H

National
Qualifications

X847/76/11

Mathematics
Paper 1 (Non-calculator)

Duration — 1 hour 15 minutes

2023 PAPER 1 - WORKED SOLUTIONS

HWallace

Total marks — 55

1. Given that $y = x^{\frac{5}{3}} - \frac{10}{x^4}$, where $x \neq 0$, find $\frac{dy}{dx}$. 3

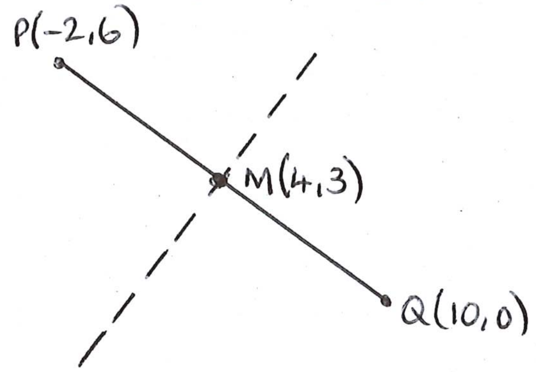
$$y = x^{5/3} - 10x^{-4}$$

$$\frac{dy}{dx} = \frac{5}{3}x^{2/3} + 40x^{-5}$$

$$\frac{dy}{dx} = \frac{5}{3}\sqrt[3]{x^2} + \frac{40}{x^5}$$

2. P and Q are the points $(-2, 6)$ and $(10, 0)$.

Find the equation of the perpendicular bisector of PQ. 4



Midpoint $m\left(\frac{-2+10}{2}, \frac{6+0}{2}\right)$

$m(4, 3)$

Gradient $m_{PQ} = \frac{0-6}{10+2} = \frac{-6}{12} = -\frac{1}{2}$

$m_{\perp} = 2$

Since $m_{PQ} \times m_{\perp} = -1$

$-\frac{1}{2} \times 2 = -1$

Equation

$y - 3 = 2(x - 4)$

$y - 3 = 2x - 8$

$y = 2x - 5$

3. Solve $\log_5 x - \log_5 3 = 2$. **3**

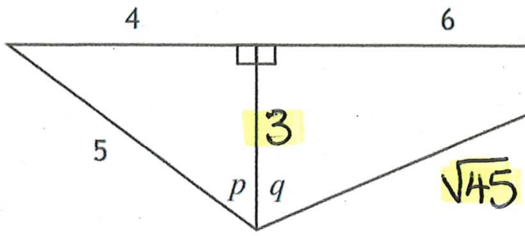
$$\log_5 x - \log_5 3 = 2\log_5 5$$

$$\log_5 \frac{x}{3} \checkmark = \log_5 5^2$$

$$\frac{x}{3} = 25 \checkmark$$

$$\underline{\underline{x = 75.}} \checkmark$$

4. The diagram shows two right-angled triangles with angles p and q as marked.



$\sin p = \frac{4}{5}$

(a) Determine the value of:

(i) $\cos p$ 1

(ii) $\cos q$. 1

$\sin q = \frac{6}{\sqrt{45}} = \frac{2}{\sqrt{5}}$

(b) Hence determine the value of $\cos(p+q)$. 3

a) $\cos p = \frac{3}{5}$ and $\cos q = \frac{3}{\sqrt{45}} = \frac{1}{\sqrt{5}}$

b) $\cos(p+q) = \cos p \cos q - \sin p \sin q$

$= \left(\frac{3}{5}\right)\left(\frac{1}{\sqrt{5}}\right) - \left(\frac{4}{5}\right)\left(\frac{2}{\sqrt{5}}\right)$

$= \frac{3}{5\sqrt{5}} - \frac{8}{5\sqrt{5}}$

$= -\frac{5}{5\sqrt{5}}$

$\cos(p+q) = -\frac{1}{\sqrt{5}}$

$ax^2 + bx + c = 0$

5. The equation $2x^2 + (3p-2)x + p = 0$ has equal roots.

Determine the possible values of p. **3**

$a = 2$

$b = (3p-2)$

$c = p$

For equal roots, $b^2 - 4ac = 0$

$$(3p-2)^2 - 4(2)(p) = 0$$

$$9p^2 - 12p + 4 - 8p = 0$$

$$9p^2 - 20p + 4 = 0$$

$$(9p-2)(p-2) = 0$$

$9p - 2 = 0$ and $p - 2 = 0$

$9p = 2$

$p = \frac{2}{9}$ and $p = 2$

6. Find $\int \left(2x^5 - 6\sqrt{x}\right) dx,\ x \geq 0$. **4**

$$\int 2x^5 - 6x^{1/2}\, dx$$

$$= \frac{2x^6}{6} - \frac{6x^{3/2}}{3/2} + c$$

$$= \frac{x^6}{3} - 4x^{3/2} + c$$

$$= \frac{x^6}{6} - 4\sqrt{x^3} + c$$

7. (a) Evaluate $\log_2 5 + \log_2 \frac{1}{40}$. 2

(b) Given that $a \in \mathbb{R}$ and that $\log_8 a$ is negative, state the range of possible values of a. 1

a) $\log_2 5 + \log_2 \frac{1}{40}$

$= \log_2 \left(\frac{5}{40}\right)$

$= \log_2 \left(\frac{1}{8}\right)$

$= \log_2 \frac{1}{2^3}$

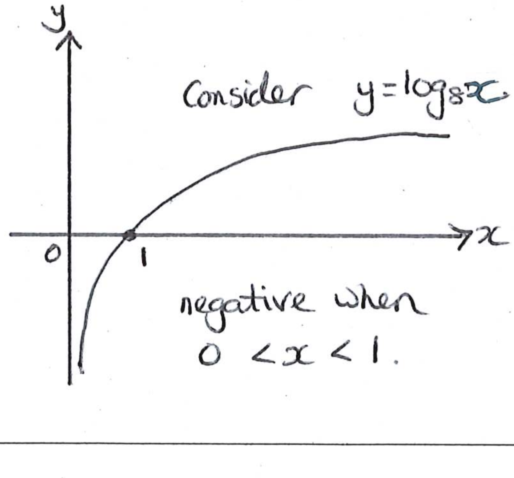
$= \log_2 2^{-3}$

$= -3\log_2 2$

$= \underline{\underline{-3.}}$

b) For $\log_8 a < 0$,

then $\underline{\underline{0 < a < 1.}}$



8. A function, f, is defined on $\mathbb{R}$, the set of real numbers, by $f(x) = x^3 + 3x^2 - 9x + 5$.

Find the coordinates of the stationary points of f and determine their nature. **6**

$f(x) = x^3 + 3x^2 - 9x + 5$

$f'(x) = 3x^2 + 6x - 9$ ✓

SP's occur when $f'(x) = 0$

$\therefore \; 3x^2 + 6x - 9 = 0$ ✓

$3(x^2 + 2x - 3) = 0$

$3(x+3)(x-1) = 0$

$x + 3 = 0$ $\qquad$ $x - 1 = 0$

$x = -3$ $\qquad$ $x = 1$ ✓

$y = (-3)^3 + 3(-3)^2 - 9(-3) + 5$

$y = 32$

SP $(-3, 32)$

$y = (1)^3 + 3(1)^2 - 9(1) + 5$

$y = 0.$ ✓

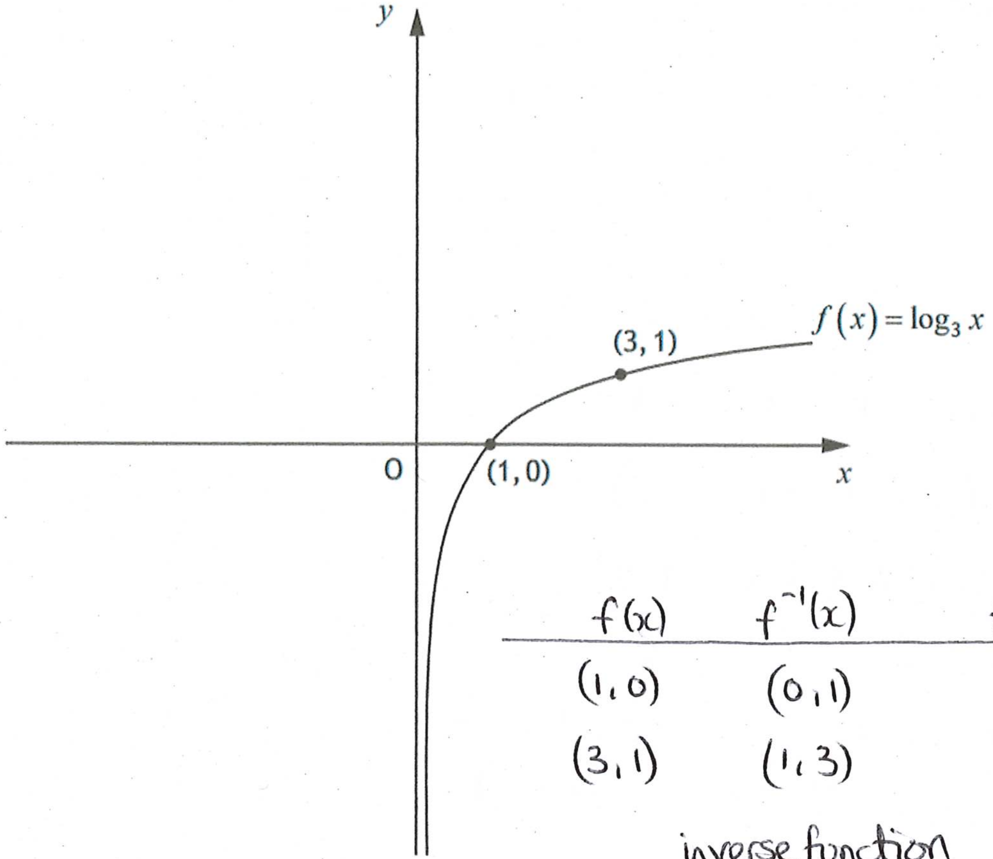
SP $(1, 0)$

Nature Table.

x	$\rightarrow$	-3	$\rightarrow$	1	$\rightarrow$
$f'(x)$	$+$	0	$-$	0	$+$
slope	/	—	\	_	/ ✓

Maximum TP $(-3, 32)$ and Minimum TP $(1, 0)$. ✓

9. The diagram shows the graph of the function $f(x) = \log_3 x$, where $x > 0$.



$f(x)$	$f^{-1}(x)$	$f^{-1}(x) - 1$
$(1,0)$	$(0,1)$	$(0,0)$
$(3,1)$	$(1,3)$	$(1,2)$

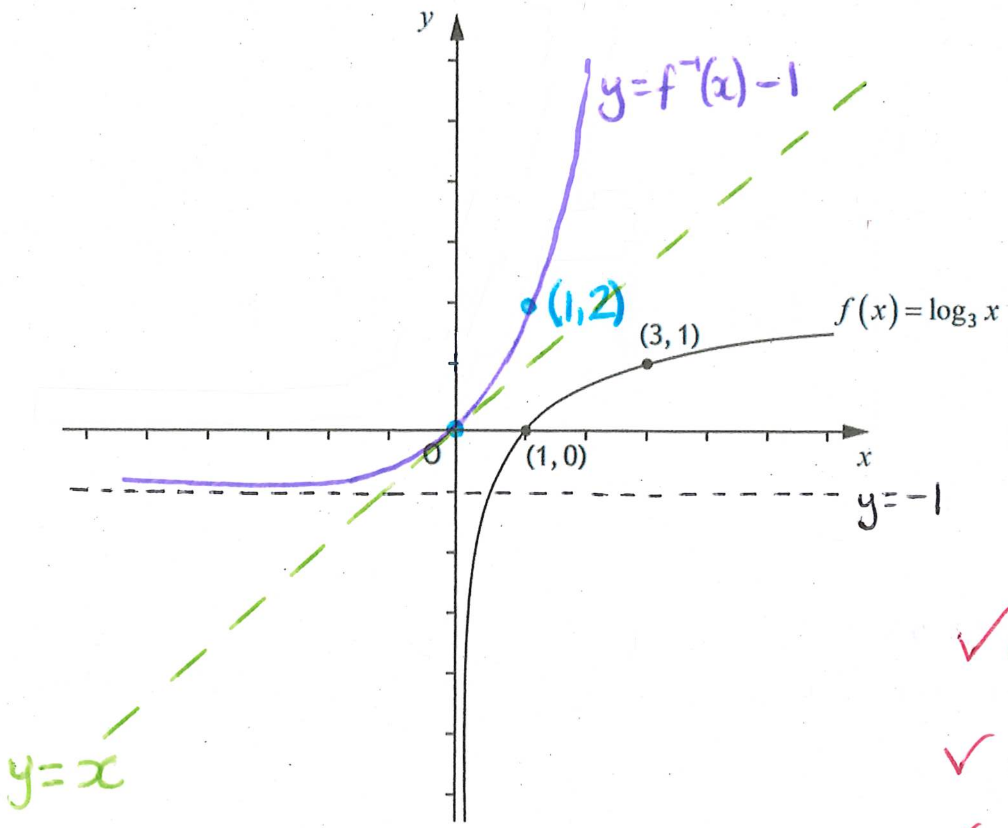
inverse function

down 1.

The inverse function, f^{-1}, exists.

On the diagram in your answer booklet, sketch the graph of $y = f^{-1}(x) - 1$. **3**

Diagram from the Answer Booklet:



✓ exponential curve

✓ points (0,0) and (1,2)

✓ approaching $y = -1$

10. (a) Show that $(x+5)$ is a factor of $x^4+3x^3-7x^2+9x-30$. **2**

(b) Hence, or otherwise, solve $x^4+3x^3-7x^2+9x-30=0$, $x \in \mathbb{R}$. **5**

a) If $(x+5)$ is a factor, then $x=-5$ is a root.

$$\begin{array}{r|rrrrr} -5 & 1 & 3 & -7 & 9 & -30 \\ & \downarrow & -5 & 10 & -15 & 30 \\ \hline & 1 & -2 & 3 & -6 & \boxed{0} \end{array}$$

Since the remainder is zero, $x=-5$ is a root and $(x+5)$ is a factor.

b) $(x+5)(x^4-2x^3+3x-6)=0$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & \downarrow & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & \boxed{0} \end{array}$$

Remainder zero $\therefore$ $(x-2)$ is also a factor.

$\hookrightarrow (x+5)(x-2)(x^2+3)=0$

$x+5=0$ $x-2=0$ $x^2+3=0$

$x=-5$ $x=2$ $x^2=-3$

no solutions

$\hookrightarrow x=-5,\ 2.$

11. (a) Evaluate $\int_{\frac{\pi}{2}}^{\pi} (5\sin x - 3\cos x)\, dx$. **3**

The diagram in your answer booklet shows the graphs with equations $y = 5\sin x$ and $y = 3\cos x$, $0 \le x \le 2\pi$.

(b) On the diagram in your answer booklet, shade the area represented by the integral in (a). **1**

a) $\int_{\frac{\pi}{2}}^{\pi} (5\sin x - 3\cos x)\, .dx$

$= \left[-5\cos x - 3\sin x\right]_{\frac{\pi}{2}}^{\pi}$ ✓

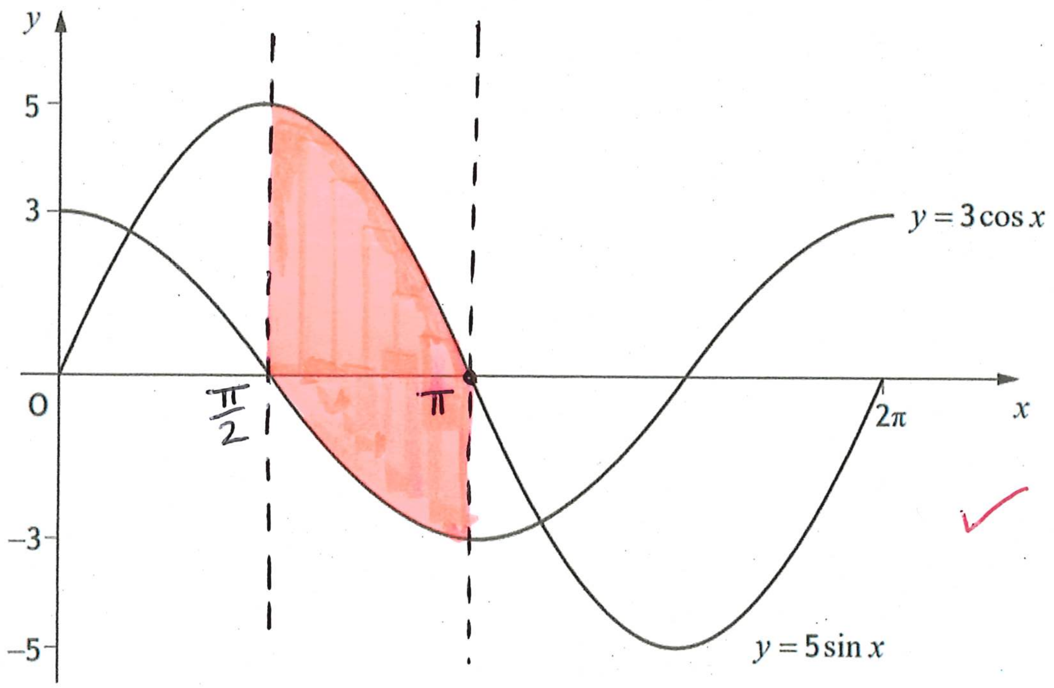
$= \left(-5\cos\pi - 3\sin\pi\right) - \left(-5\cos\frac{\pi}{2} - 3\sin\frac{\pi}{2}\right)$ ✓

$= -5(-1) - 3(0) + 5(0) + 3(1)$

$= 5 - 0 + 0 + 3$

$= \underline{\underline{8}}$ ✓

Diagram from the Answer Booklet:



✓

12. Express $-2x^2 - 12x + 7$ in the form $a(x+b)^2 + c$. **3**

$= -2[x^2 + 6x] + 7$

$= -2[(x+3)^2 - 9] + 7$

$= -2(x+3)^2 + 18 + 7$

$= \underline{\underline{-2(x+3)^2 + 25.}}$

13. Functions f and g are defined by:

- $f(x) = 2\sin x$, where $0 < x < \frac{\pi}{2}$
- $g(x) = 2x$, where $0 < x < \frac{\pi}{4}$

(a) (i) Evaluate $f\left(g\left(\frac{\pi}{6}\right)\right)$. **1**

(ii) Determine an expression for $f(g(x))$. **2**

(b) (i) Given that $f(p) = \frac{1}{3}$, determine the exact value of $\sin p$. **1**

(ii) Hence, determine the exact value of $f(g(p))$. **3**

a) $f(g(x)) = f(2x)$ ✓

$f(g(x)) = 2\sin 2x$ ✓

$f\left(g\left(\frac{\pi}{6}\right)\right) = 2\sin 2\left(\frac{\pi}{6}\right)$

$= 2\sin \frac{\pi}{3}$

$= 2\sin 60°$

$= 2\left(\frac{\sqrt{3}}{2}\right)$

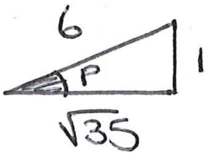
$f\left(g\left(\frac{\pi}{6}\right)\right) = \sqrt{3}.$ ✓

b) $f(p) = \frac{1}{3}$

$f(p) = 2\sin p$

$\therefore\ 2\sin p = \frac{1}{3}$

$\sin p = \frac{1}{6}$ ✓



$\cos p = \frac{\sqrt{35}}{6}$ ✓

$f(g(p)) = f(2p)$

$= 2\sin 2p$

$= 2 \times 2\sin p \cos p$ ✓

$= 4\left(\frac{1}{6}\right)\left(\frac{\sqrt{35}}{6}\right)$

$= \frac{4\sqrt{35}}{36}$

$f(g(p)) = \frac{\sqrt{35}}{9}$ ✓