H

National
Qualifications

X847/76/12

Mathematics
Paper 2

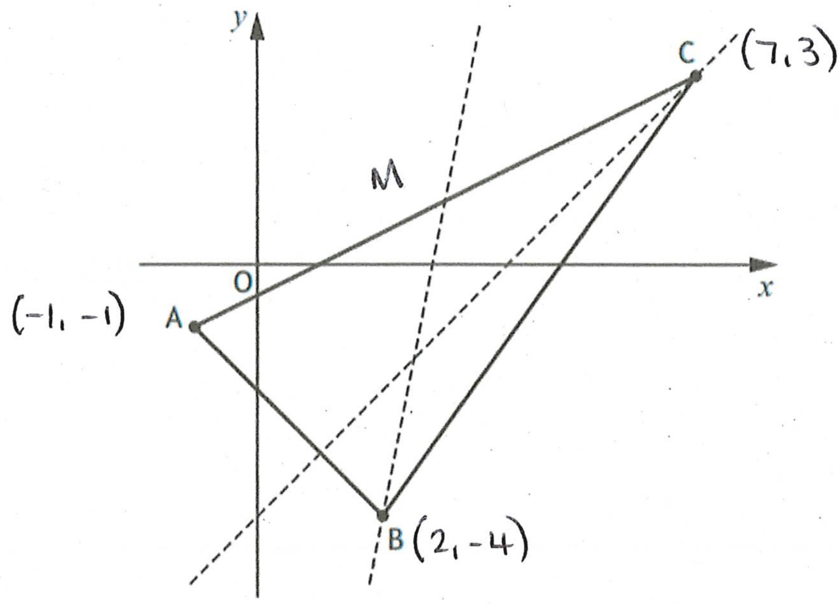
Duration — 1 hour 30 minutes

2022 PAPER 2 – WORKED SOLUTIONS

Total marks — 65

H Wallace

1. Triangle ABC has vertices A(−1, −1), B(2, −4) and C(7, 3).



(a) Find the equation of the altitude through C. **3**

(b) Find the equation of the median through B. **3**

(c) Determine the coordinates of the point of intersection of the altitude through C and the median through B. **2**

a) Gradient

$m_{AB} = \dfrac{-4-(-1)}{2-(-1)}$

$m_{AB} = \dfrac{-3}{3} = -1$ ✓

$m_{\perp} = 1$ ✓

Since $m_{AB} \times m_{\perp} = -1$

$-1 \times 1 = -1$

Equation Altitude

$m = 1$

$a = 7$

$b = 3$

$y - b = m(x - a)$

$y - 3 = 1(x - 7)$

$y - 3 = x - 7$

$\underline{\underline{y = x - 4}}$ ✓

b) Midpoint AC

$M\left(\frac{-1+7}{2}, \frac{-1+3}{2}\right)$

$M(3, 1)$ ✓

Gradient

$m_{Bm} = \frac{-4-1}{2-3} = \frac{-5}{-1} = 5$ ✓

Equation Median

$y - b = m(x - a)$

$y - 1 = 5(x - 3)$

$y - 1 = 5x - 15$

$y = 5x - 14$ ✓

c) Point of intersection

$y = x - 4$

$y = 5x - 14$

Let $y = y$

$5x - 14 = x - 4$

$4x = 10$

$x = \frac{10}{4}$

$x = \frac{5}{2} = 2.5$ ✓

$y = 2.5 - 4$

$y = -1.5$

Point $(2.5, -1.5)$ ✓

2. The equation $2x^2 - 8x + (4-p) = 0$ has two real and distinct roots.

Determine the range of values for p. 3

$$2x^2 - 8x + (4-p) = 0$$
$$ax^2 + bx + c = 0$$

$a = 2$

$b = -8$

$c = (4-p)$

$b^2 - 4ac > 0$ for two real and distinct roots.

$$(-8)^2 - 4(2)(4-p) > 0$$
$$64 - 8(4-p) > 0$$
$$64 - 32 + 8p > 0$$
$$32 + 8p > 0$$
$$8p > -32$$
$$\underline{\underline{p > -4}}$$

3. (a) Express $4\sin x + 5\cos x$ in the form $k\sin(x+a)$ where $k > 0$ and $0 < a < 2\pi$. 4

(b) Hence solve $4\sin x + 5\cos x = 5.5$ for $0 \le x < 2\pi$. 3

a) $k\sin(x+a) = k\sin x\cos a + k\cos x\sin a$ ✓

$= k\cos a.\sin x + k\sin a.\cos x$

$\rightarrow 4.\sin x + 5.\cos x$

$k\cos a = 4$

$k\sin a = 5$ ✓

$k = \sqrt{4^2+5^2}$

$k = \sqrt{41}$ ✓

$\tan a = \dfrac{k\sin a}{k\cos a}$

$\tan a = \dfrac{5}{4}$

S	A
T	C

$ra = \tan^{-1}\left(\frac{5}{4}\right) = 0.896$ (3dp)

$a = 0.896$

$\hookrightarrow$ $4\sin x + 5\cos x = \sqrt{41}\sin(x + 0.896)$ ✓

b) Solve $4\sin x + 5\cos x = 5.5$

$\hookrightarrow$ $\sqrt{41}\sin(x+0.896) = 5.5$ ✓

$\sin(x+0.896) = \dfrac{5.5}{\sqrt{41}}$

S	A
T	C

$ra = \sin^{-1}\left(5.5/\sqrt{41}\right) = 1.033$ (3dp)

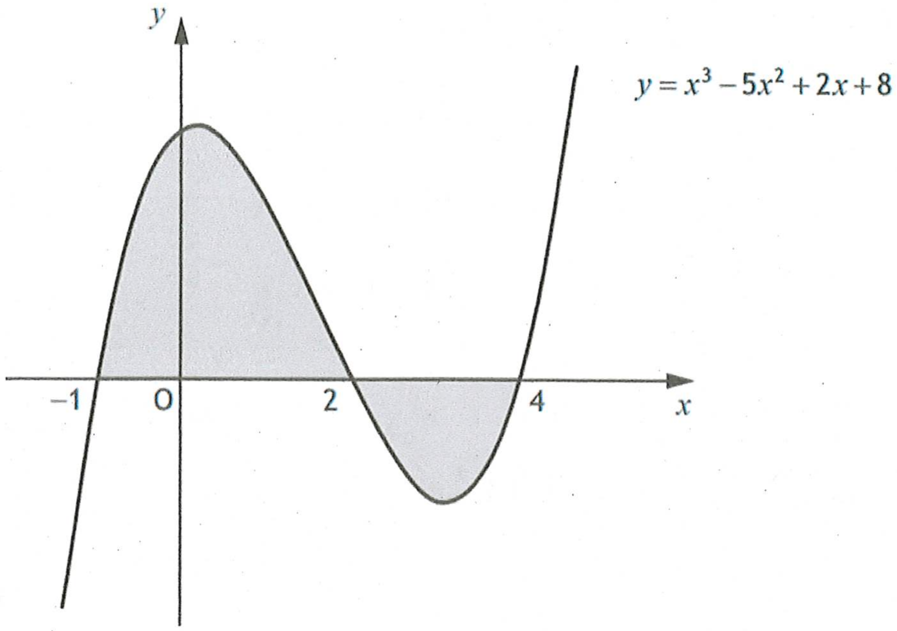
$\hookrightarrow$ $x + 0.896 = 1.033,\ \pi - 1.033$

$x + 0.896 = 1.033,\ 2.108$ ✓

$x = 0.137,\ 1.212$ ✓

4. The graph shown has equation $y = x^3 - 5x^2 + 2x + 8$.

The total shaded area is bounded by the curve and the x-axis.



(a) Calculate the shaded area above the x-axis. 4

(b) Hence calculate the total shaded area. 3

a) Area above $= \int_{-1}^{2} x^3 - 5x^2 + 2x + 8 . \, dx$

$$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + \frac{2x^2}{2} + 8x \right]_{-1}^{2}$$

$$= \left(\frac{(2)^4}{4} - \frac{5(2)^3}{3} + (2)^2 + 8(2) \right)$$

$$- \left(\frac{(-1)^4}{4} - \frac{5(-1)^3}{3} + (-1)^2 + 8(-1) \right)$$

$$= \frac{32}{3} - \left(-\frac{61}{12} \right)$$

$$= \frac{63}{4} = 15\tfrac{3}{4} \text{ units}^2.$$

4b) Below $= \int_2^4 x^3 - 5x^2 + 2x + 8 \, dx$ ✓

$= \left[\frac{x^4}{4} - \frac{5x^3}{3} + x^2 + 8x \right]_2^4$

$= \left(\frac{(4)^4}{4} - \frac{5(4)^3}{3} + (4)^2 + 8(4) \right)$

$- \left(\frac{(2)^4}{4} - \frac{5(2)^3}{3} + (2)^2 + 8(2) \right)$

$= \frac{16}{3} - \frac{32}{3}$

$= -\frac{16}{3}$ ✓

$\therefore$ Area below x-axis $= \frac{16}{3}$ units^2

↳ Total shaded Area $= \frac{63}{4} + \frac{16}{3}$

$= \frac{253}{12}$

$= 21\frac{1}{12}$ units^2. ✓

5. Functions f and g are given by $f(x) = x^2 - 2$ and $g(x) = 3x + 5$, $x \in \mathbb{R}$.

(a) Find expressions for:

(i) $f(g(x))$ and 2

(ii) $g(f(x))$. 1

(b) Determine the range of values of x for which $f(g(x)) < g(f(x))$. 4

a) $f(g(x)) = f(3x+5)$

$f(g(x)) = (3x+5)^2 - 2$

$g(f(x)) = g(x^2-2)$

$= 3(x^2-2) + 5$

$= 3x^2 - 6 + 5$

$g(f(x)) = 3x^2 - 1$

b) $f(g(x)) < g(f(x))$

$(3x+5)^2 - 2 < 3x^2 - 1$

$9x^2 + 30x + 25 - 2 < 3x^2 - 1$

$6x^2 + 30x + 24 < 0$ *

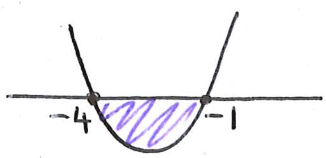
Consider roots $6x^2 + 30x + 24 = 0$

$6(x^2 + 5x + 4) = 0$

$6(x+4)(x+1) = 0$

$x = -4, -1.$

Sketch.



Solution:

$-4 < x < -1$

MARK

6. A curve with equation $y = f(x)$ is such that $\frac{dy}{dx} = 1 - \frac{3}{x^2}$, where $x > 0$.

The curve passes through the point $(3, 6)$.

Express y in terms of x. 5

$$\frac{dy}{dx} = 1 - 3x^{-2}$$

$$y = \int 1 - 3x^{-2} \, dx \checkmark$$

$$y = x \checkmark - \frac{3x^{-1}}{-1} + c \checkmark$$

$$y = x + \frac{3}{x} + c$$

Point $(3, 6)$
$x \; y$

$$6 = 3 + \frac{3}{3} + c \checkmark$$

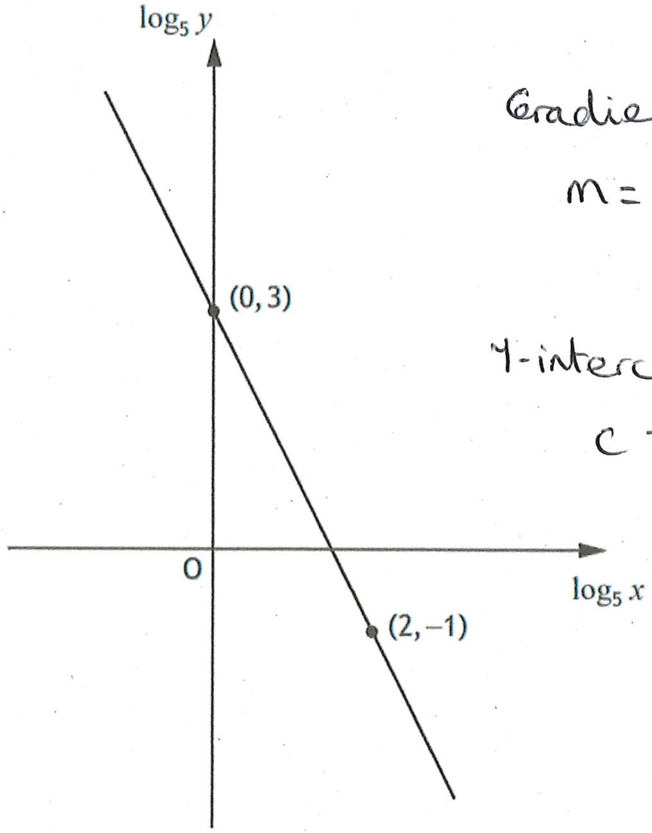
$$6 = 3 + 1 + c$$

$$c = 2$$

$$\hookrightarrow \quad y = x + \frac{3}{x} + 2. \checkmark$$

7. Two variables, x and y, are connected by the equation $y = kx^n$.

The graph of $\log_5 y$ against $\log_5 x$ is a straight line as shown.



Gradient

$m = \frac{-1-3}{2-0} = \frac{-4}{2} = -2.$

Y-intercept

$c = 3$

Find the values of k and n. **5**

$Y = mX + c$

$\log_5 y = -2\log_5 x + 3$

$\log_5 y = -2\log_5 x + 3\log_5 5$

$\log_5 y = \log_5 x^{-2} + \log_5 5^3$

$\log_5 y = \log_5 125x^{-2}$

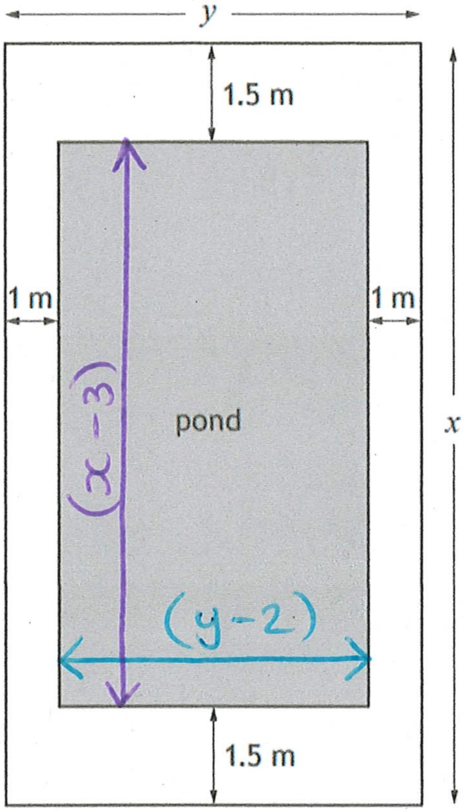
$y = 125x^{-2}$

$k = 125$ and $n = -2.$

8. A rectangular plot consists of a rectangular pond surrounded by a path.

The length and breadth of the plot are x metres and y metres respectively.

The path is 1.5 metres wide at the ends of the pond and 1 metre wide along the other sides as shown.



The total area of the **pond and path together** is 150 square metres.

(a) Show that the area of the pond, A square metres, is given by

$$A(x) = 156 - 2x - \frac{450}{x}.$$ **3**

(b) Determine the maximum area of the pond. **6**

a) Total Area $= xy$

$150 = xy$

$y = \frac{150}{x}$ ✓

Area Pond $= (x-3)(y-2)$ ✓

$A(x) = (x-3)\left(\frac{150}{x} - 2\right)$

$= 150 - 2x - \frac{450}{x} + 6$

$A(x) = 156 - 2x - \frac{450}{x}$ ✓

b) $A(x) = 156 - 2x - \frac{450}{x}$

$A(x) = 156 - 2x - 450x^{-1}$ ✓

$A'(x) = -2 + 450x^{-2}$ ✓

SP's occur at $A'(x) = 0$

$0 = -2 + \frac{450}{x^2}$ ✓

$2 = \frac{450}{x^2}$

$2x^2 = 450$

$x^2 = 225$

$x = \sqrt{225}$

$x = \pm 15$ but $x > 0$ for length.

$\therefore$ $x = 15$ m. ✓

Nature Table.

x	$\rightarrow$	15	$\rightarrow$
$A'(x)$	+	0	−
slope	/	—	\

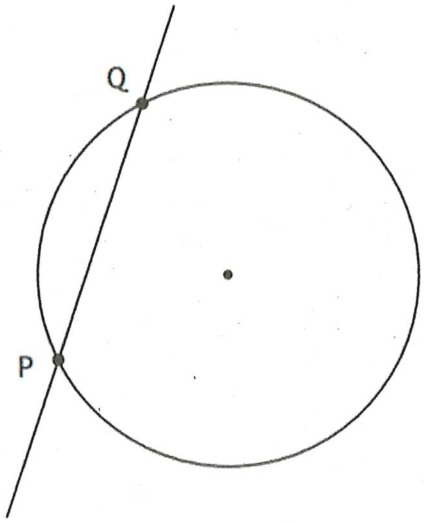
✓

Maximum Area occurs when $x = 15$m.

$A(15) = 156 - 2(15) - \frac{450}{15}$

Maximum Pond Area $= 96\text{m}^2$. ✓

9. The line $y = 3x + 7$ intersects the circle $x^2 + y^2 - 4x - 6y - 7 = 0$ at the points P and Q.



(a) Find the coordinates of P and Q. **5**

$x^2 + y^2 - 4x - 6y - 7 = 0$

$x^2 + (3x+7)^2 - 4x - 6(3x+7) - 7 = 0$ ✓

$x^2 + 9x^2 + 42x + 49 - 4x - 18x - 42 - 7 = 0$

$10x^2 + 20x = 0$ ✓

$10x(x + 2) = 0$ ✓

$10x = 0$ $x + 2 = 0$

$x = 0$ $x = -2$ ✓

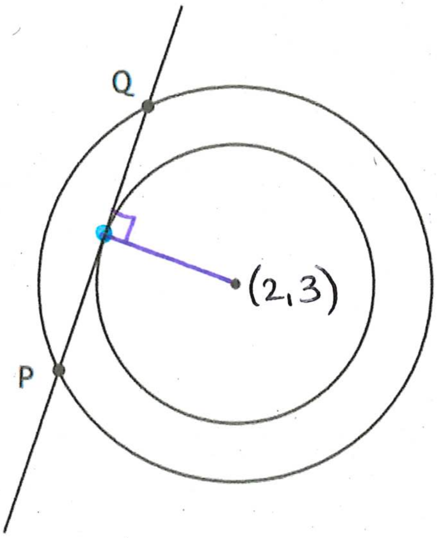
$y = 3(0) + 7$ $y = 3(-2) + 7$

$y = 7$ $y = 1$

↳ Points $P(-2, 1)$ and $Q(0, 7)$ ✓

PQ is a tangent to a second, smaller circle.

This circle is concentric with the first.



(b) Determine the equation of the smaller circle. 4

Large circle: $x^2 + y^2 - 4x - 6y - 7 = 0$

$x^2 + y^2 + 2gx + 2fy + c = 0$

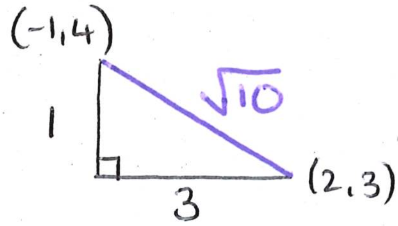
centre $(-g, -f)$ $\quad 2g = -4 \quad 2f = -6$

centre $(2, 3)$ ✓ $\quad g = -2 \quad f = -3$

Midpoint PQ $\quad M\left(\frac{0 + (-2)}{2}, \frac{7 + 1}{2}\right)$

$M(-1, 4)$ ✓

Radius



$r = \sqrt{10}$ units ✓

Equation $\quad (x - a)^2 + (y - b)^2 = r^2$

$(x - 2)^2 + (y - 3)^2 = 10.$ ✓

10. The heptathlon is an athletics contest made up of seven events.

Athletes score points for each event.

In the 200 metres event, the points are calculated using the formula

$$P = 4.99087(42.5 - T)^{1.81}$$

where P is the number of points awarded, and T is the athlete's time, in seconds.

(a) Calculate how many points would be awarded for a time of 24.55 seconds in the 200 metres event. 1

In the long jump event, the points are calculated using the formula

$$P = 0.188807(D - 210)^{k}$$

where P is the number of points awarded, D is the distance jumped, in centimetres, and k is a constant.

(b) Given that 850 points are awarded for a jump of 600 cm, calculate the value of k. 4

a) At $T = 24.55$

$P = 4.99087\,(42.5 - 24.55)^{1.81}$

$P = 929.0368\ldots$

929 points ✓

b) At $P = 850$ and $D = 600$

$850 = 0.188807\,(600 - 210)^{k}$ ✓

$850 = 0.188807\,(390)^{k}$

$\dfrac{850}{0.188807} = 390^{k}$ ✓

$\ln\left(\dfrac{850}{0.188807}\right) = \ln 390^{k}$ ✓

$k = \dfrac{\ln\left(\frac{850}{0.188807}\right)}{\ln 390} = 1.41$ 2dp. ✓