+	0	−
slope	/	—	\

$\therefore$ Volume of box maximises at $x = 20$ cm. ✓

15. The line $x+3y=17$ is a tangent to a circle at the point $(2,5)$.



The centre of the circle lies on the y-axis.

Find the coordinates of the centre of the circle. 4

x-coordinate of centre is zero $\therefore$ C(0, y)

Tangent

$x+3y=17$

$3y=-x+17$

$y=-\frac{1}{3}x+\frac{17}{3}$

$m_{tan}=-\frac{1}{3}$ ✓

Radius

$m_{rad}=3$ ✓

since $m_{tan}\times m_{rad}=-1$

$-\frac{1}{3}\times 3=-1$

Remember $m=\frac{y_2-y_1}{x_2-x_1}$

$\therefore\ 3=\frac{5-y}{2-0}$ ✓

$3=\frac{5-y}{2}$

$6=5-y$

$y=5-6$

$y=-1$

Centre $(0,-1)$ ✓